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LOCAL AND NORM BEHAVIOR OF BLOWUP SOLUTIONS TO A PARABOLIC SYSTEM OF CHEMOTAXIS

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ABSTRACT. We study a parabolic system of chemotaxis introduced by E. F. Keller and L. A. Segel. First, norm behaviors of the blow-up solution are proven. Then some kind of symmetry breaking and the concentration toward the boundary follow when the L^1 norm of the initial value is less than 8π. Meanwhile a method of rearrangement is proposed to prove an inequality of Trudinger-Moser's type.

1. Introduction

In 1970, E. F. Keller and L. A. Segel [8] proposed a system of parabolic equations to describe the aggregation of some organisms sensitive to the gradient of a chemical substance. In a simplified form of Nanjundiah [14], it is given as

$$\text{(KS)}\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla\cdot(\nabla u - \chi u \nabla v) & \text{in} \quad \Omega, \quad t>0, \\ \tau\dfrac{\partial v}{\partial t} = \Delta v - \gamma v + \alpha u & \text{in} \quad \Omega, \quad t>0, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega, \quad t>0, \\ u(\cdot,0)=u_0, \quad v(\cdot,0)=v_0 & \text{on} \quad \Omega, \end{cases}$$

where

1. τ, α, γ and χ are positive constants
2. Ω is a bounded domain of $\mathbf{R}^2$ with smooth boundary $\partial\Omega$
3. $u_0 \not\equiv 0$ and v_0 are smooth and nonnegative functions on $\overline{\Omega}$.

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Here, $u(x,t)$ and $v(x,t)$ denote the density of the organisms and the concentration of the chemical substance, respectively. In use of the semi-group theory, we can prove the existence and uniqueness of the classical solution locally in time, and its regularity and positivity also (Yagi [17], Biler [1]). On the other hand, the asymptotic behavior of the solution, particularly the blowup mechanism, has attracted interests from both mathematical and biological sides.

Simplified forms are proposed, replacing the second equation by the elliptic one, just putting $\tau = 0$ (Nagai [11]) or taking

$$0 = \Delta v + \alpha \left(u - \fint_{\Omega} u_0 \right),$$

where $\fint_{\Omega} = \frac{1}{|\Omega|}\int_{\Omega} \cdot$ (Jäger and Luckhaus [7]). In such systems, the blowup mechanism is understood better. For instance, in the former, we have the estimate

$$\begin{aligned} 2 \times \sharp\,(\text{interior blowup points}) \quad &+ \quad \sharp\,(\text{boundary blowup points}) \\ &\leq \quad \alpha\chi \|u_0\|_1 /4\pi \end{aligned} \tag{1}$$

as a result of the *chemotactic collapse* at each blowup point ([16]). Here and henceforth, $\|\cdot\|_p$ denotes the standard L^p norm. Not so much is known for the full system (KS) but the following are proven by Herrero and Velázquez [6], Nagai, Senba, and Yoshida [13], Biler [1], and Gajewski and Zacharias [5]. Let $T_{\max}$ be the blowup time.

1. The conditions

$$\|u_0\|_1 < 4\pi/(\alpha\chi) \qquad \text{and} \qquad \|u_0\|_1 < 8\pi/(\alpha\chi)$$

imply $T_{\max} = +\infty$ for the general and the radial cases, respectively.

2. There is a family of radial solutions satisfying

$$u(x,t)dx \rightharpoonup 8\pi\delta_0(dx) + f(x)\,dx$$

as $t \uparrow T_{\max} < +\infty$, where $f(x)$ is a nonnegative L^1 function.

Those results suggest inequality (1) even for the system (KS).

The first theorem of the present paper shows a fundamental, but never trivial fact. It is on the norm behavior of blowup solutions. Henceforth, we put $\tau = \alpha = \gamma = \chi = 1$ for simplicity.

THEOREM 1. *If $T_{\max} < +\infty$, we have*

$$\lim_{t\uparrow T_{\max}} \|u \log u\|_1 = \lim_{t\uparrow T_{\max}} \|uv\|_1 = \lim_{t\uparrow T_{\max}} \|\nabla v\|_2^2 = \lim_{t\uparrow T_{\max}} \int_\Omega e^{av} = +\infty, \tag{2}$$

where $a > 1$.

In particular, it follows that $\lim_{t\uparrow T_{\max}} \|u(t)\|_\infty = \lim_{t\uparrow T_{\max}} \|v(t)\|_\infty = +\infty$.

If $\lim_{t\uparrow T_{\max}}$ is replaced by $\limsup_{t\uparrow T_{\max}}$, the above conclusions are already known. They are also known for the simplified systems described before. See [12] and [16], respectively, for those facts.

We also show the following theorems supporting the validity of (1) for (KS).

THEOREM 2. *Let Ω be the unit disc, $\|u_0\|_1 < 8\pi$, and*

$$u_0(x) = u_0(-x), \quad v_0(x) = v_0(-x).$$

Then (KS) *admits a uniformly bounded classical solution, globally in time.*

THEOREM 3. *If Ω is simply connected, $\|u_0\|_1 < 8\pi$, and $T_{\max} < +\infty$, it holds that*

$$\lim_{t\uparrow T_{\max}} \int_{\partial\Omega} e^{v/2} = +\infty. \tag{3}$$

Those theorems say that if $T_{\max} < +\infty$ occurs with $\|u_0\|_1 < 8\pi$, then some kind of symmetry breaking and the concentration toward the boundary are observed to the solution.

We mention some technical facts, where (u, v) denotes the solution of (KS) . First, L^1 norm of u is preserved so that we have

$$\|u(t)\|_1 = \|u_0\|_1 \qquad (0 \le t < T_{\max}).$$

Then, the second equation gives

$$\sup_{0\le t<T_{\max}} \|v(t)\|_{W^{1,q}} < +\infty \tag{4}$$

for any $q \in [1, 2)$. Next, for

$$W(t) = \int_\Omega \{u \log u - uv\} + \frac{1}{2} \|v\|_{H^1}^2 \tag{5}$$

we have

$$\frac{dW}{dt}+\int_\Omega v_t^2+\int_\Omega u\left|\nabla\left(\log u-v\right)\right|^2=0 \qquad \bigl(t\in(0,T_{max})\bigr). \tag{6}$$

Here and henceforth,

$$\|v\|_{H^1}^2=\|\nabla v\|_2^2+\|v\|_2^2.$$

See [13], [1], [5], and [12] for the proof of those facts.

2. Proof of Theorem 1

Gagliardo-Nirenberg's inequality implies the following for $s>1$, where C is a positive constant determined only by Ω. See [16]:

$$\|u\|_2^2\le\frac{C}{\log s}\int_\Omega\left(u\log u+e^{-1}\right)\cdot\int_\Omega u^{-1}\left|\nabla u\right|^2+C\left\|u\right\|_1^2+2s^2\left|\Omega\right| \tag{7}$$

Here, the elementary inequality

$$u\log u+e^{-1}\ge 0 \qquad (u>0) \tag{8}$$

is worth noting.

Chang and Yang's inequality [3] is described as follows, where K is a constant:

$$\log\left(⨍_\Omega e^w\right)\le\frac{1}{8\pi}\left\|\nabla w\right\|_2^2+⨍_\Omega w+K \qquad \bigl(w\in H^1(\Omega)\bigr) \tag{9}$$

The following inequality is a consequence of Jensen's inequality, where $a>0$ and $M=\int_\Omega u$. See[13] for the proof:

$$a\int_\Omega uv\le\int_\Omega u\log u+M\log\left(\int_\Omega e^{av}\right)-M\log M \tag{10}$$

We are ready to give the proof of Theorem 1. In fact, because W is a Lyapunov function, it is monotone decreasing on $[0,T_{max})$. We have either

$$\inf_{0\le t<T_{\max}}W(t)>-\infty \tag{11}$$

or

$$\lim_{t\uparrow T_{\max}}W(t)=-\infty. \tag{12}$$

Assume (12). In use of (8), we have

$$W(t) \geq -|\Omega|\, e^{-1} - \int_\Omega uv.$$

This implies the second relation of (2). From (4) and (9), we have

$$\begin{aligned}\log\left(\int_\Omega e^{av}\right) &\leq \frac{a^2}{8\pi}\|\nabla v\|_2^2 + \frac{a\|v\|_1}{|\Omega|} + K \\ &\leq -\frac{a^2}{4\pi}\int_\Omega u\log u + \frac{a^2}{4\pi}\int_\Omega uv + \frac{a^2}{4\pi}W(t) + C(a+1).\end{aligned}$$

with a constant $C > 0$. Combining this with (10), we get

(13)
$$a\left(1 - \frac{aM}{4\pi}\right)\int_\Omega uv \leq \left(1 - \frac{a^2M}{4\pi}\right)\int_\Omega u\log u + M\left\{\frac{a^2}{4\pi}W(t) + C(a+1)\right\}.$$

Here, $W(t) \leq W(0)$ and $M = \|u(t)\|_1 = \|u_0\|_1$. If we take

$$0 < a < \min\left\{\frac{4\pi}{M}, \left(\frac{4\pi}{M}\right)^{1/2}\right\},$$

then the first relation of (2) follows.

Next, in use of

$$\int_\Omega u\log u \leq W + \int_\Omega uv \tag{14}$$

and Young's inequality

$$a\int_\Omega uv \leq \int_\Omega u\log u + \frac{1}{e}\int_\Omega e^{av}$$

with $a > 0$, we have

$$(a-1)\int_\Omega uv \leq \frac{1}{e}\int_\Omega e^{av} + W(0). \tag{15}$$

Therefore, the fourth relation of (2) follows, and then the third relation is a consequence of (9).

Assume (11). By (6), we have

$$\int_0^{T_{max}} dt \int_\Omega v_t^2 \leq W(0) - \inf_{0\leq t<T_{max}} W(t) \ < \ +\infty. \tag{16}$$

Multiplying $\log u$ by the first equation of (KS), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u\log u &= -\int_\Omega \nabla u\cdot\nabla\log u + \int_\Omega u\nabla v\cdot\nabla\log u \\ &= -\int_\Omega u^{-1}|\nabla u|^2 - \int_\Omega u\Delta v.\end{aligned}$$

Here, the second equation of (KS) gives

$$\begin{aligned}-\int_\Omega u\Delta v &= -\int_\Omega u(v_t+v-u) \le \int_\Omega (u^2-uv_t) \\ &\le \frac{1}{4}\int_\Omega v_t^2 + 2\int_\Omega u^2.\end{aligned}$$

Therefore, in use of (7) we have

$$\frac{d}{dt}\int_\Omega u\log u + D\int_\Omega u^{-1}|\nabla u|^2 \le \frac{1}{4}\int_\Omega v_t^2 + 2C\|u_0\|_1^2 + 4s^2|\Omega|$$

with

$$D = 1 - \frac{2C|\Omega|}{e\log s} - \frac{2C}{\log s}\int_\Omega u\log u.$$

Taking $s > 1$ as

$$s = s(t) = \exp\left(2C\int_\Omega u\log u + \frac{2C|\Omega|}{e}\right)$$

gives $D = 0$, and consequently,

$$\frac{dI}{dt} \le \frac{1}{4}\int_\Omega v_t^2 + 2C\|u_0\|_1^2 + 4|\Omega|\exp\left(4CI + \frac{4C|\Omega|}{e}\right)$$

for

$$I = I(t) \equiv \int_\Omega u\log u.$$

The standard comparison theorem for ordinary differential equations guarantees from (16) that

$$\liminf_{t\uparrow T_{\max}} I(t) < +\infty \quad \Rightarrow \quad \limsup_{t\uparrow T_{\max}} I(t) < +\infty.$$

However, as [13], [1], and [5] have shown, the latter gives $T_{\max} = +\infty$, a contradiction. We have the first relation of (2). Then (14) implies the second relation of (2). This implies the fourth relation and hence the third relation by (9). The proof is complete. □

3. Proof of Theorem 2

In this theorem, Ω denotes the unit disc. From the assumption on the initial values, $v(x,t) = v(-x,t)$ follows for $t \in [0, T_{\max})$. We show that there exist constants $C > 0$ and K satisfying

$$\log\left(\fint_\Omega e^v\right) \le \frac{1}{16\pi}\int_\Omega |\nabla v|^2 + C\fint_\Omega v + K. \tag{17}$$

To this end we make use of the following lemma by Moser [10], where $S^2 \subset \mathbf{R}^3$ denotes the unit sphere.

PROPOSITION 1. *Let f be a C^1 function satisfying $f(x) = f(-x)$ on S^2, we have*

$$\log\left(\fint_{S^2} e^f\right) \le \frac{1}{32\pi}\int_{S^2} |\text{grad } f|^2 + \fint_{S^2} f + K', \tag{18}$$

where K' is an absolute constant.

Let $P \in S^2$ be the north pole and Π the plane perpendicular to the vector $\overrightarrow{OP}$, containing the origin $O \in \mathbf{R}^3$. Then the stereographic projection, denoted by s, is defined from S^2 to $\Pi \cup \{\infty\}$. The function $f_1 = v(\cdot, t) \circ s$ satisfies

(19)

$$\int_{S^2_-} |\text{grad } f_1|^2 = \int_\Omega |\nabla v|^2, \quad \int_{S^2_-} e^{f_1} = \frac{1}{2}\int_\Omega e^v p_*, \quad \int_{S^2_-} f_1 = \frac{1}{2}\int_\Omega v p_*$$

where $S^2_- = \{x = (x_1, x_2, x_3) \in S^2 \mid \ x_3 \le 0\}$ and $p_*(x) = 8/(1+|x|^2)^2$. Setting $S^2_+ = S^2 \backslash S^2_-$, we define

$$f(x) = \begin{cases} f_1(x) & (x \in S^2_-) \\ f_1(-x) & (x \in S^2_+). \end{cases}$$

Then, $f(x)$ is a C^1 function on S^2 satisfying $f(x) = f(-x)$. Inequality (18) is applicable and it follows from (19) that

$$\log\left(\frac{1}{4\pi}\int_\Omega e^v p_*\right) \le \frac{1}{16\pi}\int_\Omega |\nabla v|^2 + \frac{1}{4\pi}\int_\Omega v p_* + K'.$$

This implies (17).

We are ready to complete the proof of Theorem 3. In fact, inequality (10) with $a = 1$ gives

$$\frac{1}{2}\|v\|_{H^1}^2 \leq W(t) + M\log\left(\int_\Omega e^v\right) - M\log M. \tag{20}$$

We have by (17) that

$$\left(\frac{1}{2} - \frac{M}{16\pi}\right)\|v\|_{H^1}^2 \leq W(0) + C$$

with a constant $C > 0$. Because $\|u_0\|_1 = M < 8\pi$, this gives

$$\sup_{0 \leq t < T_{\max}} \|v\|_{H^1} < +\infty$$

and hence $T_{\max} = +\infty$ by Theorem 1. The proof is complete.

4. Proof of Theorem 3

In this theorem, Ω denotes a simply connected domain. For the proof, it suffices to show the following.

PROPOSITION 2. *We have*

$$\log\left(\fint_\Omega e^v\right) \leq \frac{1}{16\pi}\int_\Omega |\nabla v|^2 + \frac{1}{2}\fint_{\partial\Omega} v + \log\left(\fint_{\partial\Omega} e^{v/2}\right) + K, \tag{21}$$

where K is an absolute constant.

In fact, then the right-hand side of (20) is dominated from above by

$$M\left\{\frac{1}{16\pi}\|\nabla v\|_2^2 + \frac{1}{2}\fint_{\partial\Omega} v + \log\left(\fint_{\partial\Omega} e^{v/2}\right)\right\} + W(0) - M\log M + MK.$$

In use of (4) we have

$$\left(\frac{1}{2} - \frac{M}{16\pi}\right)\|v\|_{H^1}^2 \leq C + \log\left(\fint_{\partial\Omega} e^{v/2}\right)$$

with a constant $C > 0$. Then, (3) follows from $M < 8\pi$ and Theorem 1.

For the proof of Proposition 2, we require three lemmas. The first one is due to Moser [9].

LEMMA 1. *If $w \in H_0^1(\Omega)$, we have*

$$\log\left(\fint_\Omega e^w\right) \leq \frac{1}{16\pi}\int_\Omega |\nabla w|^2 + K. \tag{22}$$

We also make use of Lebedev-Milin's inequality in the following form. It is invariant under the conformal transformation and related topics are discussed in Chang [2].

LEMMA 2. *If $\rho \in C(\overline{\Omega})$ is harmonic in Ω, it holds that*

$$\log\left(\fint_{\partial\Omega} e^\rho\right) \leq \frac{1}{4\pi}\int_\Omega |\nabla \rho|^2 + \fint_{\partial\Omega}\rho.$$

Finally, the following fact is due to Nehari [15].

LEMMA 3. *If ρ is harmonic in Ω and $\omega \subset\subset \Omega$ is a subdomain with smooth boundary, we have*

$$4\pi\int_\omega e^\rho \leq \left(\int_{\partial\omega} e^{\rho/2}\right)^2. \tag{23}$$

If ρ is constant, (23) is nothing but the standard isoperimetic inequality. It allows us to perform a rearrangement process.

Let $\rho \in C(\overline{\Omega})$ be harmonic in Ω and put

$$\rho^* = \log\left(\fint_\Omega e^\rho\right).$$

Given a measurable function w in Ω and $t \in \mathbf{R}$, we set $A_t = \{w > t\}$. Then $A_t^* \subset \mathbf{R}^2$ denotes the open disc with the center at the origin satisfying

$$a(t) \equiv \int_{A_t} e^\rho = \int_{A_t^*} e^{\rho^*}.$$

We define the symmetric decreasing rearrangement of w relative to ρ by

$$w^*(x) = \sup\ \{t \in \mathbf{R} \mid x \in A_t^*\}.$$

If Ω^* denotes the open ball with the center at the origin satisfying $|\Omega^*| = |\Omega|$, then $w^*(x)$ is regarded as a function defined in Ω^*. We have

$$\int_\Omega g \circ w \cdot e^\rho = \int_{\Omega^*} g \circ w^* \cdot e^{\rho^*} = \int_{-\infty}^{\infty} g(\xi) d\big(-a(t)\big) \tag{24}$$

for any continuous function g on $\mathbf{R}$. Similarly to the Schwarz symmetrization, if w is Lipshitz continuous then w^* is so.

We show the following lemma.

LEMMA 4. *If $w \in C^1(\overline{\Omega})$ is C^2 in $\{w > 0\}$ and satisfies*

$$w \geq 0 \quad in \quad \Omega \qquad and \qquad w = 0 \quad on \quad \partial\Omega,$$

then the inequality

$$\int_\Omega |\nabla w|^2 \geq \int_{\Omega^*} |\nabla w^*|^2 \tag{25}$$

holds.

Proof. For the moment, we specify the n-dimensional Hausdorff measure dH^n to avoid confusions. Then, co-area formula in the differential form gives

$$-\frac{da}{dt}(t) = \int_{\{w=t\}} \frac{e^\rho}{|\nabla w|} dH^1 \qquad \text{a.e.} \quad t > 0.$$

See [4] e.g., for this formula. From the assumption follows that $\partial \{w > t\} \subset \{w = t\}$ for $t \in (0, m)$, but Sard's lemma assures furthermore that

$$\partial \{w > t\} = \{w = t\} \qquad (\text{a.e. } t \in (0, m)),$$

where $m = \max_{\overline{\Omega}} w$. Moreover, Lemma 3 implies for a.e. $t \in (0, m)$ that

$$\begin{aligned} \int_{\{w=t\}} |\nabla w| \, dH^1 \ &\geq \ \left(\int_{\{w=t\}} e^{\rho/2} dH^1\right)^2 \left(\int_{\{w=t\}} \frac{e^\rho}{|\nabla w|} dH^1\right)^{-1} \\ &\geq \ -\frac{4\pi}{a'(t)} \int_{\{w>t\}} e^\rho dH^2 = -\frac{4\pi a(t)}{a'(t)}. \end{aligned} \tag{26}$$

Then, co-area formula in the integral form gives

$$\int_\Omega |\nabla w|^2 = \int_0^m dt \int_{\{w=t\}} |\nabla w| \, dH^1 \geq -4\pi \int_0^m \frac{a(t)}{a'(t)} dt. \tag{27}$$

On the other hand, the radially symmetric function w^* in Ω^* also satisfies

$$\partial \{w^* > t\} = \{w^* = t\} \qquad (\text{a.e. } t \in (0,m))$$

because $H^2(\{w = t\}) = 0$ for a.e. t again by Sard's lemma. For a.e. $t \in (0,m)$, the equalities hold at each step of (26) with ρ replaced by ρ^*. This implies

$$\int_\Omega |\nabla w^*|^2 = \int_0^m dt \int_{\{w^*=t\}} |\nabla w^*| \, dH^1 = -4\pi \int_0^m \frac{a(t)}{a'(t)} dt. \tag{28}$$

Inequality (25) is a consequence of (27) and (28).

We are ready to complete the proof of Proposition 2. Let $\rho \in C^2(\Omega) \cap C(\overline{\Omega})$ be the solution to

$$\Delta\rho = 0 \quad \text{in} \quad \Omega \qquad \text{and} \qquad \rho = v \quad \text{on} \quad \partial\Omega$$

and $v_0 = v - \rho$. It follows that

$$\begin{aligned} \log\left(\fint_\Omega e^v\right) \le \log\left(\fint_\Omega e^{|v_0|}e^\rho\right) &= \log\left(\fint_{\Omega^*} e^{|v_0|^*}e^{\rho^*}\right) \\ &= \rho^* + \log\left(\fint_{\Omega^*} e^{|v_0|^*}\right). \end{aligned}$$

by (24). In use of Lemmas 1 and 4, the right-hand side is dominated from above by

$$\begin{aligned} \rho^* + \frac{1}{16\pi}\int_{\Omega^*} |\nabla |v_0|^*|^2 + K &\le \rho^* + \frac{1}{16\pi}\int_\Omega |\nabla |v_0||^2 + K \\ &= \rho^* + \frac{1}{16\pi}\int_\Omega |\nabla v_0|^2 + K. \end{aligned}$$

Here,

$$\Delta\rho = 0 \quad \text{in} \quad \Omega \qquad \text{and} \qquad v_0 = 0 \quad \text{on} \quad \partial\Omega$$

and hence

$$\int_\Omega \nabla v_0 \cdot \nabla\rho = 0$$

follows. We have

$$\int_\Omega |\nabla v_0|^2 = \int_\Omega |\nabla v|^2 - \int_\Omega |\nabla \rho|^2$$

so that

$$\log\left(\fint_\Omega e^v\right) \le \frac{1}{16\pi}\int_\Omega |\nabla v|^2 + \left\{\log\left(\fint_\Omega e^\rho\right) - \frac{1}{16\pi}\int_\Omega |\nabla\rho|^2\right\} + K.$$

Now, Lemma 3 gives

$$\fint_\Omega e^\rho \le \frac{|\partial\Omega|^2}{4\pi|\Omega|}\left(\fint_{\partial\Omega} e^{\rho/2}\right)^2$$

and we obtain

$$\begin{aligned}
&\log\left(\fint_\Omega e^\rho\right) - \frac{1}{16\pi}\int_\Omega |\nabla\rho|^2 \\
&\le 2\log\left(\fint_{\partial\Omega} e^{\rho/2}\right) - \frac{1}{4\pi}\int_\Omega |\nabla(\rho/2)|^2 + \log\frac{|\partial\Omega|^2}{4\pi|\Omega|} \\
&\le \log\left(\fint_{\partial\Omega} e^{\rho/2}\right) + \fint_{\partial\Omega}(\rho/2) + \log\frac{|\partial\Omega|^2}{4\pi|\Omega|} \\
&= \log\left(\fint_{\partial\Omega} e^{v/2}\right) + \fint_{\partial\Omega}(v/2) + \log\frac{|\partial\Omega|^2}{4\pi|\Omega|}
\end{aligned}$$

by Lemma 2. The proof is complete. □

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