Comm. Korean Math. Soc. **15** (2000), No. 4, pp. 597–603

THE UNITS AND IDEMPOTENTS IN THE GROUP RING $K(\mathbb{Z}_m \times \mathbb{Z}_n)$

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ABSTRACT. Let K be an algebraically closed field of characteristic 0 and let $G = \mathbb{Z}_m \times \mathbb{Z}_n$. We find the conditions under which the elements of the group ring KG are units and idempotents respectively by using the represented matrix. We can see that if $\alpha = \sum r(g)g \in KG$ is an idempotent then $r(1) = 0, \frac{1}{mn}, \frac{2}{mn}, \cdots, \frac{mn-1}{mn}$ or 1.

1. Introduction

Let $G = \{g_0, g_1, g_2, \cdots, g_{n-1}\}$ be a finite group, where $g_0 = 1$. A basic group table matrix of G with a fixed order $g_0, g_1, ..., g_{n-1}$ of elements is a matrix with the diagonal entries 1 obtained from the group table matrix by elementary row operations interchanging two rows.

Let R be a ring with unity and $G = \{g_0, g_1, g_2, \cdots, g_{n-1}\}$ be a finite group, where $g_0 = 1$. From the element $\alpha = \sum_{i=0}^{n-1} r(g_i)g_i$ of the group ring RG, we obtain the following matrix M_α by putting $r(g_i)$ in the place of g_i in the basic group table matrix of G with a fixed order $g_0, g_1, ..., g_{n-1}$ of elements

$$M_\alpha = \begin{pmatrix} r(1) & r(g_1) & \cdots & r(g_{n-1}) \\ r(g_1^{-1}) & r(1) & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ r(g_{n-1}^{-1}) & \cdot & \cdots & r(1) \end{pmatrix}.$$

This matrix M_α is called the represented matrix of $\alpha = \sum_{i=0}^{n-1} r(g_i)g_i$.

Received June 7, 2000. Revised August 16, 2000.

2000 Mathematics Subject Classification: 20.13.

Key words and phrases: represented matrix.

This study was financially supported by Chonnam National University during author's sabbatical year in 1999.

Let K be a field of characteristic 0. Kaplansky and Zalesskii have shown that for any finite group G if $\alpha = \sum r(g)g \in KG$ is a nontrivial idempotent, then $r(1)$ is a rational number lying strictly between 0 and 1 (cf. [9]).

Cliff and Sehgal have shown that for any polycyclic-by-finite group G if $\alpha = \sum r(g)g \in KG$ is a nontrivial idempotent, then $r(1)$ is a rational number such that $r(1) = \frac{r}{s}$ and $(r, s) = 1$ (cf. [2]).

In [5] and [6], we found all idempotents in KG and thus have shown that $r(1) = 0, \frac{1}{|G|}, \frac{2}{|G|}, \cdots, \frac{|G|-1}{|G|}$ or 1 if $\alpha = \Sigma r(g)g \in KG$ is an idempotent where K is an algebraically closed field of characteristic 0 and G is a Klein's four group or a finite cyclic group.

In this paper, we shall find the units and idempotents in the group ring $K(\mathbb{Z}_m \times \mathbb{Z}_n)$ where K is an algebraically closed field of characteristic 0, and prove that if $\alpha = \sum r(g)g \in K(\mathbb{Z}_m \times \mathbb{Z}_n)$ is an idempotent then

$$r(1) = 0, \frac{1}{mn}, \frac{2}{mn}, \cdots, \frac{mn-1}{mn} \quad \text{or} \quad 1.$$

2. Main results

Let K be an algebraically closed field of characteristic 0. For each $(k, j) \in \mathbb{Z}_m \times \mathbb{Z}_n$, let

$$(k, j) = g_{kn+j} \qquad (0 \leq k \leq m-1,\ 0 \leq j \leq n-1).$$

Then the represented matrix M_α of the element $\alpha = \sum_{i=0}^{mn-1} r_i g_i$ of the group ring $K(\mathbb{Z}_m \times \mathbb{Z}_n)$ is as follows.

If m and n are relatively prime, then $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ and so

$$M_\alpha = \begin{pmatrix} r_0 & r_1 & r_2 & \cdots & r_{mn-1} \\ r_{mn-1} & r_0 & r_1 & \cdots & r_{mn-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_1 & r_2 & r_3 & \cdots & r_0 \end{pmatrix}.$$

If m and n are not relatively prime, then

$$M_\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ A_m & A_1 & \cdots & A_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{pmatrix},$$

where

$$A_t = \begin{pmatrix} r_{(t-1)n} & r_{(t-1)n+1} & \cdots & r_{tn-1} \\ r_{tn-1} & r_{(t-1)n} & \cdots & r_{tn-2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{tn-n+1} & r_{tn-n+2} & \cdots & r_{(t-1)n} \end{pmatrix}, \quad (t = 1, 2, ..., m).$$

In the case that m and n are relatively prime, we can obtain the units and idempotents in the group ring $K(\mathbb{Z}_m \times \mathbb{Z}_n)$ from [6] since $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$.

In the case that m and n are not relatively prime, we have

$$M_\alpha = \frac{1}{\sqrt{m}} \begin{pmatrix} I_n & I_n & \cdots & I_n \\ I_n & \theta I_n & \cdots & \theta^{m-1} I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & \theta^{m-1} I_n & \cdots & \theta I_n \end{pmatrix}$$

$$\times \begin{pmatrix} \sum_{i=1}^{m} A_i & 0 & 0 & \cdots & 0 \\ 0 & \sum_{i=1}^{m} \theta^{i-1} A_i & 0 & \cdots & 0 \\ 0 & 0 & \sum_{i=1}^{m} \theta^{2(i-1)} A_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{i=1}^{m} \theta^{(m-1)(i-1)} A_i \end{pmatrix}$$

$$\times \frac{1}{\sqrt{m}} \begin{pmatrix} I_n & I_n & \cdots & I_n \\ I_n & \theta^{-1} I_n & \cdots & (\theta^{-1})^{m-1} I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & (\theta^{-1})^{m-1} I_n & \cdots & \theta^{-1} I_n \end{pmatrix},$$

where θ is a primitive m-th root of unity 1 in K.

Let ξ be a primitive n-th root of unity 1 in K and let $V(1, \xi, \xi^2, \ldots, \xi^{n-1})$ be the Vandermonde matrix. Then

$$\sum_{i=1}^{m} \theta^{(i-1)j} A_i = \frac{1}{\sqrt{n}} V(1, \xi, \xi^2, \ldots, \xi^{n-1}) \times \operatorname{diag}\left(p_j(1), p_j(\xi), \ldots, p_j(\xi^{n-1})\right) \times \frac{1}{\sqrt{n}} V(1, \xi^{-1}, (\xi^{-1})^2, \ldots, (\xi^{-1})^{n-1}),$$

where

$$p_j(x) = \sum_{k=0}^{m-1} r_{kn}\theta^{jk} + \sum_{k=0}^{m-1} r_{kn+1}\theta^{jk}x + \cdots + \sum_{k=0}^{m-1} r_{kn+n-1}\theta^{jk}x^{n-1} \quad (j = 0, 1, 2, .., m-1).$$

Let

$$V = \frac{1}{\sqrt{n}} V(1, \xi, \xi^2, \cdots, \xi^{n-1}), \ \bar{V} = \frac{1}{\sqrt{n}} V(1, \xi^{-1}, (\xi^{-1})^2, \cdots, (\xi^{-1})^{n-1})$$

and

$$P_j = \operatorname{diag}\left(p_j(1) p_j(\xi) \ldots p_j(\xi^{n-1}) \right) \quad (j = 0, 1, 2, .., m-1).$$

Then we have

$$M_\alpha = \frac{1}{\sqrt{m}} \begin{pmatrix} I_n & I_n & \cdots & I_n \\ I_n & \theta I_n & \cdots & \theta^{m-1} I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & \theta^{m-1} I_n & \cdots & \theta I_n \end{pmatrix} \times \begin{pmatrix} VP_0\bar{V} & 0 & 0 & \cdots & 0 \\ 0 & VP_1\bar{V} & 0 & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & \cdots & VP_{m-1}\bar{V} \end{pmatrix} \times \frac{1}{\sqrt{m}} \begin{pmatrix} I_n & I_n & \cdots & I_n \\ I_n & \theta^{-1} I_n & \cdots & (\theta^{-1})^{m-1} I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & (\theta^{-1})^{m-1} I_n & \cdots & \theta^{-1} I_n \end{pmatrix}$$

$$
= \frac{1}{m}\begin{pmatrix} \bar{V} & \bar{V} & \cdots & \bar{V} \\ \bar{V} & \theta^{-1}\bar{V} & \cdots & (\theta^{-1})^{m-1}\bar{V} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{V} & (\theta^{-1})^{m-1}\bar{V} & \cdots & \theta^{-1}\bar{V} \end{pmatrix}^{-1}
\times \begin{pmatrix} P_0 & 0 & 0 & \cdots & 0 \\ 0 & P_1 & 0 & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & \cdots & P_{m-1} \end{pmatrix}
\times \begin{pmatrix} \bar{V} & \bar{V} & \cdots & \bar{V} \\ \bar{V} & \theta^{-1}\bar{V} & \cdots & (\theta^{-1})^{m-1}\bar{V} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{V} & (\theta^{-1})^{m-1}\bar{V} & \cdots & \theta^{-1}\bar{V} \end{pmatrix}.
$$

Therefore,

$$
\det M_\alpha = \frac{1}{m}\Pi_{i=0}^{n-1}\Pi_{j=0}^{m-1} p_j(\xi^i).
$$

Hence we have the following theorem.

THEOREM 1. *Let K be an algebraically closed field of characteristic 0 and let $G = \mathbb{Z}_m \times \mathbb{Z}_n$ where m and n are relatively prime. For each $(k, j) \in G$, let*

$$
(k, j) = g_{kn+j} \quad (0 \le k \le m-1, \quad 0 \le j \le n-1).
$$

Then $\alpha = \sum_{i=0}^{mn-1} r_i g_i \in KG$ is a unit if and only if

$$
p_j(\xi^i) \neq 0 \qquad (0 \le j \le m-1, \quad 0 \le i \le n-1),
$$

where ξ is a primitive n-th root of unity 1 in K and

$$
p_j(x) = \sum_{k=0}^{m-1} r_{kn}\theta^{jk} + \sum_{k=0}^{m-1} r_{kn+1}\theta^{jk}x + \cdots + \sum_{k=0}^{m-1} r_{kn+n-1}\theta^{jk}x^{n-1},
$$

where $j = 0, 1, 2, \cdots, m-1$ and θ is a primitive m-th root of unity 1 in K.

Note that $\alpha = \sum_{i=0}^{mn-1} r_i g_i \in KG$ is an idempotent if and only if $M_\alpha^2 = M_\alpha$. Therefore we have the following theorem.

THEOREM 2. *Under the same assumption as in* Theorem 1, *let* $\alpha = \sum_{i=0}^{mn-1} r_i g_i \in KG$. *Then* α *is an idempotent if and only if*

$$p_j(\xi^i) = 0 \text{ or } 1 \qquad (0 \le j \le m-1, \quad 0 \le i \le n-1),$$

where ξ *is a primitive* n*-th root of unity 1 in* K,

$$p_j(x) = \sum_{k=0}^{m-1} r_{kn}\theta^{jk} + \sum_{k=0}^{m-1} r_{kn+1}\theta^{jk}x + \cdots + \sum_{k=0}^{m-1} r_{kn+n-1}\theta^{jk}x^{n-1},$$

where θ *is a primitive* m*-th root of unity 1 in* K.

THEOREM 3. *Let* K *be an algebraically closed field of characteristic* 0 *and let* $G = \mathbb{Z}_m \times \mathbb{Z}_n$. *For each* $(k, j) \in G$, *let*

$$(k, j) = g_{kn+j} \qquad (0 \le k \le m-1, \quad 0 \le j \le n-1).$$

If $\alpha = \sum_{i=0}^{mn-1} r_i g_i \in KG$ *is an idempotent, then*

$$r_0 = 0, \frac{1}{mn}, \frac{2}{mn}, \cdots, \frac{mn-1}{mn} \quad \text{or} \quad 1.$$

PROOF. If m and n are relatively prime, then the proof is given in [6]. If m and n are not relatively prime, then by Theorem 2, we have

$$r_0 + r_n + \cdots + r_{(m-1)n} = \frac{1}{n}\sum_{i=0}^{n-1} p_0(\xi^i)$$

$$r_0 + r_n\theta + \cdots + r_{(m-1)n}\theta^{m-1} = \frac{1}{n}\sum_{i=0}^{n-1} p_1(\xi^i)$$

$$\vdots$$

$$r_0 + r_n\theta^{m-1} + \cdots + r_{(m-1)n}\theta = \frac{1}{n}\sum_{i=0}^{n-1} p_{m-1}(\xi^i).$$

Hence $r_0 = \frac{1}{mn}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} p_j(\xi^i)$. Thus, by Theorem 2, we have

$$r_0 = 0, \frac{1}{mn}, \frac{2}{mn}, \cdots, \frac{mn-1}{mn} \quad \text{or} \quad 1. \qquad \square$$

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