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# A TRANSFORMATION FORMULA ASSOCIATED WITH THE GENERALIZED HYPERGEOMETRIC SERIES

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Abstract. The authors aim at presenting a presumably new transformation formula involving generalized hypergeometric series by making use of series rearrangement technique which is one of the most effective methods for obtaining generating functions or other identities associated with (especially) the hypergeometric series. They also consider a couple of interesting special cases of their main result.

## 1. Introduction and Preliminaries

The generalized hypergeometric function with $p$ numerator and $q$ denominator parameters is defined by

$$ {}_pF_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p;\\ \beta_1,\cdots,\beta_q;\end{matrix}\ z\right]=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k(\alpha_2)_k\cdots(\alpha_p)_k}{(\beta_1)_k(\beta_2)_k\cdots(\beta_q)_k}\frac{z^k}{k!}, \tag{1.1}$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(\alpha)_n:=\begin{cases}\alpha(\alpha+1)\cdots(\alpha+n-1) & \text{if } n\in\mathbb{N}:=\{1,\ 2,\ 3,\ \cdots\},\\ 1 & \text{if } n=0.\end{cases} \tag{1.2}$$

In terms of the familiar Gamma function $\Gamma(z)$ with its fundamental functional relationship $\Gamma(z+1)=z\Gamma(z)$, $(\alpha)_n$ can be rewritten in the form:

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \tag{1.3}$$

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There are several useful equivalent expressions of $\Gamma$ whose Weierstrass canonical product form is given by

$$\{\Gamma(z)\}^{-1} = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}}, \tag{1.4}$$

$\gamma$ being the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) \cong 0.577215664901532\cdots. \tag{1.5}$$

From (1.2), it is easy to deduce the following elementary yet useful identity (in particular) in carrying out the above-mentioned series rearrangement technique:

$$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k} \quad (0 \le\ k \le\ n;\ k,\ n \in \mathbb{N}). \tag{1.6}$$

For $\alpha = 1$, we have

$$(n-k)! = \frac{(-1)^k\, n!}{(-n)_k} \quad (0 \le k \le n), \tag{1.7}$$

which may alternatively be written in the form:

$$(-n)_k = \begin{cases} \dfrac{(-1)^k\, n!}{(n-k)!} & (0 \le k \le n), \\ 0 & (k > n). \end{cases} \tag{1.8}$$

Recall the generalized binomial theorem

$$(1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k \quad (|z| < 1). \tag{1.9}$$

Two well-known series manipulation formulas were recorded in various literature (*cf.* Rainville [5, pp. 56–57]):

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A_{k,n} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{k,n-k} \tag{1.10}$$

and (in what follows, $[x]$ denotes the greatest integer part in $x$)

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]} A_{k,n-k}, \tag{1.11}$$

which were also presented in the work of Choi and Seo [3] who investigated many other such formulas rather systematically.

Srivastava and Panda [7, p. 423, Eq. (26)] presented a definition of a general double hypergeometric function:

$$\begin{aligned} &F_{l:m;n}^{p:q;k}\left[\begin{matrix}(a_p) \ : \ (b_q) \ ; (c_k) \ ; \\ (\alpha_l) \ : (\beta_m) \ ; (\gamma_n) \ ; \end{matrix}\ x, y\right] \\ &= \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{r+s} \prod_{j=1}^{q} (b_j)_r \prod_{j=1}^{k} (c_j)_s}{\prod_{j=1}^{l} (\alpha_j)_{r+s} \prod_{j=1}^{m} (\beta_j)_r \prod_{j=1}^{n} (\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!}, \end{aligned} \tag{1.12}$$

where the several cases of convergence conditions are given in [6, p. 64].

The object of this note is to present a presumably new transformation formula associated with generalized hypergeometric series expressed in terms of the above-recalled Srivastava and Panda's function (1.12), which is more general than the one defined by Kampé de Fériet [4] (*cf.* Appell and Kampé de Fériet [1, p. 150, Eq. (29)]), by making use of series rearrangement technique which is one of the most effective methods for obtaining generating functions or other identities involving (especially) the hypergeometric series. A couple of interesting special cases of our main result allied to a known identity are also considered.

## 2. A Transformation Formula

We start with defining, for convenience, by

$$\begin{aligned} S(x,t) &:= {}_pF_q\left[\begin{matrix}a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q;\end{matrix}\ t(1-x+tx)\right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n \left[(1-x)+tx\right]^n}{n! (b_1)_n (b_2)_n \cdots (b_q)_n} t^n. \end{aligned} \tag{2.1}$$

By employing the binomial theorem in (2.1), and using (1.11), (1.6) and (1.7), we obtain

(2.2)

$$S(x,t) = \sum_{n=0}^{\infty} {}_{q+2}F_p \left[ \begin{matrix} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, 1-b_1-n, 1-b_2-n, \dots, 1-b_q-n; \\ 1-a_1-n, 1-a_2-n, \dots, 1-a_p-n; \end{matrix} (-1)^{p+q} \frac{4x}{(1-x)^2} \right] \frac{(1-x)^n (a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{t^n}{n!}.$$

On the other hand, by applying the binomial theorem in (2.1) repeatedly, we find that

$$\begin{aligned} S(x,t) &= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n [1-x(1-t)]^n}{n! (b_1)_n (b_2)_n \cdots (b_q)_n} t^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{(-n)_k}{n!} \frac{(-1)^j (a_1)_n (a_2)_n \cdots (a_p)_n x^k}{j!(k-j)!(b_1)_n (b_2)_n \cdots (b_q)_n} t^{n+j}, \end{aligned}$$

which, upon making use of the following series rearrangement identity:

$$\sum_{k=0}^{n} \sum_{j=0}^{k} B_{k,j} = \sum_{j=0}^{n} \sum_{k=j}^{n} B_{k,j}, \tag{2.3}$$

immediately yields

$$S(x,t) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{k=j}^{n} \frac{(-n)_k}{n!} \frac{(-1)^j (a_1)_n (a_2)_n \cdots (a_p)_n x^k}{j!(k-j)!(b_1)_n (b_2)_n \cdots (b_q)_n} t^{n+j}. \tag{2.4}$$

If we apply (1.11) to the first double summation of (2.4) and set $k-j=k'$ in the third summation of the resulting series and then drop the prime on $k$, we readily get

$$S(x,t) = \sum_{n=0}^{\infty} S_I(x) \frac{t^n}{n!}, \tag{2.5}$$

where, for convenience,

$$S_I(x) := \sum_{j=0}^{\left[\frac{n}{2}\right]} \sum_{k=0}^{n-2j} \frac{(-n)_k\,(-n+k)_{2j}\,\prod_{i=1}^{p}\,(a_i)_{n-j}\,x^{k+j}}{j!\,k!\,\prod_{i=1}^{q}\,(b_i)_{n-j}}. \tag{2.6}$$

It is not difficult to see that

$$\frac{(-n)_{2k}\,(-n+2k)_{2j}}{(1)_{2k}} = \frac{2^{2j}\,\left(-\frac{n}{2}\right)_{k+j}\,\left(-\frac{n}{2}+\frac{1}{2}\right)_{k+j}}{\left(\frac{1}{2}\right)_k\,k!} \tag{2.7}$$

and

$$\frac{(-n)_{2k+1}\,(-n+1+2k)_{2j}}{(1)_{2k+1}} = -\frac{n\,2^{2j}\,\left(-\frac{n}{2}+\frac{1}{2}\right)_{k+j}\,\left(-\frac{n}{2}+1\right)_{k+j}}{\left(\frac{3}{2}\right)_k\,k!}. \tag{2.8}$$

In view of the familiar identity

$$\sum_{k=0}^{n} A_k = \sum_{k=0}^{\left[\frac{n}{2}\right]} A_{2k} + \sum_{k=0}^{\left[\frac{n-1}{2}\right]} A_{2k+1} \quad (n \in \mathbb{N}), \tag{2.9}$$

the inner sum of (2.6) is separated into even and odd parts so that, using (2.7), (2.8), and (1.6), $S_I(x)$ can be expressed in terms of Srivastava and Panda's function defined by (1.12):

$$S_I(x) = \frac{(a_1)_n\,(a_2)_n\,\cdots(a_p)_n}{(b_1)_n\,(b_2)_n\,\cdots(b_q)_n}\left[P(x) - nx\,Q(x)\right], \tag{2.10}$$

where, for convenience,

$$\begin{aligned} &P(x) \\ &:= F^{2:q;0}_{0:p;1}\left[\begin{matrix} -\frac{n}{2},\,-\frac{n}{2}+\frac{1}{2}\,:\,1-b_1-n,\,\cdots,\,1-b_q-n\,;\,-\,; \\ -\quad :1-a_1-n,\,\cdots,\,1-a_p-n\,;\,\frac{1}{2}\,; \end{matrix}\right. \\ &\qquad\qquad \left.(-1)^{p+q}\,4x,\,x^2\right] \end{aligned} \tag{2.11}$$

and

$$Q(x) := F^{2:q;0}_{0:p;1}\left[\begin{matrix} -\frac{n}{2}+\frac{1}{2},\, -\frac{n}{2}+1 : 1-b_1-n,\, \cdots,\, 1-b_q-n\,;\, -\,; \\ - \qquad : 1-a_1-n,\, \cdots,\, 1-a_p-n\,;\, \frac{3}{2}\,; \end{matrix}\; (-1)^{p+q}\, 4x,\, x^2\right]. \tag{2.12}$$

Now put (2.10) into (2.5) and equate the coefficients of $t^n$ in the resulting equation and (2.2). We finally obtain an (presumably new) interesting transformation formula for the generalized hypergeometric series:

$$\begin{aligned} &{}_{q+2}F_p\left[\begin{matrix} -\frac{n}{2},\, -\frac{n}{2}+\frac{1}{2},\, 1-b_1-n,\, \ldots,\, 1-b_q-n\,; \\ 1-a_1-n,\, \ldots,\, 1-a_p-n\,; \end{matrix}\; (-1)^{p+q}\,\frac{4x}{(1-x)^2}\right] \\ &\quad = (1-x)^{-n}\,\left[P(x) - nx\,Q(x)\right], \end{aligned} \tag{2.13}$$

where $P(x)$ and $Q(x)$ are defined by (2.11) and (2.12).

Choosing

$$p-1 = 1 = q, \quad b_1 = \gamma, \quad a_1 = \gamma - b, \quad \text{and} \quad a_2 = \gamma - c \tag{2.14}$$

in (2.13) and considering an identity in Rainville [5, p. 106, Example 8], we also get a simpler yet interesting transformation formula for ${}_3F_2$:

$$\begin{aligned} &\sum_{k=0}^{n} \frac{(-1)^{n-k}\,(\gamma-b-c)_{n-k}\,(\gamma-b)_k\,(\gamma-c)_k\,x^{n-k}}{k!\,(n-k)!\,(\gamma)_k} \\ &\qquad \cdot {}_3F_2\left[\begin{matrix} -k,\, b,\, c\,; \\ 1-\gamma+b-k,\, 1-\gamma+c-k\,; \end{matrix}\; x\right] \\ &\quad = \frac{(\gamma-b)_n\,(\gamma-c)_n}{n!\,(\gamma)_n}\,\left[A(x) - nx\,B(x)\right], \end{aligned} \tag{2.15}$$

where, for convenience,

$$A(x) := F^{2:1;0}_{0:2;1}\left[\begin{matrix} -\frac{n}{2},\, -\frac{n}{2}+\frac{1}{2} : \quad 1-\gamma-n\,;\, -\,; \\ - \quad : 1-\gamma+b-n,\, 1-\gamma+c-n\,;\, \frac{1}{2}\,; \end{matrix}\; -4x,\, x^2\right] \tag{2.16}$$

and

(2.17)

$$B(x) := F^{2:1;0}_{0:2;1}\left[\begin{matrix} -\frac{n}{2}+\frac{1}{2},\ -\frac{n}{2}+1 : & 1-\gamma-n\,;\,-\,; \\ - & :1-\gamma+b-n,\ 1-\gamma+c-n\,;\,\frac{3}{2}\,; \end{matrix}\ -4x,\ x^2\right].$$

It is noted that the special case of (2.15) when $\gamma = b+c$, with the left-handed member of (2.13) replaced by (2.14), yields the familiar transformation formula for ${}_3F_2$ due to Whipple [8, p. 267, Eq. (7.1)]: For $n \in \mathbb{N}_0$, and $b$ and $c$ being independent of $n$,

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} -n, & b, & c\,; \\ & 1-b-n, & 1-c-n\,; \end{matrix}\ x\right] \\ &= (1-x)^n\,{}_3F_2\left[\begin{matrix} -\frac{1}{2}n, & -\frac{1}{2}n+\frac{1}{2}, & 1-b-c-n\,; \\ & 1-b-n, & 1-c-n\,; \end{matrix}\ \frac{-4x}{(1-x)^2}\right], \end{aligned} \tag{2.18}$$

which also appears in the work of Rainville [5, p. 88] who proved it by making use of the series rearrangement technique. It is also interesting to check that the special case of the right-hand side of (2.13) when $p-1 = 1 = q$, $a_1 = b$, $a_2 = c$, and $b_1 = b+c$ readily reduces to the right-hand side of (2.18).

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