Comm. Korean Math. Soc. **15** (2000), No. 4, pp. 617–626

LINEAR FUNCTIONALS ON $\mathcal{O}_n$ ASSOCIATED TO UNIT VECTORS

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ABSTRACT. We study the vectors related to states on the Cuntz algebra $\mathcal{O}_n$ and prove that, for two states ω and ρ on $\mathcal{O}_n$ with $\omega|_{\mathrm{UHF}_n} = \rho|_{\mathrm{UHF}_n}$, if $(\omega(s_1), \cdots, \omega(s_n))$ and $(\rho(s_1), \cdots, \rho(s_n))$ are unit vectors, then they are linearly dependent. We also study the linear functional on $\mathcal{O}_n$ associated to a sequence of unit vectors in $\mathbb{C}^n$ which is the generalization of the Cuntz state. We show that if the linear functional associated to a sequence of unit vectors with a certain condition is a state, then it is just the Cuntz state.

1. Introduction

In 1977, Cuntz [3] introduced the C^*-algebra generated by $n = 2, 3, \cdots$ isometries $s_1, s_2, \cdots, s_n$ satisfying the Cuntz relations of

$$s_i^* s_j = \delta_{ij} 1 \quad \text{and} \quad \sum_{i=1}^{n} s_i s_i^* = 1$$

and denoted it by $\mathcal{O}_n$. It is called the *Cuntz algebra* and the isomorphic type of this simple C^*-algebra does not depend on the choice of isometries but on the number of isometries. A UHF_n *algebra* is a uniformly hyperfinite algebra with Glimm type n^∞(see [4]) and we understand UHF_n as a subalgebra of $\mathcal{O}_n$. We note that since the Cuntz algebra $\mathcal{O}_n$ is the universal C^*-algebra generated by isometries $s_1, s_2, \cdots, s_n$ satisfying the Cuntz relations, it is the closure of the linear span of operators of the form $s_{i_1} s_{i_2} \cdots s_{i_k} s_{j_l}^* s_{j_{l-1}}^* \cdots s_{j_1}^*$ for $k, l = 0, 1, \cdots$ and a subalgebra

Received April 2, 2000. Revised June 27, 2000.
2000 Mathematics Subject Classification: Primary 46L30, Secondary 46L35.
Key words and phrases: Cuntz algebra, UHF_n, associated linear functional, state.
*Partially supported by BSRI-99-1420.

UHF_n of $\mathcal{O}_n$ is the closure of the linear span of operators of the form $s_{i_1}s_{i_2}\cdots s_{i_k}s_{j_k}^*s_{j_{k-1}}^*\cdots s_{j_1}^*$.

One of the study of the Cuntz algebra $\mathcal{O}_n$ is about it's representations (see [1], [2], [6]). In fact, there is a correspondence between representations of $\mathcal{O}_n$ and endomorphisms of $\mathcal{B}(\mathcal{H})$ of Powers index n up to unitary action, where $\mathcal{B}(\mathcal{H})$ is the set of all bounded linear operators on a Hilbert space $\mathcal{H}$.

Thus, to study representations of the Cuntz algebra $\mathcal{O}_n$ is one of main concerns. However, the simple C^*-algebra $\mathcal{O}_n$ is the famous example whose representations are bad. There are many cases that the representations of $\mathcal{O}_n$ have infinitely many irreducible subrepresentations. But, even in finite cases, finding subrepresentations takes a lot of work(see [5]). On the other hand, since states on $\mathcal{O}_n$ give representations of $\mathcal{O}_n$ by GNS constructions, it is natural that our attention is to study states on $\mathcal{O}_n$.

For a state ρ on $\mathcal{O}_n$, we first concern the vector $\big(\rho(s_1), \rho(s_2), \cdots, \rho(s_n)\big) \in \mathbb{C}^n$ related to ρ which may not be a unit vector. But when it is a unit vector, we show that ρ satisfies $\rho(s_i s_j^*) = \rho(s_i)\overline{\rho(s_j)}$ for $i, j = 1, 2, \cdots, n$. We study the relation between states on $\mathcal{O}_n$ and vectors related to states. We prove that for two states ω and ρ on $\mathcal{O}_n$ with the same restriction $\omega|_{\mathrm{UHF}_n} = \rho|_{\mathrm{UHF}_n}$, if two vectors $(\omega(s_1), \cdots, \omega(s_n))$ and $(\rho(s_1), \cdots, \rho(s_n))$ related to ω and ρ, respectively, are unit vectors, then they are linearly dependent.

Product pure states on UHF_n can be described by sequences of unit vectors in $\mathbb{C}^n$ and the natural extensions of these states to $\mathcal{O}_n$ become linear functionals on $\mathcal{O}_n$. Since a *state* on a C^*-algebra is a positive linear functional of norm one, it is natural that our concern is the linear functionals which come from sequences of unit vectors in $\mathbb{C}^n$. For further study, we define the linear functional on $\mathcal{O}_n$ associated to a sequence of unit vectors in $\mathbb{C}^n$.

DEFINITION 1.1. For a sequence $\{\eta_m\}_m$ of unit vectors $\eta_m = (\eta_m^1, \cdots, \eta_m^n) \in \mathbb{C}^n$, the *associated linear functional* ω on $\mathcal{O}_n$ is defined by

$$\omega(s_{i_1}\cdots s_{i_k}s_{j_l}^*\cdots s_{j_1}^*) = \eta_1^{i_1}\cdots\eta_k^{i_k}\overline{\eta_l^{j_l}}\cdots\overline{\eta_1^{j_1}}.$$

In particular, for a constant sequence $\{\eta\}_m$ of a unit vector $\eta =$

$(\eta^1, \cdots, \eta^n)$ in $\mathbb{C}^n$, the associated linear functional ω_η on $\mathcal{O}_n$ is

$$\omega_\eta(s_{i_1} \cdots s_{i_k} s_{j_l}^* \cdots s_{j_1}^*) = \eta^{i_1} \cdots \eta^{i_k} \overline{\eta^{j_l}} \cdots \overline{\eta^{j_1}}.$$

We report here on the Cuntz state. In [2], Bratelli, Jorgensen, and Price defined a state ω_η on $\mathcal{O}_n$ for a unit vector $\eta = (\eta^1, \eta^2, \cdots, \eta^n)$ in $\mathbb{C}^n$ by $\omega_\eta(s_{i_1} \cdots s_{i_k} s_{j_l}^* \cdots s_{j_1}^*) = \eta^{i_1} \cdots \eta^{i_k} \overline{\eta^{j_l}} \cdots \overline{\eta^{j_1}}$ and called it *the Cuntz state* which is the pure state on $\mathcal{O}_n$. In fact, the Cuntz state ω_η is the linear functional on $\mathcal{O}_n$ associated to a constant sequence of a fixed unit vector η in $\mathbb{C}^n$. Indeed, the Cuntz state ω_η is an example of a state on $\mathcal{O}_n$ which is the natural extension of a state $\omega_\eta|_{\mathrm{UHF}_n}$ on UHF_n.

However, we give an example to show that such natural extensions turn out not to be states but linear functionals except a specific type. Thus the problem of whether there exist non-constant sequences such that the associated linear functionals are states is left open. We investigate the conditions for the associated linear functionals to be states. In this paper, we naturally generalize the Cuntz state to the linear functional on $\mathcal{O}_n$ associated to a sequence of unit vectors in $\mathbb{C}^n$. And we prove that if the linear functional associated to a simple sequence of unit vectors such as a sequence of scalar multiples of a fixed unit vector, is a state, then the sequence should be a constant sequence.

2. Vectors related to states on $\mathcal{O}_n$

In this section, we examine the vectors related to states on a simple infinite C^*-algebra $\mathcal{O}_n$.

We first recall that $M_k(\mathcal{O}_n)$ denotes the set of all $k \times k$ matrices (a_{ij}), $a_{ij} \in \mathcal{O}_n$ and for any $A = (a_{ij})$ in $M_k(\mathcal{O}_n)$, the adjoint $A^* = (b_{ij})$ of A is given by $b_{ij} = a_{ji}^* \in \mathcal{O}_n$. In particular, for a vector $\xi = (\xi^1, \xi^2, \cdots, \xi^n)$ in $\mathbb{C}^n$, we consider the adjoint ξ^* of ξ as a column vector whose i-th component is $\overline{\xi^i}$. On the other hand, the inner product $<\xi, \eta>$ of two vectors ξ and η in $\mathbb{C}^n$ is given by $<\xi, \eta> = \xi\eta^*$.

Now let ρ be a state on $\mathcal{O}_n$. We also recall that, for $k = 1, 2, \cdots$, a map $\rho_k : M_k(\mathcal{O}_n) \to M_k$ is given by $\rho_k((a_{ij})) = (\rho(a_{ij}))$ for any $(a_{ij}) \in M_k(\mathcal{O}_n)$, where M_k is an $k \times k$ matrix algebra over $\mathbb{C}$. Here we consider the vector $(\rho(s_1), \rho(s_2), \cdots, \rho(s_n))$ in $\mathbb{C}^n$ related to ρ. It turns out that even when ρ is pure, the vector $(\rho(s_1), \rho(s_2), \cdots, \rho(s_n))$

may not be a unit vector. But it is easy to check that for a unit vector $\eta \in \mathbb{C}^n$ and the Cuntz state ω_η, the vector related to ω_η is just η and ω_η satisfies $\omega_\eta(s_i s_j^*) = \omega_\eta(s_i)\overline{\omega_\eta(s_j)}$ for $i, j = 1, 2, \cdots, n$. In addition, we know that ω_η can not be a homomorphism, but $\omega_\eta(s_i s_j) = \omega_\eta(s_i)\omega_\eta(s_j)$ and $\omega_\eta(s_i^*) = \overline{\omega_\eta(s_i)}$ hold. We notice here that the vector η related to ω_η is a unit vector. Thus when the vector $(\rho(s_1), \rho(s_2), \cdots, \rho(s_n))$ related to a state ρ is a unit vector, one would naturally expect that ρ satisfies some property similar to a homomorphism.

In the following lemma, we obtain the result that the vector related to a state on $\mathcal{O}_n$ is in the unit ball of $\mathbb{C}^n$. In addition, when it is a unit vector, we investigate the property of ρ.

LEMMA 2.1. *Let ρ be a state on $\mathcal{O}_n$. Then we have $\sum_{i=1}^n |\rho(s_i)|^2 \leq 1$. In particular, if $\sum_{i=1}^n |\rho(s_i)|^2 = 1$, then we have $\rho(s_i s_j^*) = \rho(s_i)\overline{\rho(s_j)}$ for any $i, j = 1, 2, \cdots, n$.*

PROOF. For a state ρ on $\mathcal{O}_n$, consider the positive matrix $\big(\rho(s_i)\overline{\rho(s_j)}\big)$. Since ρ is positive linear functional, it is completely positive and $(\rho(s_i s_j^*))$ is also positive. For simplicity, we denote $(\rho(s_i)\overline{\rho(s_j)})$ and $(\rho(s_i s_j^*))$ by S and T, respectively.

At first, we show that $S \leq T$. To do this, consider the matrix $A = (a_{ij}) \in M_n(\mathcal{O}_n)$ given by $a_{1j} = s_j^*$ for $j = 1, 2, \cdots, n$ and $a_{ij} = 0$ for $i = 2, 3, \cdots, n$ and $j = 1, 2, \cdots, n$. Since $\begin{pmatrix} I & A \\ O & O \end{pmatrix}^* \begin{pmatrix} I & A \\ O & O \end{pmatrix} = \begin{pmatrix} I & A \\ A^* & A^*A \end{pmatrix} \in M_{2n}(\mathcal{O}_n)$ is a positive matrix, where I and O are identity matrix and zero matrix in $M_n(\mathcal{O}_n)$, respectively, and $\rho_{2n}\begin{pmatrix} I & A \\ A^* & A^*A \end{pmatrix} = \begin{pmatrix} I & \rho_n(A) \\ \rho_n(A)^* & \rho_n(A^*A) \end{pmatrix} \in M_{2n}$ is also a positive matrix.

Now let ξ and η be any two unit vectors in $\mathbb{C}^n$. We set $\xi\rho_n(A)\eta^* = re^{i\theta}$ for some real numbers r and θ. Then for any real number t, we have

$$\begin{aligned} &\left(te^{-i\theta}\xi, \quad \eta\right) \begin{pmatrix} I & \rho_n(A) \\ \rho_n(A)^* & \rho_n(A^*A) \end{pmatrix} \begin{pmatrix} te^{i\theta}\xi^* \\ \eta^* \end{pmatrix} \\ &= t^2 + 2rt + \eta\rho_n(A^*A)\eta^* \geq 0. \end{aligned}$$

Thus we obtain the inequality $r^2 \leq \eta\, \rho_n(A^*A)\, \eta^*$ and so $|\, \xi\, \rho_n(A)\, \eta^*\, |^2 \leq$

$\eta\ \rho_n\ (A^*A)\ \eta^*$ which is equivalent to $|<\xi\rho_n(A),\eta>|^2 \leq\ <\eta\rho_n(A^*A),\eta>$.

When $\eta\rho_n(A)^* \neq 0$, if we replace ξ by a unit vector $\frac{\eta\rho_n(A)^*}{\|\eta\rho_n(A)^*\|}$ in the last inequality, then we get the inequality

$$<\eta\rho_n(A)^*,\eta\rho_n(A)^*> \leq\ <\eta\rho_n(A^*A),\eta> .$$

When $\eta\rho_n(A)^* = 0$, the above inequality is also true.

Therefore, the equality $<\eta\rho_n(A)^*,\eta\rho_n(A)^*>=<\eta\rho_n(A^*)\rho_n(A),\eta>$ gives that $<\eta\rho_n(A)^*\rho_n(A),\eta> \leq\ <\eta\rho_n(A^*A),\eta>$ for any unit vector η in $\mathbb{C}^n$ which implies that $\rho_n(A)^*\rho_n(A) \leq \rho_n(A^*A)$. Thus we conclude that $S = (\rho(s_i)\overline{\rho(s_j)}) \leq (\rho(s_i s_j^*)) = T$.

On the other hand, $S \leq T$ gives $\mathrm{tr}S \leq \mathrm{tr}T$ which implies that

$$\sum_{i=1}^{n} |\rho(s_i)|^2 \leq \sum_{i=1}^{n} \rho(s_i s_i^*) = \rho(\sum_{i=1}^{n} s_i s_i^*) = \rho(1) = 1.$$

Moreover, if $\sum_{i=1}^{n} |\rho(s_i)|^2 = 1$, then we have $\mathrm{tr}S = \mathrm{tr}T$ and so $\mathrm{tr}(T - S) = 0$. Note that for a positive matrix $C = (c_{ij})$ over $\mathbb{C}$, if $\mathrm{tr}C = \sum c_{ii}$ is zero, then C should be zero matrix. Since $T - S$ is a positive matrix, we conclude that $T = S$ and so $\rho(s_i s_j^*) = \rho(s_i)\overline{\rho(s_j)},\ \ i,j = 1,2,\cdots,n.$□

We remark that the equality $\rho(s_i s_j^*) = \rho(s_i)\overline{\rho(s_j)}$ in Lemma 2.1, is equivalent to $\rho(s_i s_j^*) = \rho(s_i)\rho(s_j^*)$, but it does not imply $\rho(s_i s_j) = \rho(s_i)\rho(s_j)$.

We also need the following lemma to prove our theorem.

LEMMA 2.2. *For two unit vectors ξ and η in $\mathbb{C}^n$, we have that $\xi^*\xi = \eta^*\eta$ if and only if $\eta = \lambda\xi$ for some $\lambda \in \mathbb{C},\ |\lambda| = 1$.*

PROOF. Let ξ and η be two unit vectors in $\mathbb{C}^n$.

If $\eta = \lambda\xi$ for some $\lambda \in \mathbb{C},\ |\lambda| = 1$, then it is straightforward computations that $\xi^*\xi = \eta^*\eta$ holds.

Conversely, suppose that ξ and η satisfy $\xi^*\xi = \eta^*\eta$. If we let p be a projection $\xi^*\xi$, then it is easily verified that p is a projection of rank 1 and ξ and η are the eigenvectors corresponding to eigenvalue 1 of p. Since the set of all eigenvectors of the projection of rank 1 is a vector space of one dimension, ξ and η are linearly dependent vectors. Hence the proof is completed. □

Now we are ready to prove the following theorem which gives the concrete description of the relation of vectors related to states on $\mathcal{O}_n$ whose restrictions to UHF_n are equal.

THEOREM 2.3. *Let ω and ρ be two states on $\mathcal{O}_n$ with $\omega|_{\mathrm{UHF}_n} = \rho|_{\mathrm{UHF}_n}$. If two vectors $(\omega(s_1), \cdots, \omega(s_n))$ and $(\rho(s_1), \cdots, \rho(s_n))$ related to ω and ρ, respectively, are unit vectors, then there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$ such that $(\omega(s_1), \cdots, \omega(s_n)) = \lambda(\rho(s_1), \cdots, \rho(s_n))$.*

PROOF. For two states ω and ρ on $\mathcal{O}_n$ with $\omega|_{\mathrm{UHF}_n} = \rho|_{\mathrm{UHF}_n}$, since for $i, j = 1, 2, \cdots, n$, $s_i s_j^*$ is an element in UHF_n, we have $\omega(s_i s_j^*) = \rho(s_i s_j^*)$.

For convenience, we let ξ and η be two vectors $(\omega(s_1), \cdots, \omega(s_n))$ and $(\rho(s_1), \cdots, \rho(s_n))$, respectively. If ξ and η are unit vectors, then by Lemma 2.1, the equality $\omega(s_i s_j^*) = \rho(s_i s_j^*)$ follows $\omega(s_i)\overline{\omega(s_j)} = \rho(s_i)\overline{\rho(s_j)}$ which is equivalent to $\xi^*\xi = \eta^*\eta$.

Therefore, Lemma 2.2 gives that $\big(\omega(s_1), \cdots, \omega(s_n)\big) = \lambda \big(\rho(s_1), \cdots, \rho(s_n)\big)$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. □

3. The associated linear functionals on $\mathcal{O}_n$

For a fixed unit vector η in $\mathbb{C}^n$, the associated linear functional ω_η on $\mathcal{O}_n$ is a state which is called the Cuntz state. We have already generalized it to the linear functional on $\mathcal{O}_n$ associated to a sequence of unit vectors in $\mathbb{C}^n$. It seems, however, that the associated linear functional on $\mathcal{O}_n$ is not a state. So one would expect that if the associated linear functional is a state, then the sequence should be a constant sequence.

In the following example, we first give a linear functional ω on $\mathcal{O}_2$ associated to a simple sequence of unit vectors in $\mathbb{C}^2$ and show that ω is not a state.

EXAMPLE 3.1. Consider a sequence $\{(1,0), (-1,0), (1,0), (1,0), \cdots\}$ of unit vectors in $\mathbb{C}^2$ and the associated linear functional ω on $\mathcal{O}_2$. For an element $x = (1 + s_1 + s_1^2)s_1^*$ in $\mathcal{O}_2$, the fact of

$$x^*x = 3s_1s_1^* + 2(s_1^2s_1^* + s_1s_1^{*2}) + s_1^3s_1^* + s_1s_1^{*3}$$

gives that

$$\omega(x^*x) = 3\cdot 1\cdot 1 + 2(1\cdot -1\cdot 1 + 1\cdot 1\cdot -1) + 1\cdot -1\cdot 1\cdot 1 + 1\cdot 1\cdot -1\cdot 1$$
$$= -3,$$

which implies that ω is not positive. Hence it is not a state.

From Example 3.1, we know that there exist sequences of unit vectors and the associated linear functionals which may not be states. Thus, it is interesting to check when the associated linear functional is a state on $\mathcal{O}_n$. Here, we investigate conditions on vectors in $\mathbb{C}^n$ for the case when the associated linear functionals on $\mathcal{O}_n$ become states.

For our purpose, we consider the canonical endomorphism ψ of $\mathcal{O}_n$ defined by $\psi(x) = \sum_{l=1}^{n} s_l x s_l^*$ (see [2]). As is known, $\psi|_{\mathrm{UHF}_n}$ is an one-sided shift and we have $\psi(x)\psi(y) = \psi(xy)$ for any $x, y \in \mathcal{O}_n$. Now for $k = 0, 1, \cdots$, let ψ^k be an endomorphism on $\mathcal{O}_n$ defined by $\psi^0(x) = x$ and $\psi^k(x) = \psi(\psi^{k-1}(x))$ for all $x \in \mathcal{O}_n$. Then $\rho(\psi^k)$ is also a state on $\mathcal{O}_n$ and we can consider the vector $(\rho(\psi^k(s_1)), \rho(\psi^k(s_2)), \cdots, \rho(\psi^k(s_n))) \in \mathbb{C}^n$ related to a state $\rho(\psi^k)$. In the following theorem, we obtain some results on these vectors.

THEOREM 3.2. *Let ρ be a state on $\mathcal{O}_n$ such that $|\rho(\psi^k(s_i))| = 1$ for some $k \in \{0, 1, \cdots\}$ and $i \in \{1, 2, \cdots, n\}$, where ψ is the canonical endomorphism of $\mathcal{O}_n$. Then for any $m = k, k+1, \cdots$, we have $\rho(\psi^m(s_i)) = \rho(\psi^k(s_i))$ and $\rho(\psi^m(s_j)) = 0$ for any $j = 1, 2, \cdots, n,\ \ j \neq i$.*

PROOF. We note that for the canonical endomorphism ψ and $i \in \{1, 2, \cdots, n\}$, the definition of ψ gives $s_i^*\psi(s_i) = s_i^* \sum_{l=1}^{n} s_l s_i s_l^* = s_i s_i^*$ and $\psi(s_i^*)s_i = s_i s_i^*$. Let ρ be a state on $\mathcal{O}_n$ with $|\rho(\psi^k(s_i))| = 1$ for some $k \in \{0, 1, \cdots\}$ and $i \in \{1, 2, \cdots, n\}$.

As is well known, we have $|\rho(x)|^2 \leq \rho(x^*x)$ for all $x \in \mathcal{O}_n$ and so when $\rho(x^*x) = 0$, $\rho(x) = 0$ holds. Thus we have

$$|\rho(\psi^k(s_i)^*)|^2 \leq \rho(\psi^k(s_i)\psi^k(s_i)^*) = \rho(\psi^k(s_i)\psi^k(s_i^*)) = \rho(\psi^k(s_i s_i^*)) \leq 1,$$

which implies $\rho(\psi^k(s_i s_i^*)) = 1$ by the fact of $|\rho(\psi^k(s_i))| = 1$.

Now for $k = 0, 1, 2, \cdots$, we consider an element $y = \psi^k(s_i) - \psi^{k+1}(s_i)$ in $\mathcal{O}_n$. Then we immediately see that

$$\begin{aligned} y^*y &= \psi^k(s_i^* s_i) - \psi^k(s_i^*\psi(s_i)) - \psi^k(\psi(s_i^*)s_i) + \psi^{k+1}(s_i^* s_i) \\ &= 2 - 2\psi^k(s_i s_i^*). \end{aligned}$$

From the equality $\rho(\psi^k(s_i s_i^*)) = 1$ it follows that $\rho(y^*y) = 0$ and so $\rho(y) = 0$. Hence we obtain $\rho(\psi^{k+1}(s_i)) = \rho(\psi^k(s_i))$. By iterating this process, we obtain $\rho(\psi^m(s_i)) = \rho(\psi^k(s_i))$ for any $m = k, k+1, \cdots$.

To complete the proof, we have to show that $\rho(\psi^m(s_j)) = 0$ for all $j,\ \ j \neq i$. But from Lemma 2.1, we know $\sum_{l=1}^n |\rho(\psi^m(s_l))|^2 \leq 1$. Thus the fact of $|\rho(\psi^m(s_i))| = 1$ gives that $\rho(\psi^m(s_j)) = 0$ for all $j,\ \ j \neq i$. □

From Theorem 3.2, if a state ρ on $\mathcal{O}_n$ satisfies $|\rho(\psi^2(s_1))| = 1$, e.g.,

$$(\rho(\psi^2(s_1)), \rho(\psi^2(s_2)), \cdots, \rho(\psi^2(s_n))) = (\rho(\psi^2(s_1)), 0, \cdots, 0),$$

then it follows directly that for any $m = 2, 3, \cdots$,

$$(\rho(\psi^m(s_1)), \rho(\psi^m(s_2)), \cdots, \rho(\psi^m(s_n))) = (\rho(\psi^2(s_1)), 0, \cdots, 0).$$

As is mentioned, the linear functional associated to a constant sequence of a unit vector is a state. Our concern is the converse of it; when the linear functional associated to a sequence of vectors is a state, this sequence is constant. The following theorem is a result about our concern which says that when the linear functional associated to a certain sequence of unit vectors is a state, the sequence must be a constant one which implies that the linear functional becomes the Cuntz state.

THEOREM 3.3. *Let $\{\eta_m\}_m$ be a sequence of unit vectors in $\mathbb{C}^n$ and ω the associated linear functional on $\mathcal{O}_n$. If $| < \eta_m, \eta_{m+1} > | = 1$ for any $m = 1, 2, \cdots$ and ω is a state, then $\{\eta_m\}_m$ is a constant sequence.*

PROOF. First we consider that for $m = 1, 2, \cdots,$

$$\sum_{i=1}^{n}(\psi^{m-1}(s_i) - \psi^m(s_i))(\psi^{m-1}(s_i) - \psi^m(s_i))^*$$

$$= \sum_{i=1}^{n}\{\psi^{m-1}(s_i s_i^*) - \psi^{m-1}(\psi(s_i)s_i^*) - \psi^{m-1}(s_i\psi(s_i^*)) + \psi^m(s_i s_i^*)\}$$

$$= 2 - \psi^{m-1}(\sum_{i=1}^{n}(\psi(s_i)s_i^* + s_i\psi(s_i^*)))$$

$$= 2 - 2\psi^{m-1}(\sum_{i,l=1}^{n} s_l s_i s_l^* s_i^*).$$

Suppose that $\{\eta_m\}_m$ is a sequence of unit vectors $\eta_m = (\eta_m^1, \eta_m^2, \cdots, \eta_m^n)$ with $| < \eta_m, \eta_{m+1} > | = 1$ and let ω be the associated linear functional on $\mathcal{O}_n$. Then since for $m = 1, 2, \cdots$ and $i = 1, 2, \cdots, n$, we immediately see that $\omega(\psi^{m-1}(s_i)) = \eta_m^i$, we get

$$\omega\Big[\psi^{m-1}(\sum_{i,l=1}^{n} s_l s_i s_l^* s_i^*)\Big] = \sum_{i,l=1}^{n} \omega\Big[\psi^{m-1}(s_l s_i s_l^* s_i^*)\Big]$$

$$= \sum_{i,l=1}^{n} \eta_m^l \eta_{m+1}^i \overline{\eta_{m+1}^l \eta_m^i} = | < \eta_m, \eta_{m+1} > |^2 = 1.$$

If ω is a state, then from above equalities and $\omega(1) = 1$, we obtain that

$$\omega\Big[\sum_{i=1}^{n}(\psi^{m-1}(s_i) - \psi^m(s_i))(\psi^{m-1}(s_i) - \psi^m(s_i))^*\Big]$$

$$= \omega\Big[2 - 2\psi^{m-1}(\sum_{i,l=1}^{n} s_l s_i s_l^* s_i^*)\Big] = 2\omega(1) - 2 \cdot 1 = 0.$$

Thus we have that for any $i = 1, 2, \cdots, n$, $\omega(\psi^{m-1}(s_i)) = \omega(\psi^m(s_i))$ which implies $\eta_m^i = \eta_{m+1}^i$ and so $\eta_m = \eta_{m+1}$. This completes the proof.$\square$

In the following, we apply Theorem 3.3 to the sequence of the scalar multiples of a fixed unit vector which contains Example 3.1.

COROLLARY 3.4. *For a unit vector $\eta \in \mathbb{C}^n$, let $\{\eta_m\}_m$ be a sequence of vectors $\eta_m = \lambda_m \eta$ with $\lambda_m \in \mathbb{C}$, $|\lambda_m| = 1$. If the associated linear functional ω on $\mathcal{O}_n$ is a state, then $\lambda_m = 1$ for $m = 1, 2, \cdots$.*

PROOF. Since $| < \eta_m, \eta_{m+1} > |^2 = |\lambda_m|^2 |\lambda_{m+1}|^2 | < \eta, \eta > |^2 = 1$, the proof is completed by Theorem 3.3. □

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