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SEMI-INVARIANT MINIMAL SUBMANIFOLDS OF CODIMENSION 3 IN A COMPLEX SPACE FORM

SEONG-CHEOL LEE, SEUNG-GOOK HAN, AND U-HANG KI

ABSTRACT. In this paper we prove the following : Let M be a real (2n-1)-dimensional compact minimal semi-invariant submanifold in a complex projective space $P_{n+1}C$. If the scalar curvature $\geq 2(n-1)(2n+1)$, then M is a homogeneous type A_1 or A_2. Next suppose that the third fundamental form n satisfies $dn = 2\theta\omega$ for a certain scalar $\theta \neq \frac{c}{2}$ and $\theta \neq \frac{c}{4}\frac{4n-1}{2n-1}$, where $\omega(X,Y) = g(X, \phi Y)$ for any vectors X and Y on a semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$. Then we prove that M has constant principal curvatures corresponding the shape operator in the direction of the distinguished normal and the structure vector ξ is an eigenvector of A if and only if M is locally congruent to a homogeneous minimal real hypersurface of $M_n(c)$.

0. Introduction

A submanifold M is called a CR submanifold of a Kaehlerian manifold $\widetilde{M}$ with complex structure J if it is endowed with a pair of mutually orthogonal and complementary differentiable distribution $(T, T^{\perp})$ such that for any $x \in M$ we have $JT_x = T_x, J_x^{\perp} \subset T_x^{\perp}M$, where $T_x^{\perp}M$ denotes the normal space of M at x([1]). In particular, M is said to be a *semi-invariant submanifold* if $\dim T^{\perp} = 1$, and the unit normal in $JT^{\perp}$ is called a *distinguished normal* to M([24], [27]). In this case, M admits an induced almost contact metric structure (Φ, ξ, g). A typical example of a semi-invariant submanifold is real hypersurfaces. For the real hypersurface case, when $\widetilde{M}$ is a complex space form, many results are known. For example, we refer to ([3], [5], [6], [8], [11], [15], [18], [19],

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[20], [21], [25] and [26]) for more details and further references. Therefore we may except to generalize some results which are valid in a real hypersurface to a semi-invariant submanifold. From this point of view, a semi-invariant submanifold with higher codimension of a complex space form are investigated by several authors ([16], [23], [24]) in connection with the shape operator and the induced almost contact metric structure. In particular, a semi-invariant submanifold of codimension 3 in a complex space form are studied in ([12], [13], [28]) by using properties of the third fundamental form of M and induced almost contact metric structure.

The main purpose of the present paper is to treat a semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$ and give sufficient conditions for the submanifold to be a real hypersurface of $M_n(c)$. The model for this note can be found in [14] by Ki, Song, Takagi.

In §**1**, we give some preliminaries and derive a series of useful formulas when the ambient space is a Kaehlerian manifold. In §**2** we also derive the structure equations and other fundamental properties on the semi-invariant submanifold when $\widetilde{M}$ is a complex space form and state important known results on a real hypersurface of a complex space form without proof.

In §**3**, we prove a generalization of Lawson's theorem([17]) to codimension 3 on a compact minimal semi-invariant submanifold of a complex projective space $P_{n+1}C$ by using the reduction theorem.

Finally §**4** is devoted to study of the third fundamental forms in a minimal semi-invariant submanifold satisfying $dn = 2\theta\omega$ for a certain scalar $\theta \neq \frac{c}{2}$ and $\theta \neq \frac{c}{4}\frac{4n-1}{2n-1}$, where $\omega(X,Y) = g(X, \phi Y)$ for any vectors X and Y. Then we prove that M is a real minimal hypersurface in a complex space form $M_n(c)$.

1. Preliminaries

Let $\widetilde{M}$ be a real $2(n+1)$-dimensional almost Hermitian manifold equipped with almost Hermitian structure J and a Riemannian metric tensor G and covered by a system of coordinate neighborhoods $\{W; y^A\}$.

Let M be a real $(2n-1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; x^h\}$ and immersed isometrically in $\widetilde{M}$ by immersion $i : M \to \widetilde{M}$. Throughout the present paper the following convention on the range of indices are used, unless otherwise stated:

$$A, B, \cdots = 1, 2, \cdots, 2n+2 : i, j, \cdots = 1, 2, \cdots, 2n-1.$$

Henceforth the summation convention will be used with respect to those systems of indices. We respect the immersion i locally by $y^A = y^A(x^h)$ and $B_j = (B_j{}^A)$ are also $(2n-1)$-linearly independent local tangent vectors of M, where $B_j{}^A = \partial_j y^A$ and $\partial_j = \frac{\partial}{\partial x^j}$. Then mutually orthogonal unit normals, C, D and E may be chosen, and the induced Riemannian metric tensor g with components g_{ji} on M is given by $g_{ji} = G(B_j, B_i)$ since the immersion i is isometric.

As is well-known, a submanifold M of an almost Hermitian manifold $\widetilde{M}$ is said to be a CR submanifold ([1], [2]) if it is endowed with a pair of mutually orthogonal complementary differentiable distribution $(T, T^\perp)$ such that for any $x \in M$ we have $JT_x = T_x, J_x{}^\perp \subset T_x{}^\perp M$, where $T_x{}^\perp M$ denotes the normal space of M at x.

In particular M is said to be a *semi-invariant submanifold*([12], [13]) provided that $\dim T^\perp$=1 or to be $CR\quad submanifold$ with CR dimension $n-1$([16]). In this case the unit normal vector field in $JT^\perp$ is called a *distinguished normal* to the semi-invariant submanifold. Thus we have (see [14])

$$JB_i = \Phi_i{}^h B_h + \xi_i C, \quad JC = -\xi^h B_h, \quad JD = -E, \quad JE = D \tag{1.1}$$

in each coordinate neighborhood, where we have put $\Phi_{ji} = G(JB_j, B_i)$, $\xi_i = G(JB_i, C)$, ξ^h being associated components of ξ_h. By the Hermitian property of J, it is clear that Φ_{ji} is skew-symmetric. A tensor field of type (1.1) with components $\Phi_i{}^h$ will be denoted by Φ. Moreover, the Hermitian property of J implies

$$\begin{aligned} \Phi_i{}^r \Phi_r{}^h &= -\delta_i{}^h + \xi_i \xi^h, \\ \xi^r \Phi_r{}^h &= 0, \quad \xi_r \Phi_i{}^r = 0, \\ g_{rs} \Phi_j{}^r \Phi_i{}^s &= g_{ji} - \xi_j \xi_i, \quad \xi_r \xi^r = 1, \end{aligned} \tag{1.2}$$

that is, the aggregate (Φ, ξ, g) defines an almost contact metric structure.

Denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to the induced Riemannian metric, equation of Gauss for M of $\widetilde{M}$

$$\nabla_j B_i = A_{ji}C + K_{ji}D + L_{ji}E, \tag{1.3}$$

where A_{ji}, K_{ji} and L_{ji} are components of the second fundamental forms in the direction of normals C, D, E respectively. Equations of Weingarten are given by

$$\begin{cases} \nabla_j C = -A_j{}^h B_h + l_j D + m_j E, \\ \nabla_j D = -K_j{}^h B_h - l_j C + n_j E, \\ \nabla_j E = -L_j{}^h B_h - m_j C - n_j D, \end{cases} \tag{1.4}$$

where $A = (A_j{}^h), A_{(2)} = (K_j{}^h)$ and $A_{(3)} = (L_j{}^h)$, which are related by $A_{ji} = A_j{}^r g_{ir}, K_{ji} = K_j{}^r g_{ir}$ and $L_{ji} = L_j{}^r g_{ir}$ respectively, and l_j, m_j and n_j being components of the third fundamental forms.

In the sequel, we denote the normal components of $\nabla_j C$ by $\nabla_j{}^\perp C$. The normal vector field C is said to be *parallel* in the normal bundle if we have $\nabla_j{}^\perp C = 0$, namely, l_j and m_j vanish identically.

In what follows we specialize to the case of an ambient Kaehlerian manifold M, that is, J is parallel. Then, by differentiating (1.1) covariantly along M and by comparing the tangential and normal parts, we get (see [28])

$$\nabla_j \Phi_i{}^h = -A_{ji}\xi^h + A_j{}^h \xi_i, \tag{1.5}$$

$$\nabla_j \xi_i = -A_{jr}\Phi_i{}^r, \tag{1.6}$$

$$K_{ji} = -L_{jr}\Phi_i{}^r - m_j \xi_i, \tag{1.7}$$

$$L_{ji} = K_{jr}\Phi_i{}^r + l_j \xi_i. \tag{1.8}$$

REMARK. To write our formulas in a convention form, in the sequel we denote by $\alpha = A_{ji}\xi^j\xi^i, h = TrA, k = TrA_{(2)}, l = TrA_{(3)}, H_{(2)} = A_{ji}A^{ji}, K_{(2)} = K_{ji}K^{ji}$ and $L_{(2)} = L_{ji}L^{ji}$.

From (1.7) and (1.8) we have

$$K_{jr}\xi^r = -m_j, \quad L_{jr}\xi^r = l_j, \tag{1.9}$$

$$m_r\xi^r = -k, \quad l_r\xi^r = l, \tag{1.10}$$

where we have used (1.2). Transforming (1.8) by $\Phi_k{}^j$ and using (1.2) and (1.7), we obtain

$$-K_{ik} - m_i\xi_k = K_{rs}\Phi_i{}^r\Phi_k{}^s + \xi_i\Phi_{kr}l^r,$$

which unable us to obtain $m_k\xi_i - m_i\xi_k = \xi_i\Phi_{kr}l^r - \xi_k\Phi_{ir}l^r$. Thus, it follows that

$$\Phi_{jr}l^r = m_j + k\xi_j, \tag{1.11}$$

which together with (1.2) yields

$$\Phi_{jr}m^r = -l_j + l\xi_j. \tag{1.12}$$

Taking the inner product (1.11) with l^j and making use of (1.10), we get

$$m_rl^r = -kl. \tag{1.13}$$

If we transform (1.7) and (1.8) by $L_k{}^i$ and take account of (1.7), (1.8) itself and (1.9), then we have respectively

$$K_{jr}L_i{}^r + K_{ir}L_j{}^r = -(l_jm_i + l_im_j), \tag{1.14}$$

$$L_{ji}{}^2 - K_{ji}{}^2 = l_jl_i - m_jm_i. \tag{1.15}$$

Now, we put $U_j = \xi^r\nabla_r\xi_j$. Then U is orthogonal to the structure vector ξ. From (1.2) and (1.6), we see that

$$\Phi_{jr}U^r = A_{jr}\xi^r - \alpha\xi_j. \tag{1.16}$$

Differentiating this covariantly along M and using (1.5) and (1.6), we find

$$\xi_j(A_{kr}U^r + \nabla_k\alpha) + \Phi_{jr}\nabla_kU^r = \xi^r\nabla_kA_{jr} - A_{jr}A_{ks}\Phi^{rs} + \alpha A_{kr}\Phi_j{}^r. \tag{1.17}$$

2. Semi-invariant submanifolds of codimension 3 in a complex space form

In the rest of this paper we shall suppose that $\widetilde{M}$ is a Kaehlerian manifold of constant holomorphic sectional curvature c, which is called a *complex space form* and denoted by $M_{n+1}(c)$. The curvature tensor of $M_{n+1}(c)$ is given by

$$R_{DCBA} = \frac{c}{4}(G_{DA}G_{CB} - G_{CA}G_{DB} + J_{DA}J_{CB} - J_{CA}J_{DB} - 2J_{DC}J_{BA}).$$

Using this and (1.1), equations of the Gauss and Codazzi are given respectively by

$$\begin{aligned} R_{kjih} =& \frac{c}{4}(g_{kh}g_{ji} - g_{jh}g_{ki} + \Phi_{kh}\Phi_{ji} - \Phi_{jh}\Phi_{ki} - 2\Phi_{kj}\Phi_{ih}) \\ &+ A_{kh}A_{ji} - A_{jh}A_{ki} + K_{kh}K_{ji} - K_{jh}K_{ki} \\ &+ L_{kh}L_{ji} - L_{jh}L_{ki}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \nabla_k A_{ji} - \nabla_j A_{ki} =& l_k K_{ji} - l_j K_{ki} + m_k L_{ji} - m_j L_{ki} \\ &+ \frac{c}{4}(\xi_k \Phi_{ji} - \xi_j \Phi_{ki} - 2\xi_i \Phi_{kj}), \end{aligned} \tag{2.2}$$

$$\nabla_k K_{ji} - \nabla_j K_{ki} = l_j A_{ki} - l_k A_{ji} + n_k L_{ji} - n_j L_{ki}, \tag{2.3}$$

$$\nabla_k L_{ji} - \nabla_j L_{ki} = m_j A_{ki} - m_k A_{ji} + n_j K_{ki} - n_k K_{ji}, \tag{2.4}$$

where R_{kjih} is covariant components of the Riemann-Christoffel curvature tensor of M, and those of the Ricci by

$$\nabla_k l_j - \nabla_j l_k = A_{jr}K_k{}^r - A_{kr}K_j{}^r + n_k m_j - n_j m_k, \tag{2.5}$$

$$\nabla_k m_j - \nabla_j m_k = A_{jr}L_k{}^r - A_{kr}L_j{}^r + n_j l_k - n_k l_j, \tag{2.6}$$

$$\nabla_k n_j - \nabla_j n_k = K_{jr}L_k{}^r - K_{kr}L_j{}^r + l_j m_k - l_k m_j + \frac{c}{2}\Phi_{kj}. \tag{2.7}$$

Let S_{ji} be the components of the Ricci tensor S of M. Then the Gauss equation (2.1) gives

$$(2.8)\quad S_{ji} = \frac{c}{4}\{(2n+1)g_{ji} - 3\xi_j\xi_i\} + hA_{ji} - {A_{ji}}^2 + kK_{ji} - {K_{ji}}^2 + lL_{ji} - {L_{ji}}^2,$$

which implies that the scalar curvature ρ of M is given by

$$(2.9)\qquad \rho = c(n^2-1) + h^2 - H_{(2)} + k^2 - K_{(2)} + l^2 - L_{(2)}.$$

If we take account of (1.9) and (1.10), then the above equation can be written as

$$(2.10)\quad \rho = c(n^2-1) + h^2 - H_{(2)} - \|K_{ji} - k\xi_j\xi_i\|^2 - \|L_{ji} - l\xi_j\xi_i\|^2,$$

where $\|F\|^2 = g(F,F)$ for any tensor field F on M.

In the following we need the following definition for the later use. The normal connection of a semi-invariant submanifold of codimension 3 in a complex space form is said to be *L-flat* if it satisfies $dn = \frac{c}{2}\omega$, that is, $\nabla_j n_i - \nabla_i n_j = \frac{c}{2}\Phi_{ji}$, where $\omega(X,Y) = g(X,\Phi Y)$ for any vectors X and Y on M(p514 [30]).

From a semi-invariant submanifold with L-flat normal connection, it is known that

THEOREM A[12]. *Let M be a semi-invariant submanifold of codimension 3 with L-flat normal connection in a complex projective space $P_{n+1}C$. If the structure vector ξ is an eigenvector of the shape operator in the direction of the distinguished normal, then we have $A_{(2)} = A_{(3)} = 0$.*

On the other hand, Takagi([26]) classified all homogeneous real hypersurfaces of P_nC as six model spaces which are said to be A_1, A_2, B, C, D and E, and Cecil-Ryan([5]) and Kimura([15]) proved that they are realized as the tubes of constant radius over Kaehlerian submanifolds. Namely, he proved the following :

THEOREM B[26]. *Let M be a homogeneous real hypersurface of P_nC. Then M is locally congruent to one of the followings:*

(A_1) *a geodesic hypersphere, that is, a tube over a hyperplane* $P_{n-1}C$,
(A_2) *a tube over a totally geodesic* $P_kC(1 \leq k \leq n-2)$,
(B) *a tube over a complex quadric* Q_{n-1},
(C) *a tube over* $P_1C \times P_{\frac{(n-1)}{2}}C$ *and* $n(\geq 5)$ *is odd,*
(D) *a tube over a complex Grassman* $G_{2,5}C$ *and* $n = 9$,
(E) *tube over a Hermitian symmetric space* $SO(10)/U(5)$ *and* $n = 15$.

Also Berndt([3]) showed that all real hypersurfaces with constant principle curvatures of a complex hyperbolic space H_nC are realized as the tubes of constant radius over certain submanifolds when the structure vector ξ is principal. Nowadays in H_nC they are said to be of type A_0, A_1, A_2, and B. He proved the following:

THEOREM C[3]. *Let* M *be a real hypersurface of* H_nC. *Then* M *has constant principal curvatures and* ξ *is principal if and only if* M *is locally congruent to one of the following :*

(A_0) *a self-tube, that is, a horosphere,*
(A_1) *a geodesic hypresphere or a tube over a hyperplane* $H_{n-1}C$,
(A_2) *a tube over a totally geodesic* H_kC $(1 \leq k \leq n-2)$,
(B) *a tube over a totally real hyperbolic space* H_nR.

For a compact minimal real hypersurface of a complex projective space the following theorem by Lawson is fundamental.

THEOREM D[17]. *Let* M *be an* n*-dimensional compact, minimal real hypersurface of* $P^{\frac{n+1}{2}}C$ *with Fubini-Study metric of constant holomorphic sectional curvature 4. If the scalar curvature of* M *is greater than or equal to* $(n+2)(n-1)$, *then* M *is an* $M^c_{p,q}$ *for some* p, q *satisfying* $p+q = m-1$, *where* $m = \frac{n+1}{2}$.

3. Compact minimal semi-invariant submanifold

Let M be a minimal semi-invariant submanifold of codimension 3 in a complex projective space $P_{n+1}C$ with constant holomorphic sectional curvature 4.

From (1.5) and (1.6), we have

$$\nabla_i \nabla_j \xi^i = -A_{jr}{}^2 \xi^r - (\nabla_k A_{ji}) \Phi^{ki}. \tag{3.1}$$

Then (2.10) is reduced to

$$\rho = 4(n^2 - 1) - H_{(2)} - K_{(2)} - L_{(2)}. \tag{3.2}$$

Transvecting ξ^i to (1.14) and using (1.9) and (1.10) with $k = l = 0$, we find

$$L_{jr} l^r + K_{jr} m^r = 0. \tag{3.3}$$

On the other hand, multiplying (2.2) with Φ^{ki} and summing for k and i , and taking account of (1.7)$\sim$(1.10) and (3.3), we have

$$(\nabla_k A_{ji}) \Phi^{ki} = -2K_{jr} m^r - 2(n-1)\xi_j,$$

which together with (3.1) yields

$$\nabla_i \nabla_j \xi^i = -A_{jr}{}^2 \xi^r + 2K_{jr} m^r + 2(n-1)\xi_j.$$

Hence we obtain

$$\xi^j \nabla_i \nabla_j \xi^i = 2(n-1) - 2m_r m^r - A_{ji}{}^2 \xi^j \xi^i$$

by virtue of (1.9).

Since we have $\operatorname{div} U = (\nabla_j \xi_i)(\nabla^i \xi^j) + \xi^j \nabla_i \nabla_j \xi^i$, the above equation implies

$$\|\nabla_j \xi_i + \nabla_i \xi_j\|^2 = 2\operatorname{div} U + 2\{H_{(2)} - 2(n-1)\} + 4m_r m^r.$$

Thus we have

$$\operatorname{div} U = \frac{1}{2}\|A\Phi - \Phi A\|^2 - H_{(2)} + 2(n-1) - 2m_t m^t \tag{3.4}$$

since M is minimal.

Using (2.10), the equation (3.4) turns out to be

$$\text{(3.5)}\quad \operatorname{div} U = \frac{1}{2}\|A\Phi - \Phi A\|^2 + \rho - 2(n-1)(2n+1) - 2m_r m^r + K_{(2)} + L_{(2)}.$$

By the way, we have

$$\text{(3.6)}\qquad \|K_{ji} + m_j\xi_i + m_i\xi_j\|^2 = K_{(2)} - 2m_r m^r.$$

Further, suppose that $\rho \geq 2(n-1)(2n+1)$. Then we have from the last two relationships

$$\text{(3.7)}\qquad A\Phi = \Phi A, \quad A_{(3)} = 0,$$

$$\text{(3.8)}\qquad K_{ji} + m_j\xi_i + m_i\xi_j = 0$$

because M is compact.

Multiplying (3.8) with K^{ji} and summing for j and i and using (1.9), we find $K_{(2)} = 2m_r m^r$.

Since it is, using (1.15) and $A_{(3)} = 0$, seen that $K_{(2)} \doteq m_r m^r - l_r l^r$, it follows that $K_{(2)} = 0$ and hence $A_{(2)} = 0$, and $l_j = m_j = 0$, that is, the distinguished normal is parallel in the normal bundle.

Let $N_0(p) = \{\eta \in T_p{}^{\perp}(M) | A_\eta = 0\}$ and $H_0(p)$ the maximal J-invariant subspace of $N_0(p)$. Since we have $A_{(2)} = A_{(3)} = 0$, the orthogonal complement of $H_0(p)$ is invariant under parallel translation with respect to the normal connection because of $\nabla_j{}^{\perp}C = 0$.

Thus by the reduction theorem in ([8], [23]) we have

THEOREM 3.1. *Let M be a real $(2n-1)$-dimensional compact, minimal semi-invariant submanifold in a complex projective space $P_{n+1}C$. If the scalar curvature $\geq 2(n-1)(2n+1)$, then M is a homogeneous type* A_1 *or* A_2.

4. The third fundamental forms in a minimal semi-invariant submanifold

In this section we shall suppose that M is a minimal semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$ and that the third fundamental form n satisfies $dn = 2\theta\omega$ for a certain scalar θ on M, namely,

$$\nabla_j n_i - \nabla_i n_j = 2\theta \Phi_{ji}. \tag{4.1}$$

Then we have from (2.7)

$$K_{jr}L_i{}^r - K_{ir}L_j{}^r + l_j m_i - l_i m_j = -2(\theta - \frac{c}{4})\Phi_{ji},$$

or, using (1.14)

$$K_{jr}L_i{}^r + l_j m_i = -(\theta - \frac{c}{4})\Phi_{ji}, \tag{4.2}$$

which together with (1.9) and (1.10) gives

$$K_{jr}l^r = 0, \ L_{jr}m^r = 0. \tag{4.3}$$

Multiplying (4.2) with Φ^{ji} and summing for j and i, and taking account of (1.8), (1.9) and (1.12), we find

$$L_{(2)} = 2(n-1)(\theta - \frac{c}{4}). \tag{4.4}$$

We notice here that θ is constant if $n > 2$ ([14]). Transvecting (4.2) with m^i and l^j, and making use of (1.11), (1.12) and (4.3), we find respectively

$$(\theta - \frac{c}{4} - m_r m^r)l_j = 0, \ (\theta - \frac{c}{4} - l_r l^r)l_j = 0.$$

Now, let Ω be a set of points such that $l_r l^r \neq 0$ on M and suppose that Ω is non-empty. Then we have

$$m_r m^r = \theta - \frac{c}{4}, \ l_r l^r = \theta - \frac{c}{4} \tag{4.5}$$

because of (1.12) on Ω. In what follows we discuss our arguments on the open subset Ω of M. Transforming (4.2) by $\Phi_k{}^i$ and taking account of (1.7) and (1.12), we find

$$K_{jk}{}^2 + \xi_k(K_{jr}m^r) + l_j l_k = (\theta - \frac{c}{4})(g_{jk} - \xi_j\xi_k), \tag{4.6}$$

from which, taking the skew-symmetric part,

$$\xi_j K_{kr} m^r - \xi_k K_{jr} m^r = 0.$$

Thus, it follows, using (1.9), that $K_{jr} m^r = -(m_r m^r)\xi_j$. Therefore (4.6) turns out to be

$$K_{ji}{}^2 = (\theta - \frac{c}{4}) g_{ji} - l_j l_i \tag{4.7}$$

by virtue of (4.5). Differentiating this covariantly along Ω, we find

$$K_j{}^r \nabla_k K_{ir} + K_i{}^r \nabla_k K_{jr} + l_j \nabla_k l_i + l_i \nabla_k l_j = 0, \tag{4.8}$$

from which, taking the skew-symmetric part with respect to indices k and j and making use of (2.3) and (2.5),

$$\begin{aligned} &K_j{}^r \nabla_k K_{ir} - K_k{}^r \nabla_j K_{ir} + l_j \nabla_k l_i - l_k \nabla_j l_i \\ &+ K_i{}^r (l_j A_{kr} - l_k A_{jr} + n_k L_{jr} - n_j L_{kr}) \\ &+ l_i (A_j{}^r K_{kr} - A_k{}^r K_{jr} + n_k m_j - n_j m_k) = 0 \end{aligned}$$

for any indices k, j and i. Thus, interchanging indices k and i, we have

$$\begin{aligned} &K_j{}^r \nabla_i K_{kr} - K_i{}^r \nabla_j K_{kr} + l_j \nabla_i l_k - l_i \nabla_j l_k \\ &+ l_j A_{ir} K_k{}^r - l_i A_{jr} K_k{}^r + n_i K_k{}^r L_{jr} \\ &- n_j K_k{}^r L_{ir} + l_k (K_i{}^r A_{jr} - K_j{}^r A_{ir} + n_i m_j - n_j m_i) = 0. \end{aligned}$$

Hence, if we use (1.14), (2.3), (2.5), and (4.2), then we obtain

$$\begin{aligned} &K_j{}^r \nabla_k K_{ir} - K_i{}^r \nabla_k K_{jr} + l_j \nabla_k l_i - l_i \nabla_k l_j \\ &+ 2 l_j A_{kr} K_i{}^r - 2 l_i A_{kr} K_j{}^r + 2(\theta - \frac{c}{4}) n_k \Phi_{ji} = 0. \end{aligned}$$

Adding this to (4.8), we obtain

$$K_j{}^r \nabla_k K_{ir} + l_j (\nabla_k l_i + A_{kr} K_i{}^r) - l_i A_{kr} K_j{}^r + (\theta - \frac{c}{4}) n_k \Phi_{ji} = 0. \tag{4.9}$$

Since we have (1.8), (4.3) and (4.7), by transforming $K_h{}^j$, we have

$$(\theta - \frac{c}{4})(\nabla_k K_{hi} - n_k L_{hi} + n_k l_h \xi_i - l_i A_{hk}) - l_h (l^r \nabla_k K_{ir}) + (A_{kr} l^r) l_h l_i = 0.$$

First of all, we prove

LEMMA 4.1. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant minimal submanifold of codimension 3 in a complex space form $M_{n+1}(c)$. If it satisfies $dn = 2\theta\omega$ and $\theta \neq \frac{c}{2}$, then $\nabla_j{}^{\perp} C = 0$, namely, the distinguished normal is parallel in the normal bundle.*

PROOF. Because of (1.9), (1.10), (1.15) and (4.7), we have

$$L_{jr}l^r = (\theta - \frac{c}{4})\xi_j. \tag{4.11}$$

Differentiating the first equation of (4.3) covariantly along Ω, we find

$$l^r \nabla_k K_{jr} + K_j{}^r \nabla_k l_r = 0,$$

which together with (4.3) implies that $(\nabla_k K_{ji})l^j l^i = 0$. Thus if we transvect l^i to (4.10) and make use of (1.10), (4.5) and (4.11), then we obtain

$$(\theta - \frac{c}{4})\{(\nabla_k K_{jr})l^r + l_j A_{kr} l^r - (\theta - \frac{c}{4})(A_{jk} + n_k \xi_j)\} = 0. \tag{4.12}$$

If $\theta = \frac{c}{4}$, then $L_{ji} = 0$ by virtue of (4.4). Hence (4.2) means $l_j = m_j = 0$ and consequently $\nabla_j{}^\perp C = 0$. Thus we may only consider the case where $\theta \neq \frac{c}{4}$.

If we take the skew-symmetric part of (4.12) and use (2.3) and (4.11), then we have

$$l_j A_{kr} l^r - l_k A_{jr} l^r = 0,$$

which implies

$$A_{jr} l^r = \sigma l_j \tag{4.13}$$

for some function σ on Ω. Because of (4.12) and (4.13), the equation (4.10) is reduced to

$$(\theta - \frac{c}{4})(\nabla_k K_{ji} - n_k L_{ji} - l_i A_{jk} - l_j A_{ik}) + 2\sigma l_j l_i l_k = 0. \tag{4.14}$$

Since we have $K_{jr}\xi^r = -m_j$, by differentiating covariantly along Ω and making use of (1.1), (1.8), (1.9), (1.10) and (4.14), we find

$$\nabla_k m_j = -n_k l_j - A_{kr} L_j{}^r. \tag{4.15}$$

If we differentiate (1.10) with $k = 0$ covariantly and take account of (1.6), (1.9), (1.12) and (4.15), we obtain

$$A_{jr} l^r = 0. \tag{4.16}$$

Thus, (4.14) is reduced to

$$(4.17) \qquad \nabla_k K_{ji} = n_k L_{ji} + l_i A_{jk} + l_j A_{ik}.$$

Substituting (4.17) into (4.9), we have

$$n_k K_{jr} L_i{}^r + l_j(\nabla_k l_i + A_{kr} K_i{}^r) + (\theta - \frac{c}{4}) n_k \Phi_{ji} = 0,$$

which transvect l^j and using (1.11) with $k = 0$, (4.3) and (4.5),

$$(4.18) \qquad \nabla_k l_j = n_k m_j - A_{kr} K_j{}^r.$$

From this and the first equation of (1.9), we verify that

$$(4.19) \qquad A_{jr} m^r = 0.$$

Differentiating (4.16) covariantly along Ω and using (4.18) and (4.19), we find

$$(4.20) \qquad (\nabla_k A_{jr}) l^r - A_{jr} A_{ks} K^{rs} = 0,$$

from which, taking the skew-symmetric part and using (1.11), (2.2), (4.3), (4.11), and (4.16), $(\theta - \frac{c}{2})(m_k \xi_j - m_j \xi_k) = 0$. Since $\theta \neq \frac{c}{2}$, it follows that $m_j = 0$ and hence $l_j = 0$ because of (1.12). This completes the proof. □

We now continue, our arguments under the same hypotheses as in Lemma 4.1. Then we have $l_j = m_j = 0$. Thus, (2.2), (2.3), (2.5), (2.6), (4.2) and (4.6) turn out respectively to

$$(4.21) \qquad \nabla_k A_{ji} - \nabla_j A_{ki} = \frac{c}{4}(\xi_k \Phi_{ji} - \xi_j \Phi_{ki} - 2\xi_i \Phi_{kj}),$$

$$(4.22) \qquad \nabla_k K_{ji} - \nabla_j K_{ki} = n_k L_{ji} - n_j L_{ki},$$

$$(4.23) \qquad A_{jr} K_i{}^r - A_{ir} K_j{}^r = 0, \quad A_{jr} L_i{}^r - A_{ir} L_j{}^r = 0,$$

$$K_{jr}L_i{}^r = -(\theta - \frac{c}{4})\Phi_{ji}, \tag{4.24}$$

$$K_{ji}{}^2 = (\theta - \frac{c}{4})(g_{ji} - \xi_j\xi_i). \tag{4.25}$$

Since we have $K_{ir}\xi^r = 0$, by differentiating covariantly along M and using (1.6) and (1.8) with $l_j = 0$, we find

$$(\nabla_k K_{ir})\xi^r = -L_{ir}A_k{}^r. \tag{4.26}$$

Differentiating (4.25) covariantly along M and using (1.6), we have

$$K_j{}^r(\nabla_k K_{ir}) + K_i{}^r(\nabla_k K_{jr}) = (\theta - \frac{c}{4})(\xi_j A_{kr}\Phi_i{}^r + \xi_i A_{kr}\Phi_j{}^r). \tag{4.27}$$

Using the quite same method as that used to (4.9) from (4.8), we can derive from (4.27) the following :

$$\begin{aligned} 2K_j{}^r\nabla_k K_{ir} =& (\theta - \frac{c}{4})\{2n_k\Phi_{ij} + (A_{ir}\Phi_j{}^r - A_{jr}\Phi_i{}^r)\xi_k \\ &+ (A_{kr}\Phi_j{}^r - A_{jr}\Phi_k{}^r)\xi_i \\ &+ (A_{kr}\Phi_i{}^r + A_{ir}\Phi_k{}^r)\xi_i\}, \end{aligned} \tag{4.28}$$

where we have used (4.22) and (4.24).

In the following, we are going to prove $A_{(2)} = 0$. By means of (4.25), we may only consider the case where $\theta - \frac{c}{4} \neq 0$ because it is already seen that θ is constant. By (4.22) we can, using $k = l = 0$, verify that $\nabla_r K_j{}^r = L_{jr}n^r$. Thus, multiplying (4.28) with g^{ki} and summing for k and i, we find

$$K_j{}^r L_{rs}n^s = (\theta - \frac{c}{4})(\Phi_{rj}n^r + \xi^i A_{ir}\Phi_j{}^r),$$

which together with (4.24) implies that $\xi^i A_{ir}\Phi_j{}^r = 0$ and hence

$$A_{jr}\xi^r = \alpha\xi_j. \tag{4.29}$$

Therefore, if we transvect (4.28) with ξ^j and take account of (1.9) and (4.29), then we obtain

$$A_{jr}\Phi_i{}^r + A_{ir}\Phi_j{}^r = 0. \tag{4.30}$$

From this and (4.21), we can prove the followings (see [9]) :

$$A_{ji}{}^2 = \alpha A_{ji} + \frac{c}{4}(g_{ji} - \xi_j\xi_i), \tag{4.31}$$

$$\nabla_k A_{ji} = -\frac{c}{4}(\xi_j\Phi_{ki} + \xi_i\Phi_{kj}). \tag{4.32}$$

By means of (4.30), the equation (4.28) can be written as

$$K_j{}^r\nabla_k K_{ir} = (\theta - \frac{c}{4})(n_k\Phi_{ij} + \xi_k A_{ir}\Phi_j{}^r + \xi_i A_{kr}\Phi_j{}^r).$$

Transforming by $K_h{}^j$ and using (1.8), (4.23), (4.25) and (4.26), we obtain

$$\nabla_k K_{ji} = n_k L_{ji} - \xi_k A_{jr} L_i{}^r - \xi_i A_{kr} L_j{}^r - \xi_j A_{ir} L_k{}^r. \tag{4.33}$$

Differentiating (1.8) with $l_j = 0$ covariantly and using (1.5), (1.7) and (4.33), we have

$$\nabla_k L_{ji} = -n_k K_{ji} + \xi_k A_{jr} K_i{}^r + \xi_i A_{kr} K_j{}^r + \xi_j A_{ir} K_k{}^r, \tag{4.34}$$

which together (1.9) with $k = l = 0$ and (4.29) implies that

$$Tr(AA_{(2)}) = 0, \qquad Tr(A^2A_{(2)}) = 0 \tag{4.35}$$

because of (4.21).

On the other hand, we have $A_{(2)}\xi = 0$ and $TrA_{(2)} = 0$ and (4.25), the shape operator $A_{(2)}$ has at most three distinct constant eigenvalues $0, \sqrt{\theta - \frac{c}{4}}, -\sqrt{\theta - \frac{c}{4}}$ with multiplicities $1, n-1, n-1$ respectively.

By (4.29), (4.30) and (4.31), we also see that A has at most three distinct constant eigenvalues $\alpha, (\alpha + \sqrt{D})/2, (\alpha - \sqrt{D})/2$ with multiplicities $1, r, s$ respectively, where $D = \alpha^2 + c, r + s = 2n - 2$.

Since we have $AA_{(2)} = A_{(2)}A$, it follows that A and $A_{(2)}$ are diagonalizable at the same time. Because of (4.34), we have $(\theta - \frac{c}{4})r(\alpha^2 + c) = 0$. Thus $s = 2(n-1)$ and consequently A has two constant eigenvalues α and $(\alpha - \sqrt{D})/2$ with multiplies $1, 2(n-1)$ respectively. Since M is minimal, it follows that

$$n\alpha = (n-1)\sqrt{D}. \tag{4.36}$$

Differentiating (4.33) covariantly along M and using (1.6), (1.9), (4.31), (4.32) and (4.34), we find

$$\begin{aligned}\nabla_h \nabla_k K_{ji} = (\nabla_h n_k) L_{ji} - \frac{c}{4}(K_{ki}\xi_j\xi_h + K_{jh}\xi_k\xi_i + 2K_{ih}\xi_j\xi_k) \\ + B_{hkji} - \alpha(\xi_j\xi_h A_{kr}K_i{}^r + \xi_k\xi_i A_{jr}K_h{}^r + 2\xi_j\xi_k A_{ir}K_h{}^r) \\ + (A_{hs}\Phi_j{}^s)(A_{kr}L_i{}^r) + (A_{hs}\Phi_k{}^s)(A_{ir}L_j{}^r) + (A_{hs}\Phi_i{}^s)(A_{jr}L_k{}^r)\end{aligned}$$

where B_{hkji} is a certain tensor with $B_{hkji} = B_{khji}$.

Multiplying the last equation with Φ^{hk} and summing for h and k, and making use of (1.2), (1.7), (1.8), (4.1), (4.23), (4.30) and (4.31), we find

$$\Phi^{hk}\nabla_h \nabla_k K_{ji} = \{2(n-1)\theta - \frac{c}{2}\}L_{ji} - \alpha A_{jr}L_i{}^r,$$

or, using the Ricci identity for $A_{(2)}$,

$$\Phi^{hk}(R_{hkjr}K_i{}^r + R_{hkir}K_j{}^r) = -\{c - 4(n-1)\theta\}L_{ji} + 2\alpha A_{jr}L_i{}^r. \tag{4.37}$$

On the other hand, we have from (2.1)

$$\Phi^{kl}R_{klih} = (cn + \frac{c}{2})\Phi_{hi} - 2\alpha A_{hr}\Phi_i{}^r + 4K_{hr}L_i{}^r,$$

where we have used (1.8), (1.9), (4.30) and (4.31), which together with (1.8) and (4.25) yields

$$\Phi^{kl}(R_{klir}K_j{}^r + R_{kljr}K_i{}^r) = \{8\theta - (2n+3)c\}L_{ji} - 4\alpha A_{jr}L_i{}^r.$$

From this and (4.37), it follows that

$$3\alpha A_{jr}L_i{}^r = \{2(n+1)\theta - (n+2)c\}L_{ji}, \tag{4.38}$$

which implies

$$3\alpha(A_{ji} - \alpha\xi_j\xi_i) = \{2(n+1)\theta - (n+2)c\}(g_{ji} - \xi_j\xi_i).$$

If we take the trace of this, then we obtain

$$-3\alpha^2 = 2(n-1)\{2(n+1)\theta - (n+2)c\}, \tag{4.39}$$

which together with (4.37) implies that

$$\theta = \frac{c}{4}\frac{4n-1}{2n-1}. \tag{4.40}$$

From this and Lemma 4.1, we have

LEMMA 4.2. *Let M be a real $(2n-1)$-dimensional semi-invariant minimal submanifold of codimension 3 in a complex space form $M_{n+1}(c)$. If it satisfies $dn = 2\theta\omega$ for a certain scalar $\theta \neq \frac{c}{2}$ and $\theta \neq \frac{c}{4}\frac{4n-1}{2n-1}$, then we have $A_{(2)} = A_{(3)} = 0$.*

REMARK. It is proved in ([14]) that the following : Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 satisfying $dn = 2\theta\omega$ for a certain scalar $\theta < \frac{c}{2}$ and $\nabla_j{}^{\perp}C = 0$ in a complex projective space $P_{n+1}C$. Then the same conclusion as that Lemma 4.2 are valid.

By the reduction theorem in ([8], [23]) and by Lemma4.1 and Lemma4.2, we have

THEOREM 4.3. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant minimal submanifold in a complex space form $M_{n+1}(c)$. If the third fundamental form n satisfies $dn = 2\theta\omega$ for a certain scalar $\theta \neq \frac{c}{2}$ and $\theta \neq \frac{c}{4}\frac{4n-1}{2n-1}$ where $\omega(X,Y) = g(X, \Phi Y)$ for any vector X and Y on M, then M is a minimal real hypersurface in a complex space form $M_n(c)$.*

Owing to Theorem B, C and Theorem 4.3, we have

THEOREM 4.4. *Let M be a real $(2n-1)$-dimensional $(n>2)$ semi-invariant minimal submanifold in a complex space form $M_{n+1}(c)$ such that the third fundamental tensor satisfies $dn = 2\theta\omega$ for a certain scalar θ, $\theta \neq \frac{c}{2}$ and $\theta \neq \frac{c}{4}\frac{4n-1}{2n-1}$. Then M has constant principal curvatures corresponding the shape operator in the direction of distinguished normal and the structure vector ξ is an eigenvector of A if and only if M is locally congruent to a homogeneous minimal real hypersurface of $M_n(c)$.*

References

[1] A. Bejancu, *CR-submanifolds of a Kaherian manifold I*, Proc. Amer. Math. Soc. **69** (1978), 135–142.

[2] A. Bejancu, *Geometry of CR-submanifolds*, D. Reidel Pub. Co., 1978.

[3] J. Berndt, *Real hypersurfaces with constant principal curvatures in a hyperbolic space*, J. reine angew Math. **395** (1989), 132–141.

[4] D. E. Blair, G. D. Ludden, and K. Yano, *Semi-invariant immersion*, Kodai Math. Sem. Rep. **27** (1976), 313–319.

[5] T. E. Cecil and P. J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499.

[6] J. T. Cho and U-H. Ki, *Real hypersurfaces of a complex projective space in terms of the Jacobi operators*, Acta Math. Hungar. **80** (1998), 155–167.

[7] Y.-W. Choe and M. Okumura, *Scalar curvature of a certain CR-submanifold of a complex projective space*, Arch. Math. **68** (1997), 340–346.

[8] J. Erbacher, *Reduction of the codimension of an isometric immersion*, J. Differential Geom. **5** (1971), 333–340.

[9] U-H. Ki, *Cyclic-parallel real hypersurfaces of a complex projective space*, Tsukuba J. Math. **12** (1988), 259–268.

[10] ———, *Real hypersurfaces with parallel Ricci tensor of a complex space form*, Tsukuba J. Math. **13** (1989), 73–81.

[11] ———, *A survey on real hypersurfaces in a complex space form*, Korean Annals of Math. **15** (1998), 153–165.

[12] U-H. Ki and H. -J. Kim, *Semi-invariant submanifolds with lift-flat normal connection in a complex projective space*, Kyungpook Math. J. **40** (2000), 185–194.

[13] U-H. Ki, S. -J. Kim, S. -B. Lee, and I. -Y. Yoo, *Semi-invariant submanifolds with harmonic curvature*, J. Korean Math. Soc. **27** (1990), 157–166.

[14] U-H. Ki, H. Song, and R. Takagi, *Submanifolds of codimension 3 admitting almost contact metric structure a complex projective space*, Nihonkai Math. J. **11** (2000), 57–86.

[15] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. **296** (1986), 137–149.

[16] J. H. Kwon and J. S. Pak, *CR-submanifolds of $(n-1)$ CR-dimension in a complex projective space*, Saitama Math. J. **15** (1997), 55–65.

[17] H. B. Lawson Jr, *Rigidity theorems in rank-1 symmetric spaces*, J. Differential Geom. **4** (1970), 349–357.
[18] S. Maeda, *Ricci tensors of real hypersurfaces in a complex projective space*, Proc. Amer. Math. Soc. **122** (1994), 1229–1235.
[19] S. Montiel, *Real hypersurfaces of a complex hyperbolic space*, J. Math. Soc. Japan **37** (1985), 515–535.
[20] S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*, Geom. Dedicata **20** (1986), 245–261.
[21] M. Okumura, *On some real hypersurfaces in a complex projective space*, Trans. Amer. Math. Soc. **212** (1975), 355–364.
[22] ———, *Normal curvature and real submanifold of the complex projective space*, Geometriae Didicata **7** (1978), 509–517.
[23] ———, *Codimension reduction problem for real submanifold of complex projective space*, Collo. Math. Janos Bolyai Dih. Geom. **56** (1989), 573–585.
[24] M. Okumura and L. Vanheke, *n-dimensional real submanifolds with* $(n-1)$*-dimensional maximal holomorphic tangent subspace in complex projective spaces*, Rendiconti del circolo Mat. di Palermo **43** (1994), 233–249.
[25] R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10** (1973), 495–506.
[26] ———, *Real hypersurfaces in a complex projective space with constant principal curvatures I,II*, J. Math. Soc. Japan **27** (1975), 43-53 and 507–516.
[27] Y. Tashiro, *Relations between the theory of almost complex spaces and that of almost contact space*, Sugaku in Japaness **16** (1964), 34–61.
[28] K. Yano and U-H. Ki, *On* $(f, g, u, v, w, \lambda, \mu, \nu)$*-structure satisfying* $\lambda^2 + \mu^2 + \nu^2 = 1$, Kodai Math. Sem. Rep. **29** (1978), 285–307.
[29] K. Yano and M. Kon, *CR submanifolds of Kaehlerian and Sasakian manifolds*, Birkhäuser, 1983.
[30] ———, *Structures on manifolds*, World Scientific, Pulb. Co., Singapore, 1984.

Seong-Cheol Lee, Seung-Gook Han
Department of Mathematics
Chosun University
Kwangju 502-759, Korea

U-Hang Ki
Department of Mathematics
Kyungpook University
Taegu 702-701, Korea