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INFINITELY MANY SOLUTIONS OF A WAVE EQUATION WITH JUMPING NONLINEARITY

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ABSTRACT. We investigate a relation between multiplicity of solutions and source terms of jumping problem in a wave equation when the nonlinearity crosses an eigenvalue and the source term is generated by finite eigenfunctions. We also show that the jumping problem has infinitely many solutions when the source term is positive multiple of the positive eigenfunction.

1. Introduction

We investigate multiplicity of solutions $u(x,t)$ for a piecewise linear perturbation of the one-dimensional wave operator $u_{tt}-u_{xx}$ under Dirichlet boundary condition on the interval $(-\frac{\pi}{2},\frac{\pi}{2})$ and periodic condition on the variable t,

$$u_{tt} - u_{xx} + g(u) = f(x,t) \quad \text{in} \quad (-\frac{\pi}{2},\frac{\pi}{2}) \times R, \tag{1.1}$$

$$u(\pm\frac{\pi}{2},t) = 0, \tag{1.2}$$

$$u \text{ is } \pi - \text{ periodic in } t \text{ and even in } x, \tag{1.3}$$

where we assume that $g(u) = bu^+ - au^-$. When a string with nonuniform density vibrates up and down, the upward restoring coefficient and the downward restoring coefficient of it are different. Hence it happens a nonlinear perturbation in a wave equation. Here we assumed that the

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upward restoring coefficient and the downward one in the vibrating of the string are constant and they are different.

We let L the wave operator, $Lu = u_{tt} - u_{xx}$. Then the eigenvalue problem for $u(x,t)$

$$Lu = \lambda u \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times R$$

with (1.2) and (1.3), has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^2 - 4m^2 \qquad (m, n = 0, 1, 2, \cdots)$$

and corresponding normalized eigenfunctions $\phi_{mn}, \psi_{mn} (m, n \geq 0)$ given by

$$\begin{aligned} \phi_{0n} &= \frac{\sqrt{2}}{\pi} \cos(2n+1)x \quad \text{for} \quad n \geq 0, \\ \phi_{mn} &= \frac{2}{\pi} \cos 2mt \cos(2n+1)x \quad \text{for} \quad m > 0, n \geq 0, \\ \psi_{mn} &= \frac{2}{\pi} \sin 2mt \cos(2n+1)x \quad \text{for} \quad m > 0, n \geq 0. \end{aligned}$$

We note that all eigenvalues in the interval $(-9, 9)$ are given by

$$\lambda_{21} = -7 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{11} = 5.$$

Let Q be the square $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and H the Hilbert space defined by

$$H = \{u \in L^2(Q) : u \ \text{ is even in } \ x\}.$$

Then the set of eigenfunctions $\{\phi_{mn}, \psi_{mn}\}$ is an orthonormal base in H. Hence equation (1.1) with (1.2) and (1.3) is equivalent to

$$Lu + bu^+ - au^- = f \quad \text{in} \quad H. \tag{1.4}$$

In [2] the authors investigate multiplicity of solutions of (1.4) when the nonlinearity $-(bu^+ - au^-)$ crosses the eigenvalue λ_{10} and the source term f is a multiple of the positive eigenfunction ϕ_{00}.

Our concern is to investigate a relation between multiplicity of solutions and source terms of jumping problem (1.4) when the nonlinearity $-(bu^+ - au^-)$ crosses the eigenvalue λ_{10} and the source term f is generated by three eigenfunctions $\phi_{00}, \phi_{10}, \psi_{10}$.

Let $-1 < \theta \leq 1$ and $\phi_\theta = \theta\phi_{10} + \sqrt{1-\theta^2}\psi_{10}$, which is an eigenfunction of L corresponding to λ_{10}. Let H_θ be the subspace of H defined by

$$H_\theta = span\{\{\phi_{mn}, \psi_{mn} : \phi_{mn} \neq \phi_{10}, \psi_{mn} \neq \psi_{10}\} \cup \{\phi_\theta\}\}.$$

In Section 2, we suppose that the nonlinearity $-(bu^+ - au^-)$ crosses an eigenvalue λ_{10} and the source term f is generated by ϕ_{00} and ϕ_θ and we investigate the existence of solutions of the equation

$$Lu + bu^+ - au^- = f \quad \text{in} \quad H_\theta. \tag{1.4'}$$

In Section 3, we reveal a relation between multiplicity of solutions and source terms in equation (1.4′) when f belongs to the two dimensional space $V_\theta = span\{\phi_{00}, \phi_\theta\}$. In Section 4, we reveal a relation between multiplicity of solutions and source terms in equation (1.4) when f belongs to the three dimensional space $V_\theta = span\{\phi_{00}, \phi_{10}, \psi_{10}\}$. We also show that (1.4) has infinitely many solutions in H when the source term is positive multiple of the positive eigenfunction ϕ_{00}.

2. A variational reduction method

In this section, we investigate multiplicity of solutions $u(x,t)$, in H_θ, for a piecewise linear perturbation $-(bu^+ - au^-)$ of the one-dimensional wave operator $u_{tt} - u_{xx}$ with the nonlinearity $-(bu^+ - au^-)$ crossing the eigenvalue λ_{10}. We suppose that $-1 < a < 3$ and $3 < b < 7$. Under this assumption, we have a concern with a relation between multiplicity of solutions in H_θ and source terms of a nonlinear wave equation

$$Lu + bu^+ - au^- = f \quad \text{in} \quad H_\theta. \tag{2.1}$$

Here we suppose that f is generated by two eigenfunctions ϕ_{00} and ϕ_θ.

We shall use the contraction mapping theorem to reduce the problem from an infinite dimensional one in H_θ to a finite dimensional one. We investigate multiplicity of solutions and source terms of equation (2.1).

Let V_θ be the two dimensional subspace of H_θ spanned by $\{\phi_{00}, \phi_\theta\}$ and W be the orthogonal complement of V_θ in H_θ. Let P_θ be an orthogonal projection H_θ onto V_θ. Then every element $u \in H_\theta$ is expressed by $u = v + w$, where $v = P_\theta u$, $w = (I - P_\theta)u$. Hence equation (2.1) is equivalent to a system

$$Lw + (I - P_\theta)(b(v+w)^+ - a(v+w)^-) = 0, \tag{2.2}$$

$$Lv + P_\theta(b(v+w)^+ - a(v+w)^-) = s_1\phi_{00} + s_2\phi_\theta. \tag{2.3}$$

LEMMA 2.1. *For fixed $v \in V_\theta$, (2.2) has a unique solution $w = w_\theta(v)$. Furthermore, $w_\theta(v)$ is* Lipschitz *continuous (with respect to L^2 norm) in terms of v.*

The proof of the lemma is similar to that of Lemma 1.1 of [3]. From now, for simplicity, we denote $w(v) = w_\theta(v)$.

By Lemma 2.1, the study of multiplicity of solutions of (2.1) is reduced to the study of multiplicity of solutions of an equivalent problem

$$Lv + P_\theta(b(v + w(v))^+ - a(v + w(v))^-) = s_1\phi_{00} + s_2\phi_\theta \tag{2.4}$$

defined on the two dimensional subspace V_θ spanned by $\{\phi_{00}, \phi_\theta\}$.

If $v \geq 0$ or $v \leq 0$, then $w(v) \equiv 0$. For example, let us take $v \geq 0$ and $w(v) = 0$. Then equation (2.2) reduces to

$$L0 + (I - P_\theta)(bv^+ - av^-) = 0$$

which is satisfied because $v^+ = v$, $v^- = 0$ and $(I - P_\theta)v = 0$, since $v \in V_\theta$.

Since the subspace V is spanned by $\{\phi_{00}, \phi_\theta\}$ and $\phi_{00}(x,t) > 0$ in Q, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\phi_{00} + c_2\phi_\theta \mid c_1 \geq 0, K_1(\theta)c_1 \leq c_2 \leq K_2(\theta)c_1\}$$

for some $K_1(\theta) < 0, K_2(\theta) > 0$ so that $v \geq 0$ for all $v \in C_1$ and a cone C_3 defined by

$$C_3 = \{v = c_1\phi_{00} + c_2\phi_\theta \mid c_1 \leq 0, K_2(\theta)c_1 \leq c_2 \leq K_1(\theta)c_1\}$$

so that $v \leq 0$ for all $v \in C_3$.

We define a map $\Phi_\theta : V_\theta \to V_\theta$ given by

$$\Phi_\theta(v) = Lv + P_\theta(b(v + w(v))^+ - a(v + w(v))^-), \quad v \in V_\theta. \tag{2.5}$$

Then Φ_θ is continuous on V_θ, since $w(v)$ is continuous on V_θ and hence we have the following lemma (cf. Lemma 2.2 of [3]).

LEMMA 2.2. *$\Phi_\theta(cv) = c\Phi_\theta(v)$ for $c \geq 0$ and $v \in V_\theta$.*

Lemma 2.2 implies that Φ_θ maps a cone with vertex 0 onto a cone with vertex 0. We define the cones C_2, C_4 as follows

$$C_2 = \left\{v = c_1\phi_{00} + c_2\phi_\theta \mid c_2 \geq 0, \frac{1}{K_1(\theta)}c_2 \leq c_1 \leq \frac{1}{K_2(\theta)}c_2\right\}$$

$$C_4 = \left\{v = c_1\phi_{00} + c_2\phi_\theta \mid c_2 \leq 0, \frac{1}{K_2(\theta)}c_2 \leq c_1 \leq \frac{1}{K_1(\theta)}c_2\right\}$$

Then the union of $C_i (1 \le i \le 4)$ is V_θ. We investigate the images of the cones $C_i (1 \le i \le 4)$ under Φ_θ. First we consider the image of the cone C_1. If $v = c_1\phi_{00} + c_2\phi_\theta \ge 0$, we have

$$\begin{aligned}\Phi_\theta(v) &= L(v) + P_\theta(b(v+w(v))^+ - a(v+w(v))^-) \\ &= c_1\lambda_{00}\phi_{00} + c_2\lambda_{10}\phi_\theta + b(c_1\phi_{00} + c_2\phi_\theta) \\ &= c_1(b+\lambda_{00})\phi_{00} + c_2(b+\lambda_{10})\phi_\theta.\end{aligned}$$

Thus the images of the rays $c_1\phi_{00} + K_i(\theta)c_1\phi_\theta (c_1 \ge 0, i = 1, 2)$ can be explicitly calculated and they are

$$c_1(b+\lambda_{00})\phi_{00} + K_i(\theta)c_1(b+\lambda_{10})\phi_\theta \qquad (c_1 \ge 0).$$

Therefore Φ_θ maps C_1 onto the cone

$$R_1 = \left\{ d_1\phi_{00} + d_2\phi_\theta \mid d_1 \ge 0, K_1(\theta)\frac{b+\lambda_{10}}{b+\lambda_{00}}d_1 \le d_2 \le K_2(\theta)\frac{b+\lambda_{10}}{b+\lambda_{00}}d_1 \right\}.$$

The cone R_1 is in the right half-plane of V_θ and the restriction $\Phi_\theta|_{C_1} : C_1 \to R_1$ is bijective.

We determine the image of the cone C_3. If $v = -c_1\phi_{00} + c_2\phi_\theta \le 0$, we have

$$\begin{aligned}\Phi_\theta(v) &= L(v) + P_\theta(b(v+\theta(v))^+ - a(v+\theta(v))^-) \\ &= Lv + P_\theta(av) \\ &= -c_1(\lambda_{00}+a)\phi_{00} + c_2(\lambda_{10}+a)\phi_\theta.\end{aligned}$$

Thus the images of the rays $-c_1\phi_{00} + K_i(\theta)c_1\phi_\theta$ $(c_1 \ge 0, i = 1, 2)$ can be explicitly calculated and they are

$$-c_1(\lambda_{00}+a)\phi_{00} + K_i(\theta)c_1(\lambda_{10}+a)\phi_\theta \qquad (c_1 \ge 0).$$

Thus Φ_θ maps the cone C_3 onto the cone

$$R_3 = \left\{ d_1\phi_{00} + d_2\phi_\theta \mid d_1 \le 0,\ K_1(\theta)\frac{\lambda_{10}+a}{\lambda_{00}+a}d_1 \le d_2 \le K_2(\theta)\frac{\lambda_{10}+a}{\lambda_{00}+a}d_1 \right\}.$$

The cone R_3 is in the left half-plane of V_θ and the restriction $\Phi_\theta|_{C_3} : C_3 \to R_3$ is bijective. We note that R_1 is in the right half plane of V_θ and R_3 is in the left half plane of it.

THEOREM 2.1. (i) *If f belongs to R_1, then equation* (2.1) *has a positive solution and no negative solution.* (ii) *If f belongs to R_3, then equation* (2.1) *has a negative solution and no positive solution.*

Lemma 2.2 means that the images $\Phi_\theta(C_2)$ and $\Phi_\theta(C_4)$ are the cones in the plane V_θ. Before we investigate the images $\Phi_\theta(C_2)$ and $\Phi_\theta(C_4)$, we set

$$R'_2 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_2 \geq 0, \frac{1}{K_2(\theta)} \frac{\lambda_{00}+a}{\lambda_{10}+a} d_2 \leq d_1 \leq \frac{1}{K_2(\theta)} \frac{b+\lambda_{00}}{b+\lambda_{10}} d_2 \right\},$$

$$R'_4 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_2 \leq 0, \frac{1}{K_1(\theta)} \frac{\lambda_{00}+a}{\lambda_{10}+a} d_2 \leq d_1 \leq \frac{1}{K_1(\theta)} \frac{b+\lambda_{00}}{b+\lambda_{10}} d_2 \right\}.$$

Then the union of four cones R_1, R'_2, R_3, R'_4 is also the space V_θ.

To investigate a relation between multiplicity of solutions and source terms in the nonlinear wave equation

$$(2.1) \qquad Lu + bu^+ - au^- = f \quad \text{in} \quad H_\theta,$$

we consider the restrictions $\Phi_\theta|_{C_i}(1 \leq i \leq 4)$ of Φ to the cones C_i. Let $\Phi_{\theta i} = \Phi_\theta|_{C_i}$, i.e.,

$$\Phi_{\theta i} : C_i \to V_\theta.$$

For $i = 1, 3$, the image of $\Phi_{\theta i}$ is R_i and $\Phi_{\theta i} : C_i \to R_i$ is bijective.

From now on, our goal is to find the image of C_i under $\Phi_{\theta i}$ for $i = 2, 4$. Suppose that γ is a simple path in C_2 without meeting the origin, and end points (initial and terminal) of γ lie on the boundary ray of C_2 and they are on each other boundary ray. Then the image of one end point of γ under Φ_θ is on the ray $c_1(b+\lambda_{00})\phi_{00} + K_2(\theta)c_1(b+\lambda_{10})\phi_\theta, c_1 \geq 0$ (a boundary ray of R_1) and the image of the other end point of γ under Φ_θ is on the ray $-c_1(\lambda_{00}+a)\phi_{00} + K_1(\theta)c_1(\lambda_{10}+a)\phi_\theta, c_1 \geq 0$ (a boundary ray of R_3). Since Φ_θ is continuous, $\Phi_\theta(\gamma)$ is a path in V_θ. By Lemma 1.2, $\Phi_\theta(\gamma)$ does not meet the origin. Hence the path $\Phi_\theta(\gamma)$ meets all rays (starting from the origin) in $R_1 \cup R'_4$ or all rays (starting from the origin) in $R'_2 \cup R_3$.

Therefore it follows from Lemma 1.2 that the image $\Phi_\theta(C_2)$ of C_2 contains one of sets $R_1 \cup R'_4$ and $R'_2 \cup R_3$.

Similarly, we have that the image $\Phi_\theta(C_4)$ of C_4 contains one of sets $R_1 \cup R'_2$ and $R'_4 \cup R_3$.

LEMMA 2.3. *Let A be one of the sets $R_1 \cup R'_4$ and $R'_2 \cup R_3$ such that it is contained in $\Phi_\theta(C_2)$. Let γ be any simple path in A with end points on ∂C_2, where each ray (starting from the origin) in A intersect only one point of γ. Then the inverse image $\Phi_{\theta 2}^{-1}(\gamma)$ of γ is a simple path in C_2 with end points on ∂C_2, where any ray (starting from the origin) in C_2 intersects only one point of this path.*

Proof. We note that $\Phi_{\theta 2}^{-1}(\gamma)$ is closed since Φ_θ is continuous and γ is closed in V. Suppose that there is a ray (starting from the origin) in C_2 which intersects two points of $\Phi_{\theta 2}^{-1}(\gamma)$, say, p, $\alpha p\,(\alpha > 1)$. Then by Lemma 1.2,

$$\Phi_{\theta 2}(\alpha p) = \alpha \Phi_{\theta 2}(p),$$

which implies that $\Phi_{\theta 2}(p) \in \gamma$ and $\Phi_{\theta 2}(\alpha p) \in \gamma$. This contradicts that each ray (starting from the origin) in A intersect only one point of γ.

We regard a point p as a radius vector in the plane V_θ. Then for a point p in V_θ, we define the argument $\arg p$ of p by the angle from the positive ϕ_{00}-axis to p.

We claim that $\Phi_{\theta 2}^{-1}(\gamma)$ meets all ray (starting from the origin) in C_2. In fact, if not, $\Phi_{\theta 2}^{-1}(\gamma)$ is disconnected in C_2. Since $\Phi_{\theta 2}^{-1}(\gamma)$ is closed and meets at most one point of any ray in A, there are two points p_1 and p_2 in C_2 such that $\Phi_{\theta 2}^{-1}(\gamma)$ does not contain any point p with

$$\arg p_1 < \arg p < \arg p_2.$$

On the other hand, if we let l the segment with end points p_1 and p_2, then $\Phi_{\theta 2}(l)$ is a path in A, where $\Phi_{\theta 2}(p_1)$ and $\Phi_{\theta 2}(p_2)$ belong to γ. Choose a point q in $\Phi_{\theta 2}(l)$ that $\arg q$ is between $\arg \Phi_{\theta 2}(p_1)$ and $\arg \Phi_{\theta 2}(p_2)$. Then there exist a point q' such that $q' = \beta q$ for some $\beta > 0$. But $\Phi_{\theta 2}^{-1}(q')$ meets l and

$$\arg p_1 < \arg \Phi_{\theta 2}^{-1}(q') < \arg p_2,$$

which is a contradiction. This completes the lemma. □

Similarly, we have the following lemma.

Lemma 2.3′. *Let A be one of the sets $R_1 \cup R_2'$ and $R_4' \cup R_3$ such that it is contained in $\Phi_\theta(C_4)$. Let γ be any simple path in A with end points on ∂A, where each ray (starting from the origin) in A intersect only one point of γ. Then the inverse image $\Phi_{\theta 4}^{-1}(\gamma)$ of γ is a simple path in C_4 with end points on ∂C_4, where any ray (starting from the origin) in C_4 intersects only one point of this path.*

With Lemma 2.3 and Lemma 2.3′, we have the following theorem, which is very important to investigate a relation between the multiplicity of solutions and source terms in a nonlinear wave equation.

Theorem 2.2. *For $i = 2, 4$, if we let $\Phi_{\theta i}(C_i) = R_i$, then R_2 is one of sets $R_1 \cup R_4'$, $R_2' \cup R_3$ and R_4 is one of sets $R_3 \cup R_4'$, $R_1 \cup R_2'$. Furthermore,* *for each* $1 \le i \le 4$, *the restriction* $\Phi_{\theta i}$ *maps* C_i *onto* R_i. *In particular,* $\Phi_{\theta 1}$ *and* $\Phi_{\theta 3}$ *are bijective.*

3. Multiplicity of solutions and source terms in H_θ

In this section we reveal the relation between multiplicity of solutions and source terms in the nonlinear wave equation (2.1). Now we remember the map $\Phi_\theta : V_\theta \to V_\theta$ given by

$$\Phi_\theta(v) = Lv + P_\theta(b(v + w(v))^+ - a(v + w(v))^-), \quad v \in V_\theta,$$

where $-1 < a < 3 < b < 7$, $w(v)$ is a solution of (2.2), and V_θ is the two-dimensional subspace of H_θ spanned by two eigenfunctions ϕ_{00}, ϕ_θ. The map Φ_θ is continuous on V_θ, since $w(v)$ is continuous on V_θ.

For $f \in V_\theta$, we establish an *a priori* bound for solutions of

$$Lv + P_\theta(b(v + z(v))^+ - a(v + z(v))^-) = f \quad \text{in} \quad V_\theta. \tag{3.1}$$

LEMMA 3.1. *Let* $C = \{(a, b) : \frac{1}{\sqrt{b+1}} + \frac{1}{\sqrt{a+1}} = 1\}$. *Let* $k(\ge 16)$ *be fixed and* $f \in V_\theta$ *with* $\|f\| = k$. *Let* $\alpha, \beta, \epsilon > 0$ *be given. Let* $3 + \alpha < b < 7 - \alpha$, $-1 + \beta < a < 3 - \beta$ *satisfy the condition* $\frac{1}{\sqrt{b+1}} + \frac{1}{\sqrt{a+1}} \ne 1$ *and* $dist((a, b), C) \ge \epsilon$. *Then there exists* $R_0 > 0$ *(depending only on* k *and* α, β, ϵ*) such that the solutions* v *of* (3.1) *satisfy* $\|v\| < R_0$.

Proof. Let $-1 < a < 3 < b < 7$, $f \in V_\theta$. Let $v \in V_\theta$ be given. Then there exists a unique solution $z \in W$ of the equation

$$Lz + (I - P_\theta)[b(v + z)^+ - a(v + z)^- - f] = 0 \quad \text{in} \quad W.$$

If $z = z(v)$, then it is continuous on V_θ. In particular $z(v)$ satisfies a uniform Lipschitz in V_θ with respect to the L^2 norm (cf. [3]).

Suppose the lemma does not hold. Then there is a sequence (b_n, a_n, v_n) such that $b_n \in [-1+\alpha, 7-\alpha]$, $a_n \in [-1+\beta, 3-\beta]$ satisfy $dist((a_n, b_n), C) \ge \epsilon$, $\|v_n\| \to +\infty$, and

$$v_n = L^{-1}(f - P_\theta(b_n(v_n + z(v_n))^+ - a_n(v_n + z(v_n))^-) \quad \text{in} \quad V_\theta.$$

Let $u_n = v_n + z(v_n)$. Then the sequence (b_n, a_n, u_n) with $b_n \in [-1+\alpha, 7-\alpha]$, $a_n \in [-1 + \beta, 3 - \beta]$ satisfies $\|u_n\| \to +\infty$ and

$$u_n = L^{-1}(f - b_n u_n^+ + a_n u^-) \quad \text{in} \quad H.$$

Put $w_n = \frac{u_n}{\|u_n\|}$. Then we have

$$w_n = L^{-1}(\frac{f}{\|u_n\|} - b_n w_n^+ + a_n w_n^-).$$

The operator L^{-1} is compact. Therefore we may assume that $w_n \to w_0$, $b_n \to b_0 \in (-1, 7)$, $a_n \to a_0 \in (-1, 3)$ with $(a_0, b_0) \notin C$. Since $\|w_n\| = 1$ for all n, $\|w_0\| = 1$ and w_0 satisfies

$$w_0 = L^{-1}(-b_0 w_0^+ + a w_0^-) \quad \text{in} \quad H_0.$$

This contradicts the fact (Theorem 1.2 of [2]) that for $-1 < a, b < 7$ with the condition $\frac{1}{\sqrt{b+1}} + \frac{1}{\sqrt{a+1}} \neq 1$ the equation $Lu + bu^+ - au^- = 0$ has only the trivial solution. □

LEMMA 3.2. *Let $-1 < a < 3$, $-1 < b < 7$ satisfy*

$$\frac{1}{\sqrt{b+1}} + \frac{1}{\sqrt{a+1}} < 1. \tag{3.2}$$

Let $k(\geq b+1)$ be fixed and $f \in V_\theta$ with $\|f\| = k$. Then we have

$$d(v - L^{-1}(f - P_\theta(b(v + z(v))^+ - a(v + z(v))^-)), B_R, 0) = 1$$

for all $R \geq R_0$.

Proof. Let $b = a = 0$. Then we have

$$d(v - L^{-1}(f), B_R, 0) = 1,$$

since the map is simply a translation of the identity and since $\|L^{-1}(f)\| < R_0$ by Lemma 3.1.

In case $b, a \neq 0 (-1 < a < 3, -1 < b < 7)$ with $\frac{1}{\sqrt{b+1}} + \frac{1}{\sqrt{a+1}} < 1$, the result follows in the usual way by invariance under homotopy, since all solutions are in the open ball B_{R_0}. □

LEMMA 3.3. *Let $-1 < a < 3 < b < 7$ satisfy the condition* (3.2) *and $f = (b+1)\phi_{00}$. Then equation* (3.1) *has a positive solution in* Int C_1, *at least one sign changing solution in* Int C_2, *and at least one sign changing solution in* Int C_4.

Proof. First we compute the degree $(R > R_0)$

$$\begin{aligned} &d(v - L^{-1}(f - P_\theta(b(v + z(v))^+ - a(v + z(v))^-)), B_R \cap C_1, 0) \\ &= d(v - L^{-1}(f - bv), B_R \cap C_1, 0) = -1, \end{aligned}$$

since $v - L^{-1}(f - bv) = 0$ has a unique solution in IntC_1 and $1 + \frac{b}{\lambda_{00}} > 0$, $1 + \frac{b}{\lambda_{10}} < 0$. Since, for $f = (b+1)\phi_{00}$, equation (3.1) has no negative solution in IntC_3,

$$d(v - L^{-1}(f - P_\theta(b(v + z(v))^+ - a(v + z(v))^-)), B_R \cap C_3, 0) = 0.$$

By the domain decomposition lemma,

$$d(v - L^{-1}(f - P_\theta(b(v + z(v))^+ - a(v + z(v))^-)), B_R \cap (C_2 \cup C_4), 0) = 2.$$

Hence equation (3.1) has at least one sign changing solution in Int $(C_2 \cup C_4)$.

Suppose that (3.1) has a solution in Int C_2. Then $\Phi_\theta(C_2) \cap R_1 \neq \phi$ and hence $R_2 = \Phi_\theta(C_2) = R_1 \cup R_4'$ by Theorem 2.2. Let $B : V_\theta \to V_\theta$ be a linear map, where the matrix B is given by

$$\begin{pmatrix} \frac{b+a+2\lambda_{00}}{2} & \frac{b-a}{\sqrt{2}} \\ \frac{b-a}{2\sqrt{2}} & \frac{b+a+2\lambda_{10}}{2} \end{pmatrix}.$$

Then $B(C_2) = R_2 = \Phi_\theta(C_2)$ and $Bv = \Phi_\theta(v)$ for all $v \in \partial C_2$. Now we may assume that the solution of $Bv = f$ is in B_{R_0}. Hence if $0 \leq t \leq 1$ and $R \geq R_0$, then we have

$$tBv + (1-t)\Phi_\theta(v) \neq f, \quad v \in \partial(B_R \cap C_2).$$

So we have

$$\begin{aligned} &d(v - L^{-1}(f - P_\theta(b(v + z(v))^+ - a(v + z(v))^-)), B_R \cap C_2, 0) \\ &= d(v - L^{-1}(f - Bv + Lv), B_R \cap C_2, 0) = 1, \end{aligned}$$

since $Bv = f$ has a unique solution in Int C_2 and $\det(L^{-1}B) > 0$. Since $d(v - L^{-1}(f - P_\theta(b(v + z(v))^+ - a(v + z(v))^-)), B_R, 0) = 1$ and $d(v - L^{-1}(f - P_\theta(b(v + z(v))^+ - a(v + z(v))^-)), B_R \cap C_3, 0) = 0$,

$$d(v - L^{-1}(f - P_\theta(b(v + z(v))^+ - a(v + z(v))^-)), B_R \cap C_4, 0) = 1.$$

Therefore (3.1) has at least one solution in Int C_4.

Similarly, if we assume that (3.1) has a solution in Int C_4, then $d(v - L^{-1}(f - P_\theta(b(v + z(v))^+ - a(v + z(v))^-)), B_R \cap C_4, 0) = 1$ and hence we get

$$d(v - L^{-1}(f - P_\theta(b(v + z(v))^+ - a(v + z(v))^-)), B_R \cap C_2, 0) = 1.$$

Therefore (3.1) has at least one solution in Int C_2. □

With Theorem 2.2 and Lemma 3.3, we get the following.

LEMMA 3.4. *Let* $-1 < a < 3 < b < 7$ *satisfy the condition* (3.2). *For* $1 \leq i \leq 4$, *let* $\Phi_\theta(C_i) = R_i$. *Then* $R_2 = R_1 \cup R_4'$ and $R_4 = R_1 \cup R_2'$, where R_2', R_4' *are the same cones as in section 2.*

Proof. It follows from Lemma 3.3 that $R_2 \cap R_1 \neq \phi$. Since R_2 is one of sets $R_1 \cup R_4'$, $R_3 \cup R_2'$ (Theorem 2.2), the image R_2 of C_2 under Φ must be $R_1 \cup R_4'$.

On the other hand, it follows from Lemma 2.6 that $R_4 \cap R_1 \neq \phi$. Since R_4 is one of sets $R_1 \cup R_2'$, $R_3 \cup R_4'$ (Theorem 2.2), the image R_4 of C_4 under Φ_θ must be $R_1 \cup R_2'$. □

If a solution of (2.4) is in C_1, then it is positive. If a solution of (2.4) is in C_3, then it is negative. If a solution of (2.4) is in Int $(C_2 \cup C_4)$, then it has both signs. Therefore we have the main theorem of this paper with aid of Theorem 2.1, Theorem 2.2, and Lemma 3.4.

THEOREM 3.1. *Let* $-1 < a < 3 < b < 7$ *satisfies the condition* (3.2). *Then we have the followings.*
(i) *If* $f \in$ Int R_1, *then equation* (2.1) *has a positive solution and at least two sign changing solutions in* H_θ.
(ii) *If* $f \in \partial R_1$, *then equation* (2.1) *has a positive solution and at least one sign changing solution in* H_θ.
(iii) *If* $f \in$ Int $R_i'(i = 2, 4)$, *then equation* (2.1) *has at least one sign changing solution in* H_θ.
(iv) If $f \in$ Int R_3, then equation (2.1) *has only the negative solution.*
(v) *If* $f \in \partial R_3$, *then equation* (2.1) *has a negative solution.* □

4. A note on the existence of infinitely many solutions in H

We suppose that $-1 < a < 3$ and $3 < b < 7$. Under this codition, we investigate the existence of solutions in H of a nonlinear wave equation

$$Lu + bu^+ - au^- = f \quad \text{in} \quad H \tag{4.1}$$

in the weak sence; u is a solutions of (4.1) iff $(Lu + bu^+ - au^-, h) = (f, h)$ holds for all $h \in H$. Here we suppose that f is generated by three eigenfunctions ϕ_{00}, ϕ_{10}, ψ_{10}. We reveal a relation between multiplicity of solutions in H and source terms of equation.

Let V be the subspace of H spanned by three eigenfunctions ϕ_{00}, ϕ_{10}, ψ_{10}. Let $f \in V$. Then $f \in V_\theta$ for some θ with $-1 < \theta \leq 1$ and hence f belongs to some cone R_i in V_θ (defined in section 3).

LEMMA 4.1. *Let $f \in V_\theta$. If u_θ is a solution in H_θ of* (2.1), *then it becomes a weak solution in H of* (4.1).

Proof. If $h \in H$, then it is expressed by $h = h_\theta + h_\theta^\perp$, where $h_\theta \in H_\theta$, $h_\theta^\perp \in H_\theta^\perp$. Here $H_\theta^\perp$ is an orthogonal compliment of H_θ in H. Since $Lu_\theta + bu_\theta^+ - au_\theta^- - f \in H_\theta$, for all $h \in H$ we have

$$\begin{aligned}&(Lu_\theta + bu_\theta^+ - au_\theta^- - f, h)\\ &= (Lu_\theta + bu_\theta^+ - au_\theta^- - f, h_\theta + h_\theta^\perp)\\ &= (Lu_\theta + bu_\theta^+ - au_\theta^- - f, h_\theta) + (Lu_\theta + bu_\theta^+ - au_\theta^- - f, h_\theta^\perp)\\ &= 0.\end{aligned}$$

□

Let P be an orthogonal projection H onto V. Then every element $u \in H$ is expressed by $u = v + w$, where $v = Pu$, $w = (I - P)u$. Hence equation (4.1) is equivalent to a system

$$Lw + (I - P)(b(v + w)^+ - a(v + w)^-) = 0, \tag{4.2}$$

$$Lv + P(b(v + w)^+ - a(v + w)^-) = f. \tag{4.3}$$

For fixed $v \in V$, (4.2) has a unique solution $w = w(v)$ (cf. Lemma 2.1). We note that if $v \in V_\theta$ then $Lw + (I - P)(b(v + w)^+ - a(v + w)^-) = Lw + (I - P_\theta)(b(v+w)^+ - a(v+w)^-) = 0$ and hence $w(v) = w_\theta(v)$ for $v \in V_\theta$. Furthermore, $w(v)$ is Lipschitz continuous (with respect to L^2 norm) in terms of v. Hence the multiplicity of solutions of (4.1) is equal to that of (4.3). We investigate the multiplicity of solutions of

$$Lv + P(b(v + w(v))^+ - a(v + w(v))^-) = f. \tag{4.4}$$

Since $\phi_{00} > 0$ in Q, there is a cone $\Gamma_1 \subset V$ such that $u > 0$ for all $u \in \Gamma_1$ and a cone $\Gamma_3 \subset V$ such that $u < 0$ for all $u \in \Gamma_3$. Let $\Gamma_2 = V \backslash (\Gamma_1 \cup \Gamma_3)$. Then every element $u \in \Gamma_2$ has both sign.

We define a map $\Phi : V \to V$ given by $\Phi(v) = Lv + P(b(v + w(v))^+ - a(v + w(v))^-)$. Then $\Phi(v) = \Phi_\theta(v)$ for $v \in V_\theta$. Let $\Sigma_1 = \Phi(\Gamma_1)$, $\Sigma_3 = \Phi(\Gamma_3)$, $\Sigma_2 = V \backslash (\Sigma_1 \cup \Sigma_3)$. Then for any θ $(-1 < \theta \le 1)$ $\Sigma_1 \cap V_\theta = R_1$, $\Sigma_3 \cap V_\theta = R_3$, $\Sigma_2 \cap V_\theta = \mathrm{Int}(C_2 \cup C_4)$ in V_θ. Hence we have the following theorem.

THEOREM 4.1. *Let $-1 < a < 3 < b < 7$ satisfy condition* (3.2). *Then we have:*
(i) *If $f \in \mathrm{Int}\Sigma_1$, then equation* (4.1) *has a postive solution and at least two sign changing solutions in H.*

(ii) *If* $f \in \partial\Sigma_1$, *then equation* (4.1) *has a postive solution and at least one sign changing solution in* H.
(iii) *If* $f \in \Sigma_2$, *then equation* (4.1) *has at least one sign changing solution in* H.
(iv) *If* $f \in \mathrm{Int}\Sigma_3$, *then equation* (4.1) *has only the negative solution in* H.
(v) *If* $f \in \partial\Sigma_3$, *then equation* (4.1) *has a negative solution in* H.

In particular, for the case $f = s\phi_{00}(s > 0)$ we have

THEOREM 4.2. *Let* $-1 < a < 3 < b < 7$ *satisfy condition* (3.2). *For* $f = s\phi_{00}(s > 0)$ *equation* (4.1) *has infinitely many solutions in* H.

Proof. Let $s > 0$. For any $\theta(-1 < \theta \leq 1)$ and $f = s\phi_{00}(s > 0)$

$$Lu + P(bu^+ - au^-) = s\phi_{00} \quad \text{in} \quad H_\theta \tag{4.5}$$

has a positive solution and at least two sign changing solutions in H_θ. By Lemma 3.4, there are solutions $u_{2\theta}$, $u_{4\theta}$ of (4.5) such that $P_\theta(u_{2\theta}) \in C_2$ and $P_\theta(u_{4\theta}) \in C_4$. If $\theta_1 \neq \theta_2(-1 < \theta_1, \theta_2 \leq 1)$, then $u_{2\theta_1} \neq u_{2\theta_2}$ and $u_{4\theta_1} \neq u_{4\theta_2}$. This proves the theorem. □

REMARK. It follows from Theorem 4.2 that the reduced functional $\tilde{F}(v, s)$ $(s > 0)$ in [2, Lemma 2.2 (ii)] has infinitely many critical points. Their critical values except for $\tilde{F}(0, s)$ are equal.

References

[1] 1 A. Ambrosetti and G. Prodi, *A primer of nonlinear analysis*, Cambridge, University Press, Cambridge Studies in Advanced Math. **34** (1993).

[2] 2 Q. H. Choi and T. Jung, *An application of a variational reduction method to a nonlinear wave equation*, J. Differential Equations, **117** (1995), 390-410.

[3] 3 Q. H. Choi, S. Chun and T. Jung, *The multiplicity of solutions and geometry of a nonlinear ellliptic equation*, Studia Math. **120** (1996), 259-270.

[4] 4 A. C. Lazer and P. J. McKenna, *Some multiplicity results for a class of semi-linear elliptic and parabolic boundary value problems*, J. Math. Anal. Appl. **107** (1985), 371-395.

[5] 5 ———, *Critical points theory and boundary value problems with nonlinearities crossing multiple eigenvalues II*, Comm. in P.D.E. **11** (1986), 1653-1676.

[6] 6 K. Schmitt, *Boundary value problems with jumping nonlinearities*, Rocky Mountain Math. J. **16** (1986), 481-496.

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