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# REMARKS ON THE TOPOLOGY OF LORENTZIAN MANIFOLDS

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ABSTRACT. The purpose of this paper is to give a necessary and sufficient condition for a smooth manifold to admit a Lorentzian metric. As an application of this result, on *Lorentzian manifolds* we have shown that the existence of a 1-dimensional distribution is equivalent to the existence of a non-vanishing vector field.

## 1. Introduction

Along this paper we will assume that a manifold $M$ is of class $C^\infty$ and has a *countable basis* in its topology. Therefore a manifold is paracompact. By using a partition of unity one can show that a manifold always admits a Riemannian metric. The same assertion is not true for Lorentzian metrics. In fact, we can consider a Lorentzian metric on each coordinate open subset but it is not possible to glue, by a partition of unity, as in the positive definite case, the local Lorenzian metrics to produce a Lorentzian metric on all of $M$.

In the sense of S. Kobayashi and K. Nomizu [2], the second countability is dropped from the list of defining axioms. Now if we consider manifolds without the assumption "to have a countable basis in its topology ", it can be shown that: If a manifold (under this new definition) admits a Lorentzian metric, it is paracompact (this fact is not trivial and was published in an article by K. B. Marathe [3] or M. Spivak [7], Corollary 25). But such a topological condition of paracompactness for a smooth manifold is not sufficient enough to admit a Lorentzian metric.

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From this point we want to find out some useful geometric conditions for a smooth manifold to admit a Lorentzian metric as in section 2. In this section we prove a Main Theorem, which gives an equivalent condition between a Lorentzian metric and a smooth 1-dimensional distribution. Though in general a 1-dimensional distribution is not related to a non-vanishing vector field (Example 3.1), but they can be equivalent to each other when we consider only Lorentzian manifolds.

Moreover, according to compact or non-compact manifolds, the mutual relation between a 1-dimensional distribution and a non-vanishing vector field for such kind of manifolds will be explained much more detail throughout some remarks and examples given in sections 2 and 3.

## 2. Main Theorem

With our definition of smooth manifold in the introduction, every manifold does not necessarily admit a Lorentzian metric. In this section we want to prove our Main Theorem, which can be valid only on Lorentzian manifolds. Then it will be another new condition which is equivalent to some conditions given in a theorem of B. O'Neill [4]. Moreover, we will give some related remarks concerning with this Theorem. Now we want to prove the following:

MAIN THEOREM. *A smooth manifold $M$ admits a Lorentzian metric if and only if it admits a smooth* 1*-dimensional distribution.*

PROOF. Assume that $M$ admits a Lorentzian metric $g_L$. Now we take an arbitrary Riemannian metric $g_R$ on $M$. Define a (1,1)-tensor field $\lambda$ on $M$ by

$$g_L(u,v) = g_R(\lambda(u), v)$$

for any vectors $u, v \in T_pM$ and any point $p \in M$. At any point $\lambda$ is self-adjoint with respect to $g_R$. Hence $\lambda$ has gotten a $g_R$-orthonormal basis of $T_pM$ consisting of eigenvectors.

Note that the non-singularity of the tensor field $\lambda$ implies that all the eigenvalues are non-zero. Even more, Lorentzian metric $g_L$ implies that the tensor field $\lambda$ has only a negative eigenvalue and $n-1$ positive eigenvalues at any point where $n \geq 2$ and $n = \dim(M)$. So we have a 1-dimensional distribution $\mathfrak{D}$ by putting $\mathfrak{D}_p$ such an eigenspace corresponding to the negative eigenvalue of $\lambda$ on $T_pM$.

Note that the distribution $\mathfrak{D}$ is not uniquely determined by $g_L$. In fact, $\mathfrak{D}$ depends on the arbitrarily chosen Riemannian metric $g_R$.

Conversely, if we have a 1-dimensional distribution $\mathfrak{D}$ on $M$ and we consider again an arbitary Riemannian metric $g_R$ on all of $M$, then we can write locally

$$\mathfrak{D} = \mathrm{Span}\{X\} \text{ with } g_R(X, X) = 1.$$

On the open subset $\mathcal{U}$ of $M$ where the above equality for $\mathfrak{D}$ is true, let us consider a 1-form

$$\omega(u) := g_R(u, X_q)$$

for any vector $u \in T_qM$ and any point $q \in \mathcal{U}$ and put on $\mathcal{U}$

$$g_L(u, v) := g_R(u, v) - 2\omega(u)\omega(v)$$

for any vectors $u, v \in T_qM$ and any point $q \in \mathcal{U}$. Then $g_L$ becomes a Lorentzian metric on the open subset $\mathcal{U}$ of $M$. Moreover, $\omega(u)\omega(v)$ does not depend on the local $g_R$-orthonormal basis of $\mathfrak{D}$ (Note $g_R$ $(u, X_q)$ $g_R$ $(v, X_q) = g_R(u, -X_q)$ $g_R(v, -X_q)$). Therefore, in such a situation we conclude that $g_L$ can be extended to all of $M$ and becomes a Lorentzian metric on $M$. □

Now let us give some remarks concerned with this Theorem much more detail as follows:

REMARK 2.1. Note that if $\mathfrak{D}^\perp$ is the $g_R$-orthogonal complementary distribution of $\mathfrak{D}$, then the (1,1)-tensor field $\lambda$ defined by $g_L$ and $g_R$ is

$$\lambda(u) = u_1 - u_2$$

for any vector $u \in T_pM$ and any point $p \in M$, where $u = u_1 + u_2$ with respect to the decomposition $T_pM = \mathfrak{D}_p^\perp \oplus \mathfrak{D}_p$. Then it follows that

$g_L = g_R$ on $\mathfrak{D}^\perp$, $g_L = -g_R$ on $\mathfrak{D}$ and $g_L(u,X) = 0$ for $u \in \mathfrak{D}^\perp$ for any $X \in \mathfrak{D}$.

Now let us assume, in particular, that there exists a $X \in \mathfrak{X}(M)$ such that $X_p \neq 0$, for any $p \in M$. Clearly, the distribution $\mathfrak{D}$ defined by $\mathfrak{D}_p = \mathrm{Span}\{X_p\}$ is a 1-dimensional distribution on $M$. Now let us take an arbitrary Riemannian metric $g_R$ on $M$. Then the Lorentzian metric previously constructed $g_L$ is given by

$$g_L(u,v) = \frac{1}{g_R(X_p, X_p)} g_R(u,v) - 2\frac{g_R(u, X_p) g_R(v, X_p)}{g_R(X_p, X_p)^2}$$

for any vectors $u, v \in T_pM$ and any point $p \in M$. Note that $g'_R = \frac{1}{g_R(X,X)} g_R$ is Riemannian and satisfies $g'_R(X,X) = 1$.

On the other hand, we have

$$g'_R(u, X_p) g'_R(v, X_p) = \frac{g_R(u, X_p) g_R(v, X_p)}{g_R(X_p, X_p)^2}.$$

Therefore $\tilde{g}_L := g_R(X,X) g_L$, that is,

$$\tilde{g}_L(u,v) = g_R(u,v) - 2\frac{g_R(u, X_p) g_R(v, X_p)}{g_R(X_p, X_p)}$$

for any vectors $u, v \in T_pM$ and any point $p \in M$ is also a Lorentzian metric on $M$. Moreover, it satisfies $\tilde{g}_L(X,X) = -g_R(X,X) < 0$ on all $\mathfrak{D}$, and $\tilde{g}_L = g_R$ on $\mathfrak{D}^\perp$, $\tilde{g}_L(u,v) = 0$ for $u \in \mathfrak{D}^\perp$, $v \in \mathfrak{D}$.

Along this remark we give an example such that

EXAMPLE 2.2. We note that the Lorentzian metric of Lorentz-Minkowski space $\mathbb{L}^n$ is obtained in this way. In fact, consider on $R^n$, $n \geq 2$, the usual Riemannian metric

$$g_R = dx_1^2 + \cdots + dx_{n-1}^2 + dx_n^2$$

and the vector field $X = \frac{\partial}{\partial n}$. In this case

$$g_L = dx_1^2 + \cdots + dx_{n-1}^2 - dx_n^2$$

is the Lorentzian metric of $\mathbb{L}^n$.

As an application of Proposition and Remark 2.1, we also note that

REMARK 2.3. If $M$ is not compact, then there exists a function $f \in C^\infty(M)$ such that $(df)_p \neq 0$, for any $p \in M$. Now let us take an arbitary Riemannian metric $g_R$ on $M$ and consider $X = \nabla f$ (the gradient of $f$ with respect to $g_R$). Using the argument in Main Theorem and Remark 2.1, we can construct a Lorentzian metric on all of $M$. Hence, any non-compact smooth manifold admits a Lorentzian metric.

## 3. Some related remarks

Before going to state important remarks from our Main Theorem, we give some relation between a 1-dimensional distribution $\mathfrak{D}$ and a non-vanishing vector field on $M$.

Given a 1-dimensional distribution $\mathfrak{D}$ on $M$, we take an auxiliary Riemannian metric $g_R$ on $M$ and we can write $\mathfrak{D}_p = \text{Span}\{v\}$ with $g_R(v, v) = 1$, $v \in T_pM$. Consider $(p, v), (p, -v)$ as points of a new set $\tilde{M}$. We can define in $\tilde{M}$ a structure of smooth manifold(in a similar way to the "double orientable cover" of a non-orientable manifold). The natural projection $\pi : \tilde{M} \to M$ becomes a local diffeomorphism.

In fact $\pi$ is a double covering map. Moreover, there exists a vector field $X \in \mathfrak{X}(M)$ such that $\mathfrak{D}_p = \text{Span}\{X_p\}$, $p \in M$, if and only if $\pi$ is trivial.

Consequently, if $M$ is 1-connected(simply connected) and admits a 1-dimensional smooth distribution $\mathfrak{D}$, then there exists a $X \in \mathfrak{X}(M)$ such that $\mathfrak{D}_p = \text{Span}\{X_p\}$ for any $p \in M$ (hence $X_p \neq 0$, for any $p \in M$). This fact follows from the following well-known result: every covering of a simply connected manifold is trivial. But in general such a situation that a 1-dimensional smooth distribution to be a non-vanishing vector field can not be valid when we consider only on compact manifolds. For this let us give the following:

EXAMPLE 3.1. We now construct a 1-dimensional distribution on a compact, orientable and 4-dimensional manifold which can not be lifted to a vector field.

Let $M = S^1 \times SO(3)$. Then $M$ is parallelizable and therefore vector fields on $M$ can be identified with maps $M \to R^4$ and 1-dimensional distributions with map $M \to RP^3$ respectively.

Next let us consider a diffeomorphism $\phi : SO(3) \to RP^3$. We can define a mapping $D : M \to RP^3$ by $(e^{i\theta}, A) \to D(e^{i\theta}, A) := \phi(A)$. Let $D_* : \pi_1(M) \to \pi_1(RP^3)$ be the induced homomorphism between the fundamental groups. If $\alpha \in \pi_1(S^1)$, $\beta \in \pi_1(SO(3))$ are the generators, we have

$$(*) \qquad D_*(\alpha, \beta) = \phi_*(\beta) \neq 0.$$

Now assume $D$ lifts to a vector field $X$. Then $X$ determines a map

$$\tau : M \to R^4 - \{0\}$$

such that

$$D = \pi \circ \tau$$

where $\pi : R^4 - \{0\} \to RP^3$ is the projection. It follows that

$$D_* = \pi_* \circ \tau_*.$$

But $\tau_*(\alpha, \beta) = 0$, because $\pi_1(R^4 - \{0\}) = 0$. Hence

$$D_*(\alpha, \beta) = 0,$$

which is a contradiction with the condition (*). Thus, $D$ does not lift to a vector field.

Now the index of a 1-dimensional smooth distribution for an oriented, even dimensional, smooth manifold can be defined explicitly in W. Greub, S. Halperin and R. Vanstone [1].

In fact, under these assumptions it can be proved that a vector field $X$ with zeros $p_1, \cdots, p_k \in M$ can be deformed into a vector field without zeros $Y$. *Observe that we do not claim that a given line field can be lifted to a vector field*(See the Example).

But for a compact, orientable and connected $n$-dimensional manifold $M$ we know that $M$ admits a Lorentzian metric if and only if $\chi(M) = 0$.

Moreover, in this case it is known that there exists a non-vanishing vector field on $M$ (See B. O'Neill[4] page 149).

We remark that $\chi(M) = \sum_{k=0}^{n}(-1)^k\beta_k(M)$, where $\beta_k(M) = \dim H^k(M)$ denotes the $k$-th Betti number of $M$ and hence Poincaré duality implies that if $n$ is an odd number, then $\chi(M) = 0$.

In conclusion, combining our Main Theorem with the above facts, we assert the following:

COROLLARY 3.2. *For a compact orientable manifold $M$, the existence of a Lorentzian metric is equivalent to each one of the following conditions:*

(1) *There exists a 1-dimensional smooth distribution on $M$,*
(2) $\chi(M) = 0$ *(i.e., the Euler characteristic of $M$ is zero),*
(3) *There exists a nowhere zero (nonvanishing) vector field on $M$ (not necessary timelike).*

REMARK 3.3. The assumption "orientable" in Theorem can be dropped by passing to the double covering $\tilde{M}$ of $M$ which is orientable if $M$ is not orientable. Recall that $\chi(\tilde{M}) = 2\chi(M)$.

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