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A NOTE ON RANDOM FUNCTIONS

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Abstract. It is known that one can generate functions distributed according to r-fold Wiener measure. So we could estimate the average case errors in a similar way as in Monte-Carlo method. Hence we study the basic properties of the generator of random functions. In addition, because the r-fold Wiener process is truly infinitely dimensional and a computer can only handle finitely dimensional spaces, we study in this paper, the properties of generator for an m-dimensional approximation of the r-fold Wiener process.

1. Introduction

For some problems, it is difficult or even impossible to study the complexity analytically. For example, in the global optimization problem, where one wants to approximate the global maximum of the function $f \in C^r$, the distribution of $\max_x f(x)$ is unknown for $r \geq 1$. Hence, for this problem, we do not even know the average error of the most trivial, zero algorithm. There is difficulty even for the relatively simpler problems such as the integration problem. Indeed, as we have seen in [1], for *Simpson's quadrature* we know that the average error equals $\Theta\left(n^{-\min\{4,r+1\}}\right)$,[1] but unfortunately we do not know the constant in the Θ-notation. However, if we are able to generate random functions distributed according to the r-fold Wiener measure ω_r, we could alleviate the problem by employing random methods. Hence, in this paper, we study how to generate r-fold Wiener random functions. The general references for this paper are [2] and [5].

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[1] The Θ-notation is used for asymptotic equalities. That is, $f(n) = \Theta(g(n))$ means that there are positive constants c_1 and c_2 such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$, $\forall n$.

2. Preliminaries

We assume that the function space is equipped with the probability measure μ_r, which is a variant of the r-fold Wiener measure ω_r. We also assume that a function f, as a stochastic process, is given by

$$f(x) = f_1(x) + f_2(1-x), \qquad x \in [0,1],$$

where f_1 and f_2 are independent and distributed according to ω_r. It is a Gaussian measure with zero mean and correlation function given by

$$M_{\omega_r}(f(x)\, f(y)) = \int_F f(x) f(y)\, \omega_r(df) = \int_0^1 \frac{(x-t)_+^r}{r!} \frac{(y-t)_+^r}{r!}\, dt,$$

where $(z-t)_+ = \max\{0, (z-t)\}$. Therefore, in order to generate the random process f distributed according to μ_r (we denote this by μ_r-random function f), it is enough to generate two independent ω_r-random functions f_1 and f_2. Therefore, we provide the generator for the r-fold Wiener process only.

3. Random function

The r-fold Wiener process is truly infinitely dimensional, i.e., for every finite dimensional subspace A of $C^r[0,1]$, the probability that $f \in A$ is zero.

Consider the problem of approximating f on the average with the error measured in L_2-norm. Given m and information

$$\Gamma_m(f) = [L_1(f), \ldots, L_m(f)],$$

where L_i's are any linear functionals, the average case error explained

$$e^a(\phi, \Gamma_m, L_2; \omega_r) := \sqrt{M_{\omega_r}(\|f - \phi(\Gamma_m(f))\|_2^2)}$$

in [1] is minimized by the algorithm $\phi(y) = \psi_m^*(y)$ being the conditional mean of $\omega_r(\,\cdot\,|\Gamma_m(f) = y)$. Such a minimal error is called the *radius* of information Γ_m and is denoted by $e^a(\Gamma_m, L_2; \omega_r)$. Hence, due to linearity of ψ_m^*,

$$\psi_m^*(y)(x) = \sum_{i=1}^{m} y_i \cdot g_i(x)$$

for suitably chosen functions $g_i \in L_2$. Actually, the same ψ_m^* minimizes the average case error of approximating f in any other norm. In particular,

$$M_{\omega_r}(\|f - \psi_m^*(\Gamma_m(f))\|_p^q) = \min_{\phi} \; M_{\omega_r}(\|f - \phi(\Gamma_m(f))\|_p^q)$$

for any $q > 0$ and any $p \geq 1$ including $p = \infty$. Moreover, for any linear operator S, $S(\psi_m^*(\Gamma_m(f)))$ minimizes the average case error of approximating $S(f)$;

$$\begin{aligned} e^a(\Gamma_m, S\,;\,\omega_r) &= e^a(S \circ \psi_m^*, \Gamma_m\,;\,\omega_r) \\ &= \min_{\phi} \sqrt{M_{\omega_r}\left(\|S(f) - \phi(\Gamma_m(f))\|^2\right)}. \end{aligned}$$

We specially focus on Γ_m^r consisting of values of $f^{(r)}$ at equally spaced points,

$$(A) \qquad \Gamma_m^r(f) = \left[f^{(r)}\left(\frac{1}{m}\right), \ldots, f^{(r)}\left(\frac{m}{m}\right)\right]$$

to have the following theorem that is the main theorem of this paper.

THEOREM. *Let* $\Gamma_m^r(f)$ *be given by* (A). *Let*

$$e^a\left(\Gamma_m^r, L_2\,;\,\omega_r\right) \quad \text{and} \quad e^a\left(\Gamma_m^r, Int\,;\,\omega_r\right)$$

be the average radii of Γ_m^r *for* L_2*-approximation and integration problems, respectively. Then, for* $r \geq 1$,

$$e^a\left(\Gamma_m^r, L_2\,;\,\omega_r\right) = \Theta\left(\frac{1}{m}\right) = e^a\left(\Gamma_m^r, Int\,;\,\omega_r\right).$$

PROOF. The upper bound follows from

$$e^a\left(\Gamma_m^r, Int\ ;\ \omega_r\right) \ \le\ e^a\left(\Gamma_m^r, L_2\ ;\ \omega_r\right) \ \le\ e^a\left(\Gamma_m^0, Int\ ;\ \omega_0\right),$$

which is due to [3] and the known fact that $e^a\left(\Gamma_m^0, Int\ ;\ \omega_0\right) \ =\ \Theta(1/m)$.

Hence, to complete the proof, we only need to show the lower bound for the integration problem. Let g be distributed according to ω_r and f be distributed according to classical Wiener measure ω_0. Since $\omega_r = \omega_0 \circ D^r$, we have $f(x) = g^{(r)}(x)$, i.e.,

$$g(x) = \int_0^1 \frac{(x-t)_+^{r-1}}{(r-1)!}\, f(t)\, dt.$$

Therefore, the average radius of Γ_m^r for an integration problem satisfies

$$\begin{aligned}
&\left(e^a(\Gamma_m^r, Int\ ;\ \omega_r)\right)^2 \\
&= M_{\omega_r}\left[\left(\int_0^1 g(x)\, dx - \psi_r^*(\Gamma_m^r(g))\right)^2\right] \\
&= M_{\omega_0}\left[\left(\int_0^1 \int_0^1 \frac{(x-t)_+^{r-1}}{(r-1)!} f(t)\, dt\, dx \ -\ \psi_0^*(\Gamma_m^0(f))\right)^2\right].
\end{aligned}$$

Hence, our problem is to estimate from below

$$e(m) \ =\ \sqrt{M_{\omega_0}\left(\left(S(f) - \psi_0^*(\Gamma_m^0(f))\right)^2\right)},$$

where

$$S(f) \ =\ \int_0^1 \int_0^x \frac{(x-t)^{r-1}}{(r-1)!} f(t)\, dt\, dx.$$

Of course, ψ_0^* is the optimal algorithm for the problem of approximating $S(f)$ and f is distributed according to ω_0.

Since S is a functional, the average case error $e(m)$ is the same as the error of approximating S by ψ_0^* in the worst case setting with respect to the unit ball in the reproducing kernel Hilbert space generated by ω_0. It is known, see e.g. [4], that it is the following *Sobolev* space:

$$W_2^1([0,1]) \ :=\ \{f : f(0) = 0,\ \ f \text{ is absolutely continuous, and } \ f' \in L_2\}$$

equipped with the norm $\|f\| = \|f'\|_2$. That is,

$$e(m) = \inf_{\phi} \sup \left\{ |S(f) - \phi(\Gamma_m^0(f))| : f \in W_2^1 \text{ and } \int_0^1 (f'(t))^2 dt \le 1 \right\}$$
$$= \sup \left\{ |S(f)| : f \in W_2^1, \|f'\|_2 \le 1, \quad \text{and } f(x_i) = 0, \forall i \right\}.$$

Take

$$f(x) = \begin{cases} x - x_{i-1}, & \text{if } x \in [x_{i-1}, m_i], \\ x_i - x, & \text{if } x \in [m_i, x_i], \end{cases}$$

where $m_i = (x_i + x_{i-1})/2$. Of course, $f \in W_2^1$, $f(x_i) = 0$, for all i, and $\int (f')^2 = 1$. Then,

$$S(f) = \int_0^1 \int_0^1 \frac{(x-t)_+^{r-1}}{(r-1)!} f(t)\, dt\, dx = \int_0^1 f(t) \int_t^1 \frac{(x-t)^{r-1}}{(r-1)!} dx\, dt$$
$$= \int_0^1 f(t) \frac{(1-t)^r}{r!}\, dt = \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} f(t) \frac{(1-t)^r}{r!}\, dt.$$

The integral over $[x_{i-1}, m_i]$ equals

$$\int_{x_{i-1}}^{m_i} (t - x_{i-1}) \frac{(1-t)^r}{r!}\, dt$$
$$= -\frac{(1-m_i)^{r+1}}{(r+1)!} \frac{x_i - x_{i-1}}{2} + \frac{(1-x_{i-1})^{r+2}}{(r+2)!} - \frac{(1-m_i)^{r+2}}{(r+2)!},$$

and over $[m_i, x_i]$ equals

$$\int_{m_i}^{x_i} (x_i - t) \frac{(1-t)^r}{r!}\, dt$$
$$= \frac{(1-m_i)^{r+1}}{(r+1)!} \frac{x_i - x_{i-1}}{2} - \frac{(1-m_i)^{r+2}}{(r+2)!} + \frac{(1-x_i)^{r+2}}{(r+2)!}.$$

Thus

$$S(f) = \sum_{i=1}^{m} \left(\zeta(x_{i-1}) + \zeta(x_{i-1} + h) - 2\zeta\left(x_{i-1} + \frac{h}{2} \right) \right),$$

where $\zeta(x) = (1-x)^{r+2}/(r+2)!$ and $h = 1/m$. Note that

$$\zeta(x_{i-1}+h) = \zeta(x_{i-1}) + h\,\zeta'(x_{i-1}) + \frac{h^2}{2}\,\zeta''(\tilde{x}_{i-1}),$$
$$\zeta\left(x_{i-1}+\frac{h}{2}\right) = \zeta(x_{i-1}) + \frac{h}{2}\,\zeta'(x_{i-1}) + \frac{h^2}{8}\,\zeta''(\tilde{\tilde{x}}_{i-1}),$$

where $\tilde{x}_{i-1} \in [x_{i-1}, x_i]$ and $\tilde{\tilde{x}}_{i-1} \in [x_{i-1}, x_i]$. Note also that $\zeta''(x) = (1-x)^r/(r!)$. Hence

$$\begin{aligned} &\zeta(x_{i-1}) + \zeta(x_{i-1}+h) - 2\zeta\left(x_{i-1}+\frac{h}{2}\right) \\ =&\frac{h^2}{4\,r!}\left(2(1-\tilde{x}_{i-1})^r - (1-\tilde{\tilde{x}}_{i-1})^r\right) \\ \geq&\frac{h^2}{4\,r!}\left(2(1-x_i)^r - (1-x_{i-1})^r\right). \end{aligned}$$

Therefore,

$$\begin{aligned} S(f) &\geq \frac{h^2}{4\,r!}\left(2\sum_{i=1}^{m}(1-x_i)^r - \sum_{i=0}^{m-1}(1-x_i)^r\right) \\ &= \frac{h^2}{4\,r!}\left(\sum_{i=1}^{m}(1-x_i)^r - 1\right) = \frac{h^2}{4\,r!}\left(\sum_{j=0}^{m-1} x_j^r - 1\right) \\ &\geq \frac{h}{4\,r!}\left(\int_0^{1-h} x^r\,dx - h\right) = \frac{h}{4\,r!}\left(\frac{(1-h)^{r+1}}{r+1} - h\right) \\ &= \frac{h}{4\,(r+1)!}\,(1+\Theta(h)). \end{aligned}$$

This completes the proof. □

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