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ANOTHER NEW HYPERGEOMETRIC GENERATING RELATION CONTIGUOUS TO THAT OF EXTON

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ABSTRACT. Very recently Professor Exton derived an interesting hypergeometric generating relation. The authors aim at deriving another hypergeometric generating relation by using the same technique developed by Exton. Some interesting special cases have also been given.

1. Introduction

The Pochhammer symbol $(a, n) = \Gamma(a+n)/\Gamma(a)$ is used frequently in this study as is the generalized hypergeometric function

$$\begin{aligned}
{}_AF_B\,[a_1, \ldots, a_A; b_1, \ldots, b_B; x] &= {}_AF_B\,[(a)\,;\,(b)\,;\,x] \\
&= \sum_n [(a_1, n) \ldots (a_A, n)\, x^n] \,/\, [(b_1, n) \ldots (b_B, n)\, n!]\,,
\end{aligned} \tag{1.1}$$

in which (throughout this paper) all indices of summation are taken to run over all of nonnegative integers and any values of parameters leading to results which do not make sense are tacitly excluded. For a detailed exposition of the properties of this function, see (for example) Exton [3], Slater [5], or Srivastava and Manocha [6].

With the help of the known result (*cf.* [2, p. 101, Eq. (5)]; see also [1, Eq. (2)]):

$$ {}_2F_1\left[a,\, a+\frac{1}{2};\, \frac{1}{2};\, x^2\right] = \frac{1}{2}\left[(1+x)^{-2a} + (1-x)^{-2a}\right]. \tag{1.2}$$

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Exton [4, p. 55, Eq. (8)] very recently obtained the following interesting and useful result:

$$\begin{aligned}(1.3)\qquad &\sum_n \left[(d,n)\left(d+\frac{1}{2},n\right)x^{2n}\right]\Big/\left[\left(\frac{1}{2},n\right)n!\right]\\ &\quad\times {}_{A+2}F_H\left[(a),\,\frac{1}{2}-n,\,-n;\,(h);\,y\right]\\ &= \frac{(1+x)^{-2d}}{2}\,{}_{A+2}F_H\left[(a),\,d,\,d+\frac{1}{2};\,(h);\,x^2y(1+x)^{-2}\right]\\ &\quad+\frac{(1-x)^{-2d}}{2}\,{}_{A+2}F_H\left[(a),\,d,\,d+\frac{1}{2};\,(h);\,x^2y(1-x)^{-2}\right].\end{aligned}$$

Many authors have been paying attention to the study of generating relations for which a general overview and detailed discussion can, among others, be found in the work of Srivastava and Manocha [6]. Here the authors aim also at deriving another interesting hypergeometric generating relation closely related to (1.3) with the aid of Bailey's identity [1, Eq. (3)]:

$$(1.4)\quad {}_2F_1\left[a,\,a+\frac{1}{2};\,\frac{3}{2};\,x^2\right]=\frac{1}{2x(1-2a)}\left[(1+x)^{1-2a}-(1-x)^{1-2a}\right]$$

by employing the same technique developed by Exton [4]. Some interesting special cases are also considered.

2. A new hypergeometric generating relation and its special cases

Consider the double series (assumed to be absolutely convergent)

$$\begin{aligned}(2.1)\qquad S &:= \sum_{m,n}\left[A_n\,(d,m+n)\left(d+\frac{1}{2},m+n\right)\left(x^2\,y\right)^n x^{2m}\right]\\ &\quad\cdot\frac{1}{\left(\frac{3}{2},m\right)m!\,n!}=\sum_n\left[A_n\,(d,n)\left(d+\frac{1}{2},n\right)\left(x^2\,y\right)^n/n!\right]\\ &\quad\cdot{}_2F_1\left[d+n,\,d+\frac{1}{2}+n;\,\frac{3}{2};\,x^2\right],\end{aligned}$$

where A_n is a general coefficient.

Replacing m by $N-n$ on the left-hand side of (2.1), we obtain

$$S=\sum_{n,N}\left[A_n\,(d,N)\,\left(d+\frac{1}{2},N\right)\,\left(x^2\,y\right)^n\,x^{2N-2n}\right] \cdot\frac{1}{\left(\frac{3}{2},N-n\right)\,(N-n)!\,n!}. \tag{2.2}$$

If we apply the well-known identities:

$$(\lambda,n-k)=\frac{(-1)^k\,(\lambda,n)}{(1-\lambda-n,k)}\quad\text{and}\quad(n-k)!=\frac{(-1)^k\,n!}{(-n,k)}\quad(0\le k\le n) \tag{2.3}$$

to (2.2), and replace N and n by n and m, respectively, in the resulting equation, we, in view of (2.1), have

$$\begin{aligned}&\sum_n\left[A_n\,(d,n)\,\left(d+\frac{1}{2},n\right)\,\left(x^2\,y\right)^n/n!\right]\,{}_2F_1\left[d+n,\,d+\frac{1}{2}+n;\,\frac{3}{2};\,x^2\right]\\&=\sum_n\left[(d,n)\,\left(d+\frac{1}{2},n\right)\,x^{2n}\right]\Big/\left[\left(\frac{3}{2},n\right)\,n!\right]\\&\quad\times\sum_m A_m\,\left(-\frac{1}{2}-n,m\right)\,(-n,m)\,y^m/m!.\end{aligned} \tag{2.4}$$

Now applying (1.4) to the inner Gauss's function on the left-hand side of (2.4) and observing the elementary identity:

$$\frac{\left(d+\frac{1}{2},n\right)}{1-2d-2n}=\frac{1}{1-2d}\left(d-\frac{1}{2},n\right) \tag{2.5}$$

we find that

(2.6)

$$\sum_n \left[(d,n)\left(d+\frac{1}{2},n\right)x^{2n}\right] \Big/ \left[\left(\frac{3}{2},n\right)n!\right] \times \sum_m A_m \left(-\frac{1}{2}-n,m\right)(-n,m)\,y^m/m!$$
$$= \frac{(1+x)^{1-2d}}{2x(1-2d)} \sum_n \left[A_n\,(d,n)\left(d-\frac{1}{2},n\right)/n!\right]\left[x^2\,y(1+x)^{-2}\right]^n$$
$$- \frac{(1-x)^{1-2d}}{2x(1-2d)} \sum_n \left[A_n\,(d,n)\left(d-\frac{1}{2},n\right)/n!\right]\left[x^2\,y(1-x)^{-2}\right]^n.$$

Putting $A_n = \{(a_1,n)\cdots(a_A,n)\}/\{(h_1,n)\cdots(h_H,n)\}$ in (2.6), we obtain the desired another new hypergeometric generating relation:

(2.7)

$$\sum_n \left[(d,n)\left(d+\frac{1}{2},n\right)x^{2n}\right] \Big/ \left[\left(\frac{3}{2},n\right)n!\right] \times {}_{A+2}F_H\left[(a),\,-\frac{1}{2}-n,\,-n\,;\,(h)\,;\,y\right]$$
$$= \frac{1}{2x(1-2d)}\left\{(1+x)^{1-2d}\,{}_{A+2}F_H\left[(a),\,d,\,d-\frac{1}{2}\,;\,(h)\,;\,x^2\,y\,(1+x)^{-2}\right]\right.$$
$$\left. -(1-x)^{1-2d}\,{}_{A+2}F_H\left[(a),\,d,\,d-\frac{1}{2}\,;\,(h)\,;\,x^2\,y\,(1-x)^{-2}\right]\right\}.$$

Finally consider some interesting special cases. Let $A_m = 1/(h,m)$ in (2.6) and we obtain

(2.8)

$$\sum_n \left[(d,n)\left(d+\frac{1}{2},n\right)x^{2n}\right] \Big/ \left[\left(\frac{3}{2},n\right)n!\right] {}_2F_1\left[-\frac{1}{2}-n,\,-n;\,h\,;\,y\right]$$
$$= \frac{1}{2x(1-2d)}\left\{(1+x)^{1-2d}\,{}_2F_1\left[d,\,d-\frac{1}{2}\,;\,h\,;\,x^2\,y\,(1+x)^{-2}\right]\right.$$
$$\left. -(1-x)^{1-2d}\,{}_2F_1\left[d,\,d-\frac{1}{2}\,;\,h\,;\,x^2\,y\,(1-x)^{-2}\right]\right\},$$

where the inner ${}_2F_1$-function on the left can be interpreted as a special Jacobi polynomial. If we let $h = d$ in (2.8), we, in view of the binomial theorem [6, p. 44, Eq. (8)], find a more compact expression:

(2.9)
$$\sum_n \left[(d,n)\left(d+\frac{1}{2},n\right)x^{2n}\right] \Big/ \left[\left(\frac{3}{2},n\right)n!\right] {}_2F_1\left[-\frac{1}{2}-n,\ -n;\ d;\ y\right]$$
$$= \frac{1}{2x(1-2d)}\left\{(1+x)^{1-2d}\left[1-x^2\,y\,(1+x)^{-2}\right]^{\frac{1}{2}-d} - (1-x)^{1-2d}\left[1-x^2\,y\,(1-x)^{-2}\right]^{\frac{1}{2}-d}\right\}.$$

On the other hand, putting $y = 1$ in (2.8) and applying the Gauss's summation theorem (*cf.* [6, p. 30, Eq. (7)]) to ${}_2F_1(1)$ on the left-hand side of (2.8), we obtain, after some simplification, an interesting identity:

(2.10)
$${}_4F_3\left[\begin{matrix} d,\ d+\frac{1}{2},\ \frac{h}{2}+\frac{1}{4},\ \frac{h}{2}+\frac{3}{4}; \\ h,\ h+\frac{1}{2},\ \frac{3}{2}; \end{matrix}\ 4x^2\right]$$
$$= \frac{1}{2x(1-2d)}\left\{(1+x)^{1-2d}\,{}_2F_1\left[d,\ d-\frac{1}{2};\ h;\ x^2\,(1+x)^{-2}\right] - (1-x)^{1-2d}\,{}_2F_1\left[d,\ d-\frac{1}{2};\ h;\ x^2\,(1-x)^{-2}\right]\right\}.$$

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