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A SIMPLE CONSTRUCTION FOR THE SPARSE MATRICES WITH ORTHOGONAL ROWS

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Abstract. We obtain a simple construction for the sparse $n \times n$ connected orthogonal matrices which have a row of p nonzero entries with $2 \le p \le n$. Moreover, we study the analogous sparsity problem for an $m \times n$ connected row-orthogonal matrices.

1. Introduction

Many contexts in computational linear algebra and numerical optimization require the computation of the sparse orthogonal (or row-orthogonal) matrices under various restrictions which arise in the number of nonzero entries of a row or column. This is to find orthogonal (or row-orthogonal) matrices with the least nonzero entries under given restrictions.

For positive integers m and n with $m \le n$, an $m \times n$ matrix A is *disconnected* if the rows and columns of A can be permuted to obtain a matrix of the form

$$\begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}.$$

Here, either of the matrices A_{11} or A_{22} may be vacuous by virtue of having no rows or no columns. But neither A_{11} nor A_{22} is allowed to be the 0×0 matrix. A matrix which is not disconnected is *connected.* If A is an $n \times n$ orthogonal matrix, then it is easy to show that if A contains a zero submatrix whose dimensions sum to n then the submatrix complementary to it is also a zero submatrix. Hence an $n \times n$ orthogonal matrix A is connected if and only if A does not contain a $p \times q$ zero submatrix with $p + q = n$ up to row and column permutations.

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In [1] (also see [8]), it is shown that each $n \times n$ connected orthogonal matrix has at least $4n-4$ nonzero entries, and that for $n \geq 2$ there exist such orthogonal matrices with exactly $4n-4$ nonzero entries. This result is extended in [2] to connected $m \times n$ *row-orthogonal* matrices whose rows are pairwise orthogonal. Recently, a construction method for the $n \times n$ connected orthogonal matrices with exactly $4n-4$ nonzero entries is obtained in [3]. And in [4], it is shown that an $n \times n$ orthogonal matrix with a full row (or column) has at least

$$(1.1) \qquad (\lfloor \log_2 n \rfloor + 3)n - 2^{\lfloor \log_2 n \rfloor + 1}$$

nonzero entries where a vector is called *full* if each of its entries is nonzero. The $n \times n$ orthogonal matrices with a full row (or column) which achieve the sparseness in (1.1) are closely related to the discrete Haar wavelet (see [4]). We call such matrix by the *Haar wavelet matrix*. This result is extended in [5] to $m \times n$ row-orthogonal matrices. That is, if A is an $m \times n$ connected row-orthogonal matrix with a full row then A has at least

$$(1.2) \quad f(m,n) := \left(\left\lfloor \log_2 \frac{n}{n-m+1} \right\rfloor + 3 \right) n - (n-m+1) 2^{\lfloor \log_2 \frac{n}{n-m+1} \rfloor + 1}$$

nonzero entries. In [7], a construction method of the $n \times n$ Haar wavelet matrices is given. Moreover, recently in [5], it is shown that an $n \times n$ connected orthogonal matrix with a row (or column) of p nonzero entries such that $2 \leq p \leq n$ has at least

$$(1.3) \qquad g(n,p) := (\lfloor \log_2 p \rfloor + 3)p - 2^{\lfloor \log_2 p \rfloor + 1} + 4(n-p)$$

nonzero entries, which extends the works in [1, 4].

In this paper, we obtain a simple construction for the $n \times n$ connected orthogonal matrices which have a row (or column) of p nonzero entries, $2 \leq p \leq n$, and have exactly $g(n,p)$ nonzero entries. Moreover, we study the sparsity problem for an $m \times n$ connected row-orthogonal matrix with a row of p nonzero entries, which generalizes (1.2) and (1.3).

2. A simple construction for the sparse orthogonal matrices

We begin this section by describing a few results from [3, 7] which we will need. The following theorem is an immediate consequence of Theorem 2.1 in [1] and Theorem 2.4 in [3].

THEOREM 2.1. *If A is an $n \times n$ connected orthogonal matrix with exactly $4n-4$ nonzero entries, then there exist permutation matrices P and Q such that for $n = 2k$*

(2.1)
$$PAQ = \Big(R(\theta_1) \oplus R(\theta_2) \oplus \cdots R(\theta_k)\Big)\Big(I_1 \oplus R(\theta_1') \oplus \cdots \oplus R(\theta_{k-1}') \oplus I_1\Big)$$

and, for $n = 2k+1$

$$PAQ = \Big(R(\theta_1) \oplus \cdots \oplus R(\theta_k) \oplus I_1\Big)\Big(I_1 \oplus R(\theta_1') \oplus \cdots \oplus R(\theta_k')\Big),$$

where for $\theta = \theta_k$ or θ_k',

$$I_1 = [1], \quad R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad or \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix},$$
$$(-\frac{\pi}{2} < \theta < \frac{\pi}{2},\ \theta \neq 0).$$

For each integer $n \geq 2$, we define the class $\mathcal{B}_n$ to be the family of all $n \times n$ connected orthogonal matrices which have exactly $4n-4$ nonzero entries and the class $\mathcal{H}_n$ to be the family of all $n \times n$ Haar wavelet matrices.

REMARK 2.2. From [1] and [7], it is easy to see that for each $n \geq 2$, a matrix in $\mathcal{B}_n$ has a column with exactly two nonzero entries and a matrix in $\mathcal{H}_n$ has a row with exactly two nonzero entries up to transpose.

Let
$$X = \begin{bmatrix} \widehat{X} \\ \mathbf{x}^T \end{bmatrix} \in \mathcal{H}_r \quad \text{and} \quad Y = \begin{bmatrix} \mathbf{y}^T \\ \widehat{Y} \end{bmatrix} \in \mathcal{H}_s,$$

where $\mathbf{x}^T$, $\mathbf{y}^T$ are full, and $r + s = n$. Theorem 2.2 of [7] asserts that if

$$2^{\lfloor \log_2 n \rfloor - 1} \leq r,\ s \leq 2^{\lfloor \log_2 n \rfloor}.$$

then

$$A = \begin{bmatrix} \widehat{X} & O \\ \frac{\mathbf{x}^T}{\sqrt{2}} & \frac{\mathbf{y}^T}{\sqrt{2}} \\ O & \widehat{Y} \\ \frac{-\mathbf{x}^T}{\sqrt{2}} & \frac{\mathbf{y}^T}{\sqrt{2}} \end{bmatrix} \in \mathcal{H}_n. \tag{2.2}$$

LEMMA 2.3. *Let*

$$U = \begin{bmatrix} U' \\ \mathbf{y}^T \end{bmatrix} \quad \text{and} \quad V = [\,\mathbf{x} \quad V'\,]$$

be a $p \times p$ orthogonal matrix and a $q \times q$ orthogonal matrix respectively. Then

$$A = \begin{bmatrix} U' & O \\ \mathbf{x}\mathbf{y}^T & V' \end{bmatrix} \tag{2.3}$$

is a $(p+q-1) \times (p+q-1)$ orthogonal matrix.

Proof. Since U is orthogonal, ${U'}^T U' + \mathrm{y}\mathrm{y}^T = I_p$, and since V is orthogonal, $\mathrm{x}^T\mathrm{x} = 1$, ${V'}^T\mathrm{x} = 0$, ${V'}^T V' = I_{q-1}$. Thus

$$\begin{aligned} A^T A &= \begin{bmatrix} {U'}^T & \mathrm{y}\mathrm{x}^T \\ O & {V'}^T \end{bmatrix} \begin{bmatrix} U' & O \\ \mathrm{x}\mathrm{y}^T & V' \end{bmatrix} = \begin{bmatrix} {U'}^T U' + \mathrm{y}\mathrm{x}^T\mathrm{x}\mathrm{y}^T & \mathrm{y}\mathrm{x}^T V' \\ {V'}^T \mathrm{x}\mathrm{y}^T & {V'}^T V' \end{bmatrix} \\ &= \begin{bmatrix} I_p & O \\ O & I_{q-1} \end{bmatrix} = I_{p+q-1}, \end{aligned}$$

showing that A is a $(p+q-1) \times (p+q-1)$ orthogonal matrix. □

Now, we are ready to give a construction method of the sparse orthogonal matrices which have a row of p nonzero entries such that $2 \le p \le n$. Throughout this paper, the number of nonzero entries in A is denoted by $\#(A)$.

THEOREM 2.4. *For an integer p with $2 \le p \le n$, if $U = \begin{bmatrix} U' \\ \mathbf{y}^T \end{bmatrix}$ is a matrix in $\mathcal{H}_p$ where $\#(\mathbf{y}^T) = 2$, and $V = [\,\mathbf{x} \quad V'\,]$ is a matrix in $\mathcal{B}_{n-p+1}$ where $\#(\mathbf{x}) = 2$, then A is an $n \times n$ connected orthogonal matrix in (2.3) with exactly $g(n,p)$ nonzero entries.*

Proof. By Lemma 2.3 and the connectivity of U and V, clearly A is an $n \times n$ connected orthogonal matrix. Furthermore,

$$\begin{aligned}
\#(A) &= \#(U') + \#(V') + \#(\mathbf{x}\mathbf{y}^T) \\
&= [(\lfloor \log_2 p \rfloor + 3)p - 2^{\lfloor \log_2 p \rfloor + 1} - 2] + [4(n-p) - 2] + 4 \\
&= (\lfloor \log_2 p \rfloor + 3)p - 2^{\lfloor \log_2 p \rfloor + 1} + 4(n-p) \\
&= g(n,p).
\end{aligned}$$

□

For example, let

$$U = \begin{bmatrix}
\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\
-\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}
\end{bmatrix}$$

and

$$V = \begin{bmatrix}
\frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 & 0 \\
-\frac{1}{2} & \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & 0 & 0 & 0 \\
0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & 0 \\
0 & \frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & 0 \\
0 & 0 & 0 & \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \\
0 & 0 & 0 & -\frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2}
\end{bmatrix}.$$

Then

$$A=\begin{bmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

is a 10×10 connected orthogonal matrix with a row of 5 nonzero entries which has $f(10,5) = 37$ nonzero entries. Note that U and V in the above example are obtained from the construction method (2.2) and (2.1) respectively.

3. The sparsity problem for row-orthogonal matrices

One can ask the analogous question for the sparsity of a connected $m \times n$ row-orthogonal matrix which has a row with p nonzero entries such that $2 \leq p \leq n$.

We begin by describing a way to build row-orthogonal matrices from [2] which we will need. Let

$$A = \begin{bmatrix} \widehat{A} \\ \mathrm{a}^T \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \mathrm{b}^T \\ \widehat{B} \end{bmatrix}$$

be a $p \times q$ and a $s \times t$ row-orthogonal matrix with the rows a^T and b^T, respectively. Define $A\Diamond B$ to be the $(p+s-1)\times(q+t)$ matrix

$$A\Diamond B = \begin{bmatrix} \widehat{A} & O \\ \mathrm{a}^T & \mathrm{b}^T \\ O & \widehat{B} \end{bmatrix}.$$

Certainly, $A\Diamond B$ is a row-orthogonal matrix and $A\Diamond B$ is connected if and only if both A and B are connected. We can extend this construction to use any number of row-orthogonal matrices by defining $A\Diamond B\Diamond C$ as $(A\Diamond B)\Diamond C$.

The following Lemma follows from [2].

LEMMA 3.1. *Let A be an $m \times n$ connected row-orthogonal matrix. Then*

$$\#(A) \geq \begin{cases} n+2m-2 & \text{if } n > 2m-2, \\ 4m-4 & \text{if } n \leq 2m-2. \end{cases}$$

Moreover, the equality holds if and only if up to column and row permutations, if $n > 2m-2$ then $A = J\Diamond B_1\Diamond\cdots\Diamond B_{m-1}$, where J is the $1\times(n-2m+2)$ matrix of all ones and $B_i \in \mathcal{B}_2$ for each $i = 1,\ldots,m-1$; if $m < n \leq 2m-2$ then $A = B_{k_1}\Diamond B_{k_2}\Diamond\cdots\Diamond B_{k_{n-m+1}}$, where $B_{k_i} \in \mathcal{B}_{k_i}$ for each $i = 1,2,\ldots,n-m+1$ such that $k_1+k_2+\cdots+k_{n-m+1} = n$ and $k_1 \geq k_2 \geq \cdots \geq k_{n-m+1} \geq 2$; if $n = m \geq 2$ then $A \in \mathcal{B}_n$.

Let m, n, and p be positive integers with $2 \leq m \leq n$ and $2 \leq p \leq n$, and let $h(m,n,p)$ denote the least number of nonzero entries in an $m\times n$ connected row-orthogonal matrix which has a row with p nonzero entries.

Note that it is shown in [1] (also see [3]) that the number of nonzero entries in each row of a connected orthogonal matrix in $\mathcal{B}_n$ is 2, 3 or 4 up to its transpose. Thus from Lemma 3.1, it is easy to show that if $n > 2m-2$ and $p = n-2m+4$, $p = 2$ or 4 then $h(m,n,p) = n+2m-2$, and if $n \leq 2m-2$ and $2 \leq p = 2,3,\ldots,k_1+k_2 \leq 8$ then $h(m,n,p) = 4m-4$.

Now, we consider the case $n \geq p \geq n-m+2$ and $n \geq m \geq 2$. We need the following lemma which follows from the proof of Lemma 2.3.

LEMMA 3.2. *If U and V in* Lemma 2.3 *are connected $p \times q$ and connected $s \times t$ row-orthogonal matrices, respectively, then A in* (2.3) *is a connected $(p+s-1) \times (q+t-1)$ row-orthogonal matrix.*

The construction in Lemma 3.2 can be used to construct sparse $m \times n$ connected row-orthogonal matrices which have a row with p nonzero entries such that $n \geq p \geq n-m+2$.

Let $U = \begin{bmatrix} U' \\ \mathbf{y}^T \end{bmatrix}$ be a $(m-n+p) \times p$ row-orthogonal matrix whose first row is full, and whose the number of nonzero entries is $f(m-n+p, p)$ in (1.2). Then U has a row with 2 nonzero entries from [5]. We may assume that such row of U is $\mathbf{y}^T$. And let $V = [\mathbf{x} \quad V']$ be a matrix in $\mathcal{B}_{n-p+1}$ whose first column $\mathbf{x}$ has 2 nonzero entries. Then from Lemma 3.2, A in (2.3) is an $m \times n$ connected, row-orthogonal matrix whose first row has p nonzero entries. Furthermore,

$$\begin{aligned} \#(A) &= \#(U') + \#(V') + \#(\mathbf{x}\mathbf{y}^T) \\ &= \left\{ \left(\left\lfloor \log_2 \frac{p}{n-m+1} \right\rfloor + 3 \right) p - (n-m+1) 2^{\left\lfloor \log_2 \frac{p}{n-m+1} \right\rfloor + 1} - 2 \right\} \\ &\quad + \{4(n-p) - 2\} + 4 \\ &= \left(\left\lfloor \log_2 \frac{p}{n-m+1} \right\rfloor + 3 \right) p - (n-m+1) 2^{\left\lfloor \log_2 \frac{p}{n-m+1} \right\rfloor + 1} + 4(n-p). \end{aligned}$$

Thus we get the following theorem.

THEOREM 3.3. *If $n \geq p \geq n-m+2$ and $n \geq m \geq 2$ then*

(2.4)

$$h(m,n,p) \leq \left(\left\lfloor log_2 \frac{p}{n-m+1} \right\rfloor + 3 \right) p - (n-m+1) 2^{\left\lfloor \log_2 \frac{p}{n-m+1} \right\rfloor + 1} + 4(n-p).$$

We conjecture the inequality (2.4) is an equality for each $p = n-m+2, \ldots, n$. Note that if $p = n$ then $f(m,n) = h(m,n,p)$, and if $m = n$ then $g(n,p) = h(m,n,p)$. Thus this is a generalization of (1.2) and (1.3).

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