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FIXED POINTS OF BETTER ADMISSIBLE MAPS ON GENERALIZED CONVEX SPACES

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ABSTRACT. We obtain generalized versions of the Fan–Browder fixed point theorem for G-convex spaces. We define the class $\mathfrak{B}$ of better admissible multimaps on G-convex spaces and show that any closed compact map in $\mathfrak{B}$ from a locally G-convex uniform space into itself has a fixed point.

1. Introduction

In 1991, the author [11] showed that any compact acyclic multimap from a nonempty convex subset of a locally convex Hausdorff topological vector space into itself has a fixed point, where an acyclic multimap is an upper semicontinuous map with compact acyclic values. This result generalizes historically well-known fixed point theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Hukuhara, Himmelberg, and others. Note that all of those authors were concerned with single-valued maps or convex-valued multimaps. For the literature, see [11, 12, 21].

Since then the theorem is extended further in several directions in the framework of topological vector spaces. The main trend of these extensions is to show that the theorem holds for more general multimaps than acyclic ones and for topological vector spaces not necessarily locally convex. In fact, the author [13-15] showed that any closed

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compact multimap in the "better" admissible class $\mathfrak{B}$ from an admissible (in the sense of Klee) convex subset of a Hausdorff topological vector space into itself has a fixed point.

Along with the above development, many features of the concept of convex sets are extended to some general convexities; for example, convex spaces of Lassonde, C-spaces of Horvath, and some others due to Pasicki, Komiya, Bielawski, Joó, and others. These general convexities are all subsumed to the concept of G-convex spaces due to the author; see [16-31]. Moreover, there have appeared a number of fixed point theorems for multimaps in the framework of such general convexities; for example, Horvath [5, 6], Tarafdar [34], Tan and Zhang [33], Kim [7], Yuan [35], Ben-El-Mechaiekh et al. [1], Park [19-24], and others.

In this paper, firstly, we obtain generalized versions of the Fan–Browder fixed point theorem for G-convex spaces. Secondly, we extend the class $\mathfrak{B}$ of better admissible multimaps to G-convex spaces and show that any closed compact map in $\mathfrak{B}$ from a locally G-convex uniform space into itself has a fixed point. Some related results and several consequences of our main results are added.

2. G-convex spaces

A *multimap* or *map* $F : X \multimap Y$ is a function from X into the power set of Y with *values* $F(x)$ for $x \in X$ and *fibers* $F^-(y) = \{x \in X : x \in F(y)\}$ for $y \in Y$. For topological spaces X and Y, a map $F : X \multimap Y$ is *closed* if its graph $\mathrm{Gr}(F) = \{(x, y) : x \in X,\ y \in F(x)\}$ is closed and *compact* if its range $F(X) = \{y \in Y : y \in F(x) \text{ for some } x \in X\}$ is contained in a compact subset of Y.

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each $A = \{a_0, a_1, \cdots, a_n\} \subset D$ with the cardinality $|A| = n + 1$, there exist a subset $\Gamma(A)$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \subset \{0, 1, \cdots, n\}$ implies $\phi_A(\Delta_J) \subset \Gamma(\{a_j : j \in J\})$, where Δ_n is an n-simplex with vertices $v_0, v_1, \cdots, v_n$ and $\Delta_J = \mathrm{co}\{v_j : j \in J\}$.

Let, $\langle D \rangle$ denote the set of all nonempty finite subsets of D. We may write $\Gamma_A := \Gamma(A)$ for each $A \in \langle D \rangle$ and $(X; \Gamma) := (X, X; \Gamma)$. In case $D \subset X$ in a G-convex space $(X, D; \Gamma)$, a subset C of X is said to be Γ-*convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$. For details on

G-convex spaces, see [16-24, 26-31], where basic theory was extensively developed.

There are a lot of examples of G-convex spaces:

EXAMPLES 2.1. If X is a convex subset of a vector space, $D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology, then $(X, D; \Gamma)$ becomes a *convex space* generalizing the one due to Lassonde. Note that any convex subset of a topological vector space is a convex space, but not conversely.

EXAMPLES 2.2. If $X = D$ and Γ_A is assumed to be contractible or, more generally, infinitely connected (that is, n-connected for all $n \geq 0$), and if for each $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$, then $(X; \Gamma)$ becomes a *C-space* (or an *H-space*) due to Horvath [5, 6]; see also [25].

EXAMPLES 2.3. For other major examples of G-convex spaces are metric spaces with Michael's convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joó's pseudoconvex spaces, and so on. For the literature, see [26-29]. Recently, further examples of G-convex spaces were given by the author [18] as follows: L-spaces and B'-simplicial convexity of Ben-El-Mechaiekh et al. [1], continuous images of G-convex spaces, Verma's or Stachó's generalized H-spaces, Kulpa's simplicial structures, $P_{1,1}$-spaces of Forgo and Joó, mc-spaces of Llinares, hypeconvex metric spaces due to Aronszajn and Panitchpakdi, and Takahashi's convexity in metric spaces.

EXAMPLES 2.4. Any hyperbolic space X in the sense of Kirk [8] and Reich–Shafrir [32] is a G-convex space, since the closed convex hull of any $A \in \langle X \rangle$ is contractible [32, p.542]. This class of metric spaces contains all normed vector spaces, all Hadamard manifolds, the Hilbert ball with the hyperbolic metric, and others. Note that an arbitrary product of hyperbolic spaces is also hyperbolic; see [32].

Now, we introduce a KKM theorem for G-convex spaces.

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if $\Gamma_A \subset F(A)$ for each $A \in \langle D \rangle$.

The following is known [22-24, 28, 29]:

THEOREM 0. *Let* $(X, D; \Gamma)$ *be a* G*-convex space and* $F : D \multimap X$ *a multimap with closed [resp. open] values. Suppose that* F *is a* KKM map. *Then*

(i) $\{F(z)\}_{z \in D}$ *has the finite intersection property; and*

(ii) *if* $\bigcap_{z \in N} \overline{F(z)}$ *is contained in a compact subset* K *of* X *for some* $N \in \langle D \rangle$, *then we have* $\bigcap_{z \in D} \overline{F(z)} \neq \emptyset$.

3. The Φ-maps on G-convex spaces

For any topological space E and a G-convex space $(X, D; \Gamma)$, a map $T : E \multimap X$ is called a Φ*-map* if there exists a map $S : E \multimap D$ such that

(i) for each $y \in E$, $M \in \langle S(y) \rangle$ implies $\Gamma_M \subset T(y)$; and

(ii) $E = \bigcup\{\operatorname{Int} S^-(x) : x \in D\}$.

The concept of Φ-maps is originated from Horvath [5] and motivated by the works of Fan and Browder; see [2, 12, 21].

From the KKM Theorem 0, we obtain our main result in this section:

THEOREM 3.1. *Let* $(X, D; \Gamma)$ *be a* G*-convex space, and* $S : D \multimap X$, $T : X \multimap X$ *multimaps. Suppose that*

(1.1) $S(z)$ *is open [resp. closed] for each* $z \in D$;

(1.2) *for each* $y \in X$, $M \in \langle S^-(y) \rangle$ *implies* $\Gamma_M \subset T^-(y)$; *and*

(1.3) $X = S(N)$ *for some* $N \in \langle D \rangle$.

Then T *has a fixed point* $x_0 \in X$; *that is,* $x_0 \in T(x_0)$.

Proof. Define a map $F : D \multimap X$ by $F(z) := X \backslash S(z)$ for $z \in D$. Then each $F(z)$ is closed [resp. open] by (1.1). Moreover, we have

$$\bigcap_{z \in N} F(z) = X \backslash \bigcup_{z \in N} S(z) = X \backslash X = \emptyset$$

by (1.3). Therefore, the family $\{F(z)\}_{z \in D}$ does not have the finite intersection property, and hence, F is not a KKM map by Theorem 0. Thus, there exists an $M \in \langle D \rangle$ such that $\Gamma_M \not\subset F(M) = \bigcup\{X \backslash S(z) : z \in M\}$. Hence, there exists an $x_0 \in \Gamma_M$ such that $x_0 \in S(z)$ or $z \in S^-(x_0)$ for all $z \in M$; that is, $M \in \langle S^-(x_0) \rangle$. Therefore, we have

$\Gamma_M \subset T^-(x_0)$ by (1.2) and hence, $x_0 \in \Gamma_M \subset T^-(x_0)$. This implies $x_0 \in T(x_0)$ and completes our proof. □

REMARKS 1. It is easy to reformulate Theorem 3.1 for a Φ-map $T : X \multimap X$.

2. Note that (1.2) generalizes the following:

(1.2)′ *$S^-(y) \subset T^-(y)$ and $T^-(y)$ is Γ-convex for each $y \in X$ whenever $D \subset X$.*

3. In Theorem 3.1, condition (1.3) can be replaced by the following without affecting the conclusion:

(1.3)′ *there exists an $A \in \langle D \rangle$ such that $S^-(y) \cap A \neq \emptyset$ for each $y \in X$.*

In fact, if for each $y \in X$, there exists a $z \in A$ such that $z \in S^-(y)$ or $y \in S(z)$, then $X = S(A)$. Therefore, (1.3)′ implies (1.3).

We give some consequences of Theorem 3.1 for the case when S has *open* values:

COROLLARY 3.2. *Let $(X, D; \Gamma)$ be a G-convex space, and $S : D \multimap X$, $T : X \multimap X$ multimaps. Suppose that*

(2.1) *$S(z)$ is open for each $z \in D$;*
(2.2) *for each $y \in X$, $M \in \langle S^-(y) \rangle$ implies $\Gamma_M \subset T^-(y)$; and*
(2.3) *one of the following holds:*
 (i) *X is compact.*
 (ii) *$X \backslash S(M)$ is compact for some $M \in \langle D \rangle$.*

Then either

 (a) *there exists an $x_0 \in X$ such that $S^-(x_0) = \emptyset$; or*
 (b) *T has a fixed point $x_1 \in X$.*

Proof. Suppose that, contrary to (a), $S^-(x) \neq \emptyset$ for all $x \in X$. Then $X = \bigcup\{S(z) : z \in D\}$ and hence, we have the following:

Case (i). The compact set X is covered by a finite number of $S(z)$'s.

Case (ii). The compact set $X \backslash S(M)$ is covered by a finite number of $S(z)$'s.

Note that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (1.3). Therefore, by Theorem 3.1, T has a fixed point. □

REMARK. Particular forms of Corollary 3.2 were applied in [33] to equilibrium existence theorems for qualitative games and for generalized games as well as to the existence theorem for loose saddle points on G-convex spaces.

COROLLARY 3.3. *Let $(X, D; \Gamma)$ be a G-convex space, and $S : D \multimap X$, $T : X \multimap X$ multimaps. Suppose that*

(3.1) $X = \bigcup\{\operatorname{Int} S(z) : z \in D\}$;
(3.2) *for each* $y \in X$, $M \in \langle S^-(y)\rangle$ *implies* $\Gamma_M \subset T^-(y)$; *and*
(3.3) $X \backslash \bigcup\{\operatorname{Int} S(z) : z \in M\}$ *is compact for some* $M \in \langle D\rangle$.

Then T has a fixed point $x_0 \in X$.

Proof. Consider the map $\operatorname{Int} S : D \multimap X$ instead of S in Corollary 3.2. Note that, for each $y \in X$, by (3.2),

$$M \in \langle(\operatorname{Int} S)^-(y)\rangle \subset \langle S^-(y)\rangle \text{ implies } \Gamma_M \subset T^-(y),$$

and hence (2.2) holds. Moreover, by (3.1), for each $x \in X$, we have a $z \in D$ such that $x \in \operatorname{Int} S(z) \subset S(z)$, and hence $S^-(x) \neq \emptyset$. Therefore, the conclusion (a) of Corollary 3.2 can not occur, and hence, we have the conclusion. □

Corollaries 3.2 and 3.3 have a lot of variations. One of the simplest forms is the following:

COROLLARY 3.4. *Let $(X; \Gamma)$ be a compact G-convex space and $T : X \multimap X$ a map such that*

(4.1) *$T(x)$ is nonempty Γ-convex for each $x \in X$; and*
(4.2) *$T^-(y)$ is open for each $y \in X$.*

Then T has a fixed point.

Proof. Put $X = D$ in Theorem 3.1 and consider (T, T) instead of (S^-, T^-). Then (1.1) and (1.2) follows from (4.2) and (4.1), respectively. Since for each $x \in X$, by (4.1) there exists a $y \in X$ such that $y \in T(x)$ or $x \in T^-(y)$, X is covered by open sets $T^-(y)$. Since X is compact, we have $X = T^-(N)$ for some $N \in \langle X\rangle$. This implies the validity of (1.3). Therefore, by Theorem 3.1, we have an $x_0 \in X$ such that $x_0 \in T^-(x_0)$. This completes our proof. □

For the case when X itself is a compact convex subset of a topological vector space, Corollary 3.4 reduces to the Fan–Browder fixed point theorem due to Browder [2].

Only one of the simplest forms of Theorem 3.1 for the case S has *closed* values is known as follows; see Park [16]:

COROLLARY 3.5. *Let* $(X;\Gamma)$ *be a* G*-convex space and* $T : X \multimap X$ *a map such that*

(5.1) $T(x)$ *is* Γ*-convex for each* $x \in X$*;*
(5.2) $T^{-}(y)$ *is closed for each* $y \in X$*; and*
(5.3) *there exists an* $A \in \langle X \rangle$ *such that* $T(x) \cap A \neq \emptyset$ *for each* $x \in X$.

Then T *has a fixed point.*

Proof. Put $X = D$ and replace (S,T) by (T^{-},T^{-}) in Theorem 3.1 with condition $(1.3)'$ instead of (1.3). □

The following is given in [26] implicitly and in [16, 17] explicitly:

LEMMA 3.6. *Let* K *be a Hausdorff compact space,* $(X,D;\Gamma)$ *a* G*-convex space, and* $T : K \multimap X$ *a* Φ*-map. Then* T *has a continuous selection* $f : K \to X$*; that is,* $f(y) \in T(y)$ *for all* $y \in K$*. More precisely, there exist two continuous functions* $p : K \to \Delta_n$ *and* $\phi_N : \Delta_n \to \Gamma_N$ *such that* $f = \phi_N \circ p$ *for some* $N \in \langle D \rangle$ *with* $|N| = n+1$.

4. Better admissible multimaps

Let $(X,D;\Gamma)$ be a G-convex space and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of multimaps from X into Y as follows:

$F \in \mathfrak{B}(X,Y) \Longleftrightarrow F : X \multimap Y$ is a map such that for any $N \in \langle D \rangle$ with $|N| = n+1$ and any continuous map $p : F(\Gamma_N) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point.

We give some subclasses of $\mathfrak{B}$ as follows:

EXAMPLES 4.1. For topological spaces X and Y, an *admissible* class $\mathfrak{A}_c^{\kappa}(X,Y)$ of maps $F : X \multimap Y$ is one such that, for each F and each nonempty compact subset K of X, there exists a map $G \in \mathfrak{A}_c(K,Y)$ satisfying $Gx \subset Fx$ for all $x \in K$; where $\mathfrak{A}_c$ consists of finite compositions of maps in a class $\mathfrak{A}$ of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $T \in \mathfrak{A}_c$ is upper semicontinuous (u.s.c.) with nonempty compact values; and
(iii) for any polytope P, each $T \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

Here, a polytope P is a homeomorphic image of a standard simplex. There are lots of examples of $\mathfrak{A}$ and $\mathfrak{A}_c^\kappa$; see [12-15, 25-28].

Note that for a G-convex space $(X, D; \Gamma)$ and any space Y, an admissible class $\mathfrak{A}_c^\kappa(X, Y)$ is a subclass of $\mathfrak{B}(X, Y)$ and that a Φ-map $T : Y \multimap X$ belongs to $\mathbb{C}_c^\kappa(Y, X)$ if Y is Hausdorff, by Lemma 3.6.

EXAMPLES 4.2. For a convex space $(X, D; \Gamma)$, where $\Gamma = \text{co}$ and ϕ_N is a homeomorphism, the class $\mathfrak{B}(X, Y)$ is originally given in [13] and investigated in [14, 15].

EXAMPLES 4.3. For a convex space X and a topological space Y, Chang and Yen [3] defined the class of maps $T : X \multimap Y$ having the KKM property as follows:

$T \in \mathfrak{K}(X, Y) \Longleftrightarrow$ the family $\{S(x) : x \in X\}$ has the finite intersection property whenever $S : X \multimap Y$ has closed values and $T(\text{co}\, N) \subset S(N)$ for each $N \in \langle X \rangle$.

For a convex space X and a Hausdorff space Y, it is known that $\mathfrak{A}_c^\kappa(X, Y) \subset \mathfrak{K}(X, Y)$ and we observed that two subclasses $\mathfrak{B}$ and $\mathfrak{K}$ coincide in the class of all closed compact maps $T : X \multimap Y$ [13].

Generalizations of the class $\mathfrak{K}$ to G-convex spaces are possible; see [10].

EXAMPLES 4.4. Recently, Ben-El-Mechaiekh et al. [1] introduced the class $\mathbb{A}(X, Y)$ of approachable maps $F : X \multimap Y$ for uniformizable spaces X and Y. It is shown that if $(X, D; \Gamma)$ is a G-convex uniform space with $D \subset X$ and Y is a uniform space, then any closed compact map $F \in \mathbb{A}(X, Y)$ belongs to $\mathfrak{B}(X, Y)$; see [19, Lemma 3]. It is known that for a compact uniform space X and a G-convex uniform space $(Y, D; \Gamma)$ with $D \subset X$, every u.s.c. map $F : X \multimap Y$ with nonempty Γ-convex values is approachable; see [1, Proposition 3.9]. Therefore, we have the following:

LEMMA 4.5. *Let $(X, D; \Gamma)$ be a G-convex uniform space with $D \subset X$. Then any u.s.c. map $F : X \multimap X$ with nonempty closed Γ-convex values belongs to $\mathfrak{B}(X, X)$.*

Here we have the following important fact:

REMARK. The basic coincidence theorem in [26, 27, Theorem 1] holds for any closed compact map $F \in \mathfrak{B}(X, Y)$ instead of $F \in \mathfrak{A}_c^\kappa(X, Y)$. Therefore, many of the author's works on the admissible class $\mathfrak{A}_c^\kappa$ may hold for the closed compact maps in the "better" admissible class $\mathfrak{B}$.

For a particular type of G-convex spaces, we can establish fixed point theorems for the class $\mathfrak{B}$.

A G-convex space $(X, D; \Gamma)$ is a Φ-*space* if X is a separated uniform space and for each entourage V there is a Φ-map $T : X \multimap X$ such that $\mathrm{Gr}(T) \subset V$. This concept is originated from Horvath [5], where a number of examples are given.

The following is our main result in this section:

THEOREM 4.6. *Let $(X, D; \Gamma)$ be a Φ-space and $F \in \mathfrak{B}(X, X)$. If F is closed and compact, then F has a fixed point.*

Proof. Let $\mathcal{V} = \{V_\lambda\}_{\lambda \in I}$ be a basis of the separated uniform structure of X. Let $K := \overline{F(X)}$ be the closure of the range of F. Since $(X, D; \Gamma)$ is a Φ-space, for each $\lambda \in I$, there is a Φ-map $T_\lambda : X \multimap X$ such that $\mathrm{Gr}(T_\lambda) \subset V_\lambda$. Since K is compact, by Lemma 3.6, $T_\lambda|_K$ has a continuous selection $f_\lambda : K \to \Gamma_N$ for some $N \in \langle D \rangle$ such that $f_\lambda = \phi_N \circ p$, where $p = p_\lambda : K \to \Delta_n$ is a continuous function. Since $F \in \mathfrak{B}(X, K)$, the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \subset K \xrightarrow{p} \Delta_n$$

has a fixed point $a_\lambda \in \Delta_n$; that is, $a_\lambda \in (p \circ F \circ \phi_N)(a_\lambda)$. Hence,

$$x_\lambda := \phi_N(a_\lambda) \in (\phi_N \circ p \circ F)(x_\lambda) = (f_\lambda \circ F)(x_\lambda)$$

and there exists a $y_\lambda \in F(x_\lambda) \subset K$ such that $x_\lambda = f_\lambda(y_\lambda) \in T_\lambda(y_\lambda)$; that is, $(x_\lambda, y_\lambda) \in \mathrm{Gr}(T_\lambda) \subset V_\lambda$. Therefore

$$(x_\lambda, y_\lambda) \in V_\lambda \cap \mathrm{Gr}(F) \subset X \times K.$$

Since K is compact, we may assume that $\{y_\lambda\}_{\lambda\in I}$ converges to some $x_0 \in K$. Since $(x_\lambda, y_\lambda) \in V_\lambda$ for all $\lambda \in I$, $\{x_\lambda\}_{\lambda \in I}$ also converges to $x_0 \in K$. Since F is closed and $(x_\lambda, y_\lambda) \in \mathrm{Gr}(F)$, we should have $(x_0, x_0) \in \mathrm{Gr}(F)$. Therefore, F has a fixed point $x_0 \in F(X) \subset K$. □

Particular forms of Theorem 4.6 were known by Horvath [5] and Park and Kim [25]. More recently, Ben-El-Mechaiekh et al. [1, Theorem (4.2)] obtained a particular form of Theorem 4.6 for approachable multimaps; see [19].

In view of Lemma 4.5 and Theorem 4.6, we have the following:

COROLLARY 4.7. *Let $(X, D; \Gamma)$ be a Φ-space with $D \subset X$. Then any compact u.s.c. map $F : X \multimap X$ with nonempty closed Γ-convex values has a fixed point.*

Note that Corollary 4.7 extends [19, Theorem 4].

For a non-closed map, we have the following:

THEOREM 4.8. *Let $(X, D; \Gamma)$ be a Φ-space and $F \in \mathfrak{A}_c^\kappa(X, X)$. If F is compact, then F has a fixed point.*

Proof. As in the proof of Theorem 4.6, for each V_λ, we have the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \subset K \xrightarrow{p} \Delta_n.$$

Since $L := \phi_N(\Delta_n)$ is a compact subset of X and $F \in \mathfrak{A}_c^\kappa(X, X)$, there exists a map $G \in \mathfrak{A}_c(L, X)$ satisfying $G(x) \subset F(x)$ for all $x \in L$. Since ϕ_N and p are continuous, we have

$$p \circ G \circ \phi_N \in \mathfrak{A}_c(\Delta_n, \Delta_n)$$

and it has a fixed point $a_\lambda \in \Delta_n$; that is, $a_\lambda \in (p \circ G \circ \phi_N)(a_\lambda)$. Hence,

$$x_\lambda := \phi_N(a_\lambda) \in (\phi_N \circ p \circ G)(x_\lambda) = (f_\lambda \circ G)(x_\lambda)$$

and there exists a $y_\lambda \in G(x_\lambda) \subset F(x_\lambda) \subset K$ such that $x_\lambda = f_\lambda(y_\lambda) \in T_\lambda(y_\lambda)$; that is, $(x_\lambda, y_\lambda) \in \mathrm{Gr}(T_\lambda) \subset V_\lambda$. Therefore

$$(x_\lambda, y_\lambda) \in V_\lambda \cap \mathrm{Gr}(G) \subset L \times K.$$

Note that $G \in \mathfrak{A}_c(L, K)$ and hence G has closed graph. Then, as in the proof of Theorem 4.6, there exists an $(x_0, x_0) \in \mathrm{Gr}(G)$. Since $\mathrm{Gr}(G) \subset \mathrm{Gr}(F)$, F has a fixed point $x_0 \in K$. □

5. Some consequences

In this section, we give some consequences of Theorem 4.6.

A *locally G-convex uniform space* is a G-convex space $(X, D; \Gamma)$ such that

(1) X is a separated uniform space with the basis $\mathcal{V}$ for symmetric entourages;
(2) D is a dense subset of X; and
(3) for each $V \in \mathcal{V}$ and each $x \in X$,

$$V[x] = \{x' \in X : (x, x') \in V\}$$

is Γ-convex.

Particular types of locally G-convex uniform spaces are treated recently by Yuan [35].

LEMMA 5.1. *A locally G-convex uniform space $(X, D; \Gamma)$ is a Φ-space.*

Proof. Let $\{V_\lambda\}_{\lambda \in I}$ be an (open) basis of the uniform structure of $(X, D; \Gamma)$. For each $\lambda \in I$, define $T_\lambda : X \multimap X$ and $S_\lambda : X \multimap D$ by

$$T_\lambda(x) := \{y \in X : (x, y) \in V_\lambda\}$$

and

$$S_\lambda(x) := \{y \in D : (x, y) \in V_\lambda\}$$

for $x \in X$. Since D is dense in X, for each $x \in X$, ther is a $y \in D$ such that $(x, y) \in V_\lambda$; hence

$$x \in S_\lambda^-(y) = T_\lambda^-(y).$$

Since $T_\lambda^-(y)$ is open, we have $X = \bigcup\{\operatorname{Int} S_\lambda^-(y) : y \in D\}$. Moreover, for each $x \in X$, if $M \in \langle S_\lambda(x) \rangle \subset D$, then by (3)

$$\Gamma_M \subset \{y \in X : (x, y) \in V_\lambda\} = T_\lambda(x).$$

Therefore S_λ and T_λ satisfy conditions (i) and (ii) of the definition of a Φ-map. □

From Lemma 5.1 and Theorem 4.6, we have the following:

THEOREM 5.2. *Let $(X, D; \Gamma)$ be a locally G-convex uniform space. Then any closed compact map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

For acyclic maps, Yuan [35] obtained a particular form of Theorem 5.2 for particular types of locally G-convex spaces. For single-valued continuous maps on locally G-convex metrizable spaces, particular forms of Theorem 5.2 are due to Rassias, Park, Park-Kim (see [25]), and Kulpa [9].

A G-convex space $(X, D; \Gamma)$ is called an *LG-space* if X is a separated uniform space such that D is a dense subset of X and if there exists a basis $\mathcal{V}$ for the symmetric entourages such that for each $V \in \mathcal{V}$, $V[C] := \{x \in X : C \cap V[x] \neq \emptyset\}$ is Γ-convex whenever $C \subset X$ is Γ-convex.

For a C-space $(X; \Gamma)$, the concept of LG-spaces reduces to that of *LC-spaces* due to Horvath [5, 6] (which are called *locally H-convex spaces* by Tarafdar [34]).

LEMMA 5.3. *Every LG-space $(X, D; \Gamma)$ is a locally G-convex uniform space if $\Gamma_{\{x\}} = \{x\}$ for each $x \in D$.*

Proof. For each basis element V of the uniformity and each $x \in D$,

$$\begin{aligned} V[x] &= \{x' \in X : (x, x') \in V\} \\ &= \{x' \in X : x \in V^{-}[x']\} \\ &= \{x' \in X : \{x\} \cap V^{-}[x'] \neq \emptyset\}. \end{aligned}$$

Since $\{x\}$ is Γ-convex and $(X, D; \Gamma)$ is an LG-space, $V[x]$ is Γ-convex. This completes our proof. □

From Lemma 5.3 and Theorem 4.6, we have the following:

THEOREM 5.4. *Let $(X, D; \Gamma)$ be an LG-space such that $\Gamma_{\{x\}} = \{x\}$ for each $x \in D$. Then any closed compact map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

In view of Lemma 4.5, Theorem 5.4 holds for an u.s.c. map with nonempty closed Γ-convex values. This fact was established by Ben-El-Mechaiekh et al. [1, Corollary 4.7] for a particular case of a paracompact LC-space $(X; \Gamma)$.

Any nonempty convex subset X of a locally convex Hausdorff topological vector space (l.c.s.) E is an obvious example of an LC-space $(X;\Gamma)$ with $\Gamma_A = \operatorname{co} A$ for $A \in \langle X \rangle$. For other examples of LC-spaces, see [5, 6, 34].

From Theorem 4.6 or Theorem 5.4, we immediately have

THEOREM 5.5 [13, Theorem 5]. *Let X be a nonempty convex subset of a l.c.s. Then any closed compact map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Note that Theorem 5.5 extends and unifies a large number of historically well-known fixed point theorems on l.c.s.; see [13-15].

A G-convex space $(X;\Gamma)$ is called an *LG-metric space* if X is equipped with a metric d such that for any $\varepsilon > 0$, the set $\{x \in X : d(x, C) < \varepsilon\}$ is Γ-convex whenever $C \subset X$ is Γ-convex and open balls are Γ-convex. This concept generalizes that of *LC-metric spaces* due to Horvath [5], who gave a number of examples.

In our previous work [20], we obtained several results on fixed points of lower semicontinuous multimaps on complete LC-metric spaces.

Note that the hyperconvex metric spaces due to Aronszajn and Panitchpakdi in 1959 are LC-metric spaces, and hence some of the results in this paper are applicable to hyperconvex metric spaces; see [19].

Finally, in this section, we were mainly concerned with consequences of Theorem 4.6 and some further consequences can be found in [19]. Similarly, we can obtain consequences of Theorem 5.4 generalizing some of known fixed point theorems due to the author; see [11-19].

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