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FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIVE MAPPINGS ON GENERALIZED BOUNDED METRIC SPACES

HAN GOO JUNG AND JEONG SHEOK UME

ABSTRACT. In this paper, we shall prove a fixed point theorem which is more general than that of Ćirić, Kannan and Rhoades.

1. Introduction

Dhage [2] introduced a new structure of a generalized metric space, that is, D-metric space and obtained fixed point theorems on this space. Dhage [2] and Rhoades [4] have extended some fixed point theorems satisfying certain contractive conditions on the D-metric space. In this paper, we shall prove a fixed point theorem which is more general than the results of Ćirić [1], Kannan [3], and Rhoades [4].

2. Preliminaries

Before proving our main theorems, we will introduce some definitions and lemmas. Throughout this paper, let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}^+$ be the set of all nonnegative real numbers and let $\mathbb{N}$ be the set of all positive integer.

DEFINITION 2.1. Let X be any set. A D-metric for X is a function $D : X \times X \times X \longrightarrow \mathbb{R}$ such that
(i) $D(x, y, z) \geq 0$ for all $x, y, z \in X$, and equality holds if and only if $x = y = z$,
(ii) $D(x, y, z) = D(p\{x, y, z\})$, where p denotes a permutation function of $\{x,\ y,\ z\}$,

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(iii) $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)$ for all $x,y,z,a \in X$.
If a function D is D-metric for X, then the ordered pair (X,D) is called a D-metric space or the set X together with D-metric is called a D-metric space.

DEFINITION 2.2. A sequence $\{x_n\}$ of points of a D-metric space X converge to a point $x \in X$ if for an arbitrary $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n, m \geq n_0, D(x_m, x_n, x) < \varepsilon$.

DEFINITION 2.3. A sequence $\{x_n\}$ of points of a D-metric space X is a Cauchy sequence if for an arbitrary $\varepsilon > 0$, there exists a positive integer n_0 such that for all $p, n, m \geq n_0, D(x_m, x_n, x_p) < \varepsilon$.

DEFINITION 2.4. D-metric space X is complete if Cauchy sequence $\{x_n\}$ in X converge in X.

DEFINITION 2.5. A set $S \subset X$ is said to be bounded if there exists a constant $k > 0$ such that $D(x,y,z) \leq k$ for all $x,y,z \in S$ and the constant k is called a D-bound of S.

DEFINITION 2.6. Let $x_0 \in X$ and $\varepsilon > 0$ be given. Then we define the open ball $B(x_0,\varepsilon)$ in X centered at x_0 of radius of ε by

$$B(x_0,\varepsilon) = \{y \in X \mid D(x_0,y,y) < \varepsilon \text{ if } y = x_0 \text{ and } \sup_{z \in X} D(x_0,y,z) < \varepsilon \text{ if } y \neq x_0\}.$$

Then the collection of all open balls $\{B(x,\varepsilon) : x \in X\}$ define the topology on X denoted by τ. Throughout this paper we assume that the D-metric space X is equipped with the topology τ.

DEFINITION 2.7. Let T be a mapping of a metric space M into itself. For $A \subset M$, let $\delta(A) = \sup\{d(x,y) : x,y \in A\}$ and for each $x \in M$, let

$$o(x,n) = \{x, Tx, \cdots, T^n x\}, \quad n = 1, 2, \cdots,$$
$$o(x,\infty) = \{x, Tx, \cdots\}.$$

DEFINITION 2.8. A space M is said to be *T-orbitally complete* if and only if every Cauchy sequence which is contained in $o(x,\infty)$ for some $x \in M$ converges in M.

Using the definition of a D-metric for X and topology τ on X, we have the following Lemma.

LEMMA 2.9. *The D-metric D is a continuous function on $X \times X \times X$ in the topology τ on X.*

Let Φ denote the class of all functions $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

(i) ϕ is continuous,

(ii) ϕ is nondecreasing,

(iii) $\phi(t) < t$ for $t > 0$,

(iv) $\sum\limits_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t \in \mathbb{R}^+$, and

(iv) $g(t) = \frac{t}{t-\phi(t)}$ is decreasing on $[0,\infty)$. In this paper, if T satisfies (3.1), then T is said to be a generalized contractive mapping on D-metric space, and if T satisfies (2.5), then T is said to be a generalized contractive mapping on metric space.

LEMMA 2.10. *Let X be a D-metric space, $T : X \longrightarrow X$ and $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that*

$$D(Tx,Ty,Tz) \le \phi\big(\max\{D(x,y,z), D(x,Tx,z), D(y,Ty,z), D(x,Ty,z), D(y,Tx,z)\}\big)$$

for all $x,y,z \in X$, $\phi \in \Phi$. Then

$$D(x_n, x_{n+p}, x_{n+p+t}) \le \phi^j \left[\max_{\substack{n-j\le\alpha\le n+p+t \\ n-j+1\le\beta\le n+p+t+1 \\ \gamma=n+p+t-j}} \{D(x_\alpha,\ x_\beta,\ x_\gamma)\} \right] \tag{2.1}$$

where $1 \le j \le n$ and $n,p,t \in \mathbb{N}$.

PROOF. Let $x_{n+1} = Tx_n$, for all $n \in \{0\} \cup \mathbb{N}$ and $x_0 \in X$ be given. We prove the inequality (2.1) by mathematical induction. It is easy to see that inequality (2.1) holds for j=1. Suppose that (2.1) holds for $j = k(k \geq 1)$, that is,

$$\begin{aligned}(2.2)\qquad & D(x_n, x_{n+p}, x_{n+p+t}) \\ & \leq \phi^k \left[\max_{\substack{n-k \\ \leq \alpha \leq n+p+t \\ n-k+1 \leq \beta \leq n+p+t+1 \\ \gamma = n+p+t-k}} \{D(x_\alpha,\ x_\beta,\ x_\gamma)\} \right].\end{aligned}$$

By induction, it remains to show that

$$\begin{aligned}(2.3)\qquad & D(x_n, x_{n+p}, x_{n+p+t}) \\ & \leq \phi^{k+1} \left[\max_{\substack{n-k-1 \\ \leq \alpha \leq n+p+t \\ n-k \leq \beta \leq n+p+t+1 \\ \gamma = n+p+t-k-1}} \{D(x_\alpha,\ x_\beta,\ x_\gamma)\} \right].\end{aligned}$$

Using hypothesis and (2.2),

(2.4)

$$\begin{aligned} & D(x_\alpha, x_\beta, x_\gamma) = D(Tx_{\alpha-1}, Tx_{\beta-1}, Tx_{\gamma-1}) \\ & \leq \phi \left[\max_{\substack{\min(\alpha-1,\beta-1) \leq p \leq \max(\alpha-1,\beta-1) \\ \min(\alpha,\beta-1) \leq q \leq \max(\alpha,\beta) \\ r=\gamma-1}} \{D(x_p,\ x_q,\ x_r)\} \right].\end{aligned}$$

From (2.2) and (2.4), we obtain condition (2.3). Therefore Lemma 2.8 is proved. □

LEMMA 2.11. *Let (M, d) be a metric space, $T : M \longrightarrow M$, and let n be any positive integer and $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that*

$$\begin{aligned}(2.5)\qquad d(Tx, Ty) \leq \phi\Big(& \max\{d(x, y), d(x, Tx), d(y, Ty), \\ & d(x, Ty), d(y, Tx)\}\Big)\end{aligned}$$

for all $x, y \in M, \phi \in \Phi$. Then the followings hold;
(a) *For each $x \in M$ and all positive integers i and j, $i, j \in \{1, 2, \cdots, n\}$ implies $d(T^i x, T^j x) \leq \phi(\delta[o(x,n)])$.*
(b) *For each $x \in M$ and every positive integer n, there exists a positive integer $k \leq n$ such that $d(x, T^k x) = \delta[o(x,n)]$.*

PROOF. Let $x \in M$, let n be any positive integer and let i, j and T satisfy conditions of Lemma 2.11.
Then $T^i x, T^j x$, $T^{i-1}x$, $T^{j-1}x \in o(x,n)$, where $T^0 x = x$. We have

$$\begin{aligned} d(T^i x, T^j x) &\leq \phi\Big(\max\{d(T^{i-1}x, T^{j-1}x), d(T^{i-1}x, T^i x), d(T^{j-1}x, T^j x\Big), \\ &\qquad\qquad d(T^{i-1}x, T^j x), d(T^{j-1}x, T^i x)\}) \\ &\leq \phi(\delta[o(x,n)]) \\ &< \delta[o(x,n)], \end{aligned}$$

which proves Lemma (a) and (b). □

LEMMA 2.12. *If T satisfies condition* (2.5) *in* Lemma 2.11. *Then*

$$\delta[o(x,\infty)] \leq g(\delta[o(x,1)])d(x,Tx)$$

for all $x \in M$.

PROOF. Let $x \in M$ be arbitrary. Since $\delta[o(x,1)] \leq \delta[o(x,2)] \leq \cdots$, we know that $\delta[o(x,\infty)] = \sup\{\delta[o(x,n)] : n \in \mathbb{N}\}$. The lemma will follow if we show that $\delta[o(x,n)] \leq g(\delta[o(x,1)])d(x,Tx)$ for all $n \in \mathbb{N}$ and $x \in M$. Let n be any positive integer. From (b) of Lemma 2.11, there exists $T^k x \in o(x,n)$ $(1 \leq k \leq n)$ such that $d(x,Tx) = \delta[o(x,n)]$. We get

$$\begin{aligned} \delta[o(x,n)] &= d(x, T^k x) \\ &\leq d(x,Tx) + d(Tx, T^k x) \\ &\leq d(x,Tx) + \phi(\delta[o(x,n)]). \end{aligned}$$

Therefore, $\delta[o(x,n)] \leq g(\delta[o(x,n)])d(x,Tx) \leq g(\delta[o(x,1)])d(x,Tx)$. Since n is arbitrary, the proof is completed. □

3. Main result

THEOREM 3.1. *Let X be a complete bounded D-metric space, T a selfmap of X satisfying*

(3.1)

$$D(Tx,Ty,Tz) \le \phi\Big(\max\{D(x,y,z), D(x,Tx,z), D(y,Ty,z), D(x,Ty,z), D(y,Tx,z)\}\Big)$$

for all $x,y,z \in X, \phi \in \Phi$. Then T has a unique fixed point u in X, and T is continuous at u.

PROOF. Let $x_0 \in X$ and define $x_{n+1} = T_{x_n}$. If $x_{n+1} = x_n$ for some n, then T has fixed point. Assume that $x_{n+1} \neq x_n$ for each n. In (3.1), setting $x = x_{n-1}, y = x_n, z = x_{n+p-1}$, we have

$$D(x_n, x_{n+1}, x_{n+p}) \le \phi\Big(\max\Big\{D(x_{n-1}, x_n, x_{n+p-1}), D(x_n, x_{n+1}, x_{n+p-1}), D(x_{n-1}, x_{n+1}, x_{n+p-1}), D(x_n, x_n, x_{n+p-1})\Big\}\Big).$$

By Lemma 2.8, we obtain

$$D(x_n, x_{n+1}, x_{n+p}) \le \phi^n \left[\max_{\substack{0\le a\le n\\ 1\le b\le n+1\\ c=p}} D(x_a, x_b, x_c)\right]. \qquad \square$$

Let $k = \sup_{x,y,z\in X} D(x,y,z)$. Then we obtain that

$$D(x_n, x_{n+1}, x_{n+p}) \le \phi^n(k). \tag{3.2}$$

Using Definition 2.1-(iii) and (3.2),

$$\begin{aligned} &D(x_n, x_{n+p}, x_{n+p+t}) - D(x_{n+1}, x_{n+p}, x_{n+p+t}) \\ &\le D(x_n, x_{n+1}, x_{n+p}) - D(x_n, x_{n+1}, x_{n+p+t}) \\ &\le 2\phi^n(k). \end{aligned}$$

For $p \geq 2,\ n \leq j \leq n+p-2$, we have

$$D(x_j, x_{n+p}, x_{n+p+t}) - D(x_{j+1}, x_{n+p}, x_{n+p+t}) \leq \phi^j(k),$$

$$\sum_{j=n}^{n+p-2} \{D(x_j, x_{n+p}, x_{n+p+t}) - D(x_{j+1}, x_{n+p}, x_{n+p+t})\} \leq \sum_{j=n}^{n+p-2} \phi^j(k),$$

$$D(x_n, x_{n+p}, x_{n+p+t}) - D(x_{n+1}, x_{n+p}, x_{n+p+t}) \leq 2 \sum_{j=n}^{n+p-2} \phi^j(k)$$

and

$$D(x_n, x_{n+p}, x_{n+p+t}) \leq 2 \sum_{j=n}^{n+p-2} \phi^j(k) + \phi^{n+p-1}(k)$$

$\longrightarrow 0$ as $n \longrightarrow \infty$. Therefore $\{x_n\}$ is D-Cauchy. Since X is complete, $\{x_n\}$ converges. Call the limit u in X.
From (3.1),

$$D(x_n, x_{n+1}, Tu) \leq \phi(\max\{D(x_{n-1}, x_n, u), D(x_n, x_{n+1}, u), D(x_{n-1}, x_{n+1}, u), D(x_n, x_n, u)\}).$$

Taking the limit as $n \longrightarrow \infty$ and using the Lemma 2.10 yield $D(u, u, Tu) \leq 0$, which implies that $u = Tu$.
To prove uniqueness, assume that $w \neq u$ is also fixed point of T.
From (3.1),

$$\begin{aligned} D(u,w,u) &= D(Tu,Tw,Tu) \\ &\leq \phi(\max\{D(u,w,u), D(u,Tu,u), D(w,Tw,u), D(u,Tw,u), D(w,Tu,u)\}) \\ &= \phi(\max\{D(u,w,u), D(w,w,u)\}) \\ &= \phi[D(w,w,u)]. \end{aligned} \tag{3.3}$$

But

$$\begin{aligned} D(w,w,u) = D(w,u,w) &= D(Tw,Tu,Tw) \\ &\leq \phi(\max\{D(w,w,u), D(u,u,w)\}). \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4) yields $D(u,w,u) \leq \phi^2 D(u,w,u)$, a contradiction. Therefore $w = u$. To show that T is continuous at u, let $\{y_n\} \subseteq X$ with $\lim_{n\longrightarrow\infty} y_n = u$. Then substituting in (3.1) with $x = z = u$, $y = y_n$, we obtain

(3.5)

$$\begin{aligned} D(Tu, Ty_n, Tu) \leq \phi(\max\{&D(u, y_n, u), D(u, Tu, u), \\ &D(y_n, Ty_n, u), D(u, Ty_n, u), \\ &D(y_n, Tu, u)\}. \end{aligned}$$

Taking the lim sup of (3.4), we obtain

$$\limsup\ D(Tu, Ty_n, u) \leq \phi(\max\{0, 0, \limsup D(u, Ty_n, u), 0\}),$$

which implies that $\lim Ty_n = u = Tu$, and T is continuous at u.

COROLLARY 3.2. *Let X be a complete bounded D-metric space, m a positive integer, T a selfmap of X satisfying*

$$\begin{aligned} D(T^m x, T^m y, T^m z) \leq \phi\Big(\max\{&D(x,y,z), D(x, T^m x, z), D(y, T^m y, z), \\ &D(x, T^m y, z), D(y, T^m x, z)\}\Big) \end{aligned}$$

for all $x, y, z \in X$, $\phi \in \Phi$. Then T has a unique fixed point u in X and T^m is continuous at u.

PROOF. From Theorem 3.1, T^m has a unique fixed point u and T^m is continuous at u. But $Tu = T(T^m u) = T^m(Tu)$, and Tu is also a fixed point of T^m. Since the fixed point of T^m is unique, $u = Tu$. □

COROLLARY 3.3 [4]. *Let X be a complete bounded D-metric space, T a selfmap of X satisfying*

$$\begin{aligned} D(Tx, Ty, Tz) \leq q\Big(\max\{&D(x,y,z), D(x, Tx, z), D(y, Ty, z), \\ &D(x, Ty, z), D(y, Tx, z)\}\Big) \end{aligned}$$

for all $x, y, z \in X$, $0 \leq q < 1$. Then T has a unique fixed point u in X, and T is continuous at u.

PROOF. Let $\phi(x) = qx$, $0 \le q < 1$, $\phi \in \Phi$. Then the hypotheses in Theorem 3.1 are satisfied. Therefore Corollary 3.3 follows from Theorem 3.1. $\square$

COROLLARY 3.4 [2]. *Let X be a complete bounded D-metric space, T a selfmap of X satisfying*

$$D(Tx, Ty, Tz) \le q[D(x, y, z)]$$

for all $x, y, z \in X$, $0 \le q < 1$. Then T has a unique fixed point u in X, and T is continuous at u.

PROOF. The hypothesis in Corollary 3.3 is satisfied, from which the assertion follows. $\square$

COROLLARY 3.5. *Let X be a complete bounded D-metric space, T a selfmap of X satisfying*

$$[D(Tx, Ty, Tz)]^2 \le q \cdot \max\{[D(x, y, z)]^2, D(x, Ty, z)D(y, Tx, z), D(x, Tx, z), D(y, Ty, z)\}$$

for all $x, y, z \in X$, $0 \le q < 1$. Then T has a unique fixed point u in X, and T is continuous at u.

PROOF. Since for any $a \ge 0$, $b \ge 0$, we have $ab \le \max\{a^2, b^2\}$ and so condition of Corollary 3.5 implies Theorem 3.1 with $\phi(x) = \sqrt{q}\, x$. $\square$

COROLLARY 3.6. *Let X be a complete bounded D-metric space, T a selfmap of X satisfying*

$$[D(Tx, Ty, Tz)]^3 \le q \cdot \max\{[D(x, y, z)]^3, D(x, Tx, z), D(y, Ty, z), D(x, Ty, z), D(y, Ty, z), D(x, Ty, z), D(y, Tx, z)\}$$

for all $x, y, z \in X$, $0 \le q < 1$. Then T has a unique fixed point u in X, and T is continuous at u.

PROOF. Since for any $a \geq 0,\ b \geq 0, c \geq 0$, we have $abc \leq \max\{a^3, b^3, c^3\}$, and so condition of Corollary 3.6 implies Theorem 3.1 with $\phi(x) = \sqrt[3]{q}\, x$. □

THEOREM 3.7. *Let (M, d) be a metric space and M be T-orbitally complete, and let $T : M \longrightarrow M$ be a mapping satisfying (2.5), for all $x, y \in M, \phi \in \Phi$. Then*
(a) *T has a unique fixed point u in M,*
(b) *$\lim_{n \longrightarrow \infty} T^n x = u$, and*
(c) *$d(T^n x, u) \leq \phi^n(g(\delta[o(x, 1)])d(x, Tx))$ for every $x \in M$.*

PROOF. Let x be an arbitrary point of M. We shall show that the sequence of $\{T^n x\}$ is a Cauchy sequence. Let $n, m\,(n < m)$ be any positive integers. Since T satisfies (2.5), it follows from (a) of Lemma 2.11 that

$$\begin{aligned} d(T^n x, T^m x) =& d(TT^{n-1}x, TT^{m-n+1}x) \\ \leq& \phi(\delta[o(T^{n-1}x, m-n+1)]). \end{aligned} \tag{3.6}$$

According to the Lemma 2.11-(b), there exists an integer k_1, $1 \leq k_1 \leq m - n + 1$, such that

$$\delta[o(T^{n-1}x, m-n+1)] = d(T^{n-1}x, T^{k_1}T^{n-1}x).$$

Again, by Lemma 2.11, we have

$$\begin{aligned} d(T^{n-1}x, T^{k_1}T^{n-1}x) &= d(TT^{n-2}x, T^{k_1+1}T^{n-2}x) \\ &\leq \phi(\delta[o(T^{n-2}x, k_1+1)]) \\ &\leq \phi(\delta[o(T^{n-2}x, m-n+2)]). \end{aligned} \tag{3.7}$$

By (3.6) and (3.7), we have

$$d(T^n x, T^m x) \leq \phi^2(\delta[o(T^{n-2}x, m-n+2)]).$$

By inductive method, we obtain

$$\begin{aligned} d(T^n x, T^m x) &\le \phi(\delta[o(T^{n-1}x, m-n+1)]) \\ &\le \phi^2(\delta[o(T^{n-2}x, m-n+2)]) \\ &\vdots \\ &\le \phi^n(\delta[o(x,m)]). \end{aligned}$$

Then it follows from Lemma 2.12 that

$$d(T^n x, T^m x) \le \phi^n(g(\delta[o(x,1)])d(x,Tx)). \tag{3.8}$$

Since $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t \in R^+$, we have $\lim_{n\longrightarrow\infty} \phi^n(t) = 0$. Therefore, $\{T^n x\}$ is a Cauchy sequence. Again, M being T-orbitally complete, $\{T^n x\}$ has a limit u in M. To prove that $Tu = u$, let us consider the following inequalities.

$$\begin{aligned} d(u,Tu) &\le d(u, T^{n+1}x) + d(T^{n+1}x, Tu) \\ &\le d(u, T^{n+1}x) + \phi(\max\{d(T^n x, u), d(T^n x, T^{n+1}x), \\ &\qquad d(u,Tu), d(T^n x, Tu), d(u, T^{n+1}x)\}). \end{aligned}$$

Since $\lim_{n\longrightarrow\infty} T^n x = u$, this shows that $d(u,Tu) = 0$, i.e., u is a fixed point under T. We shall prove uniqueness. Let $w \neq u$ be also a fixed point of T.

$$\begin{aligned} d(u,w) = d(Tu,Tw) &\le \phi\big(\max\{d(u,w), d(u,Tu), d(w,Tw), \\ &\qquad d(u,Tw), d(w,Tu)\}\big) \\ &\le \phi(d(u,w)) \\ &< d(u,w). \end{aligned}$$

This is a contradiction. Therefore $w = u$. So we have proved (a) and (b), as x is arbitrary. Letting m tend to infinity in (3.8), we obtain the inequality (c). This completes the proof of Theorem. □

Corollary 3.8. *Let (X,d) be a complete metric space, m a positive integer, T a selfmap of X satisfying*

$$d(T^m x, T^m y) \le \phi(\max\{d(x,y), d(x,T^m x), d(y,T^m y), \\ d(x,T^m y), d(y,T^m x)\})$$

for all $x,y \in X$, $\phi \in \Phi$. Then T has a unique fixed point u in X.

Proof. From Theorem 3.7, T^m has a unique fixed point u. But $Tu = T(T^m u) = T^m(Tu)$, and Tu is also a fixed point of T^m. Since the fixed point of T^m is unique, $u = Tu$. □

Corollary 3.9 [1]. *Let (X,d) be a complete metric space, T a selfmap of X satisfying*

$$d(Tx,Ty) \le q \cdot \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

for all $x,y \in X$, $0 \le q < 1$. Then T has a unique fixed point u in X.

Proof. Let $\phi(x) = qx$, $0 \le q < 1$. Then $\phi \in \Phi$. By Theorem 3.7, the result follows. □

Corollary 3.10. *Let (X,d) be a complete metric space, m a positive integer, T a selfmap of X satisfying*

$$d(T^m x, T^m y) \le q \cdot \max\{d(x,y), d(x,T^m x), d(y,T^m y), \\ d(x,T^m y), d(y,T^m x)\}$$

for all $x,y \in X$, $0 \le q < 1$. Then T has a unique fixed point u in X.

Proof. Let $\phi(x) = qx$, $0 \le q < 1$. Then $\phi \in \Phi$. By Corollary 3.8, the result follows. □

COROLLARY 3.11. *Let (X, d) be a complete metric space, T a selfmap of X satisfying*

$$d(Tx, Ty) \leq \phi[d(x, y)]$$

for all $x, y \in X$, $\phi \in \Phi$. Then T has a unique fixed point u in X.

PROOF. Since the conditions in Theorem 3.7 are satisfied, the result follows. □

COROLLARY 3.12. *Let (X, d) be a complete metric space, T a selfmap of X satisfying*

$$d(Tx, Ty) \leq q \cdot d(x, y)$$

for all $x, y \in X$, $0 \leq q < 1$. Then T has a unique fixed point u in X.

PROOF. Let $\phi(x) = qx$, $0 \leq q < 1$. Then $\phi \in \Phi$. Thus by Corollary 3.11 and Theorem 3.7, the result follows. □

COROLLARY 3.13. *Let (X, d) be a complete metric space, T a selfmap of X satisfying*

$$d(Tx, Ty) \leq \phi(\max\{d(x, Tx), d(y, Ty)\})$$

for all $x, y \in X$, $\phi \in \Phi$. Then T has a unique fixed point u in X.

PROOF. By Theorem 3.7, the result follows. □

COROLLARY 3.14 [3]. *Let (X, d) be a complete metric space, and let T be a mapping from X into itself. Suppose T is a Kannan mapping, i.e. there exists $q \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq q \cdot [d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$. Then T has a unique fixed point u in X.

PROOF. Let $\phi(x) = 2qx,\ 0 \leq q < \frac{1}{2}$. Then $\phi \in \Phi$. Thus by Corollary 3.13, the result follows. □

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References

[1] L. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), no. 2, 267–273.

[2] B. C. Dhage, *Generalized metric spaces and mappings with fixed point*, Bull. Cal. Math. Soc. **84** (1992), 329–336.

[3] R. Kannan, *Some results on fixed points-II*, Amer. Math. Monthly **76** (1969), 405–408.

[4] B. E. Rhoades, *A fixed point theorem for generalized metric spaces*, Internet. J. Math. Sci. **19** (1996), no. 3, 457–460.

Department of Applied Mathematics
Changwon National University
Changwon 641-773, Korea
E-mail: jsume@sarim.changwon.ac.kr