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DICKSON INVARIANTS HIT BY THE STEENROD SQUARE

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ABSTRACT. Let D_3 be the Dickson invariant algebra of $\mathbb{F}_2[X_1, X_2, X_3]$ by $\mathrm{GL}(3, \mathbb{F}_2)$. In this paper, we provide an elementary proof of Theorem 3.2 of [2]: each element in D_3 is hit by the Steenrod square in $\mathbb{F}_2[X_1, X_2, X_3]$.

1. Introduction

Throughout this paper, we work on the polynomials and homology groups over $\mathbb{F}_2$. The objective of this paper is to give an elementary proof of the special case of the conjecture that each element in D_n is hit by the Steenrod square, where $n > 2$. This conjecture is closely related [2] to the famous spherical conjecture, which says that the image of the Hurewitz map: $\pi_*(Q_0S^0) \to H_*(Q_0S^0)$ can only be elements of Hopf invariant one and of Kervaire invariant one.

Let $\mathrm{GL}(n, \mathbb{F}_2)$ be the $n \times n$-general linear group over $\mathbb{F}_2$. Then the standard group action:

$$\mathrm{GL}(n, \mathbb{F}_2) \times \mathbb{F}_2[X_1, X_2, \dots, X_n] \to \mathbb{F}_2[X_1, X_2, \dots, X_n]$$

gives rise to the algebra of Dickson invariants

$$\mathbb{F}_2[X_1, X_2, \dots, X_n]^{\mathrm{GL}(n,\mathbb{F}_2)},$$

which is isomorphic to the polynomial ring $\mathbb{F}_2[Q_{n,0}, Q_{n,1}, ..., Q_{n,n-1}]$, where each generator $Q_{n,i}$ $(i = 0, ..., n-1)$ is a certain polynomial over $\mathbb{F}_2$.

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DEFINITION 1.1. A polynomial $f(X_1, X_2, \ldots, X_n)$ is *hit* if it can be written as

$$f(X_1, X_2, , \ldots , X_n) = \sum_{i\geq 1} \mathrm{Sq}^i f_i(X_1, X_2, \ldots , X_n)$$

for some polynomials $f_i(X_1, X_2, \ldots, X_n)$.

QUESTION 1.2. *Is each* Dickson *invariant hit?*

The motivation of presenting this question was illustrated in detail in [2] and in an excellent expository paper [9], p501. When $n = 1$ and 2, the answer to the above question is negative. For example, when $n = 1$, the Dickson invariant X_1 is not hit; when $n = 2$, $X_1^2 + X_2^2 + X_1X_2$ is a Dickson invariant but is not hit.

CONJECTURE 1.3 (Hung [2]). *When $n \geq 3$, all* Dickson *invariants are hit.*

When $n = 3$, Hung has proved that the conjecture is true [2] by using many advanced tools. The main goal of this paper is to give an elementary proof of the above conjecture when $n = 3$.

Write $V_1 = X_1$, $V_2 = X_2(X_2 + X_1)$, and $V_3 = X_3(X_3 + X_2)(X_3 + X_1)(X_3 + X_2 + X_1)$. Then $Q_{3,0} = V_3V_2V_1$, $Q_{3,1} = Q_{2,0}^2 + V_3Q_{2,1} = (V_2V_1)^2 + V_3(V_1^2 + V_2)$, and $Q_{3,2} = Q_{2,1}^2 + V_3 = (V_1^2 + V_2)^2 + V_3$. Clearly, $\deg(Q_{3,0}) = 7$, $\deg(Q_{3,1}) = 6$, and $\deg(Q_{3,2}) = 4$. The Steenrod operation acts on the Dickson invariants in the following way.

THEOREM 1.4 (Hung [1]). *The* Steenrod *operation acts trivially on $Q_{3,0}$, $Q_{3,1}$, and $Q_{3,2}$ except for the following cases,*

$$\begin{aligned}
\mathrm{Sq}^4 Q_{3,0} &= Q_{3,0}Q_{3,2}, & \mathrm{Sq}^6 Q_{3,0} &= Q_{3,0}Q_{3,1}, \\
\mathrm{Sq}^7 Q_{3,0} &= Q_{3,0}^2, & \mathrm{Sq}^1 Q_{3,1} &= Q_{3,0}, \\
\mathrm{Sq}^4 Q_{3,1} &= Q_{3,1}Q_{3,2}, & \mathrm{Sq}^5 Q_{3,1} &= Q_{3,0}Q_{3,2}, \\
\mathrm{Sq}^6 Q_{3,1} &= Q_{3,1}^2, & \mathrm{Sq}^2 Q_{3,2} &= Q_{3,1}, \\
\mathrm{Sq}^3 Q_{3,2} &= Q_{3,0}, & \mathrm{Sq}^4 Q_{3,2} &= Q_{3,2}^2.
\end{aligned}$$

Now we define a function $\gamma : \mathbb{N} \cup \{0\} \to \mathbb{N}$ as follows: $\gamma(0) = 0$ and for $k \geq 1$, $\gamma(k) = 2^k - 1$. Write $\mu(n)$ for

$$\min\left\{m \in \mathbb{N} \mid n = \sum_{i=1}^{m} \gamma(k_i) \quad \text{for some } k_i \in \mathbb{N}\right\}.$$

Let E and F be homogeneous polynomials of degrees e and f, respectively. Then we have the following.

THEOREM 1.5 (Silverman [5]). *Suppose that $e < (2^{k+1} - 1)\mu(f)$ for some $k \geq 0$. Then the polynomial $EF^{2^{k+1}}$ is hit.*

For an integer t, let $\alpha(t)$ be the number of digits 1 in the binary expansion of t and let U be an n-variable homogeneous polynomial. Then we have the following theorem, known as the Peterson Conjecture.

THEOREM 1.6 (Wood [8]). *If $\alpha(\deg(U) + n) > n$, then U is hit.*

John Milnor showed that there is an anti-automorphism χ from the Steenrod algebra to itself, satisfying the following condition,

$$\chi(\mathrm{Sq}^k) = \sum_{i=1}^{k} \mathrm{Sq}^i \chi(\mathrm{Sq}^{k-i}) \quad \text{for} \quad k \geq 1.$$

We will frequently use the so-called *χ-trick* in the following sections, which is based on the following.

PROPOSITION 1.7 (see for example, [9]). *For any polynomials E_1 and E_2,*

$$E_1[\mathrm{Sq}^m(E_2)] - [\chi(\mathrm{Sq}^m)(E_1)]E_2$$

is hit, where m is any non-negative integer.

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2. Proofs

Clearly, to prove our result, we only need show that $Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2}$ is hit for any non-negative integers n_0, n_1 and n_2. The proof will split into the following cases.

(1) At least two of n_0, n_1 and n_2 are even;
(2) Exactly one of n_0, n_1 and n_2 is even;
(3) n_0, n_2 are odd and $n_1 \equiv 1 \mod 4$;
(4) n_0 is odd, $n_1 \equiv 3 \mod 4$ and $n_2 \equiv 1 \mod 4$;
(5) $n_0 \equiv 1 \mod 4$, $n_1 \equiv 3 \mod 4$ and $n_2 \equiv 3 \mod 4$;
(6) $n_0 \equiv 3 \mod 4$, $n_1 \equiv 3 \mod 4$ and $n_2 \equiv 3 \mod 4$.

Now we will prove each of the above six cases.

REMARK 2.1. The proof of Cases (1)-(4) is extracted from the first author's MSc thesis [7].

NOTATION 2.2. *For simplicity, we use the following notation: for any polynomials E_1 and E_2, $E_1 \equiv E_2$ means that $E_1 - E_2$ is hit.*

2.1. Proof of Case (1)

If n_0, n_1, and n_2 are all even, then the result is a direct consequence of the definition of the Steenrod square.

Suppose that n_1, n_2 are even and n_0 is odd. Then putting $n_0 = 2l_0+1$, $n_1 = 2l_1$, and $n_2 = 2l_2$, we have the following,

$$\begin{aligned} Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} &= Q_{3,0}^{2l_0+1}Q_{3,1}^{2l_1}Q_{3,2}^{2l_2} \\ &= [\mathrm{Sq}^1 Q_{3,1}](Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 \\ &= \mathrm{Sq}^1[Q_{3,1}(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2]. \end{aligned}$$

Suppose that n_0, n_2 are even and n_1 is odd. Then by letting $n_0 = 2l_0$, $n_1 = 2l_1 + 1$, and $n_2 = 2l_2$, we have the following,

$$\begin{aligned} Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} &= Q_{3,0}^{2l_0}Q_{3,1}^{2l_1+1}Q_{3,2}^{2l_2} \\ &= \mathrm{Sq}^2[Q_{3,2}(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2] + Q_{3,2}\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 \\ &\equiv Q_{3,2}\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2. \end{aligned}$$

Notice that

(2.1.1) $$Q_{3,2} = Q_{2,1}^2 + V_3 \quad \text{and} \quad V_3 = \mathrm{Sq}^1(Q_{2,1}X_3 + X_3^3).$$

Then

$$\begin{aligned} Q_{3,2}\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 &= (Q_{2,1}^2 + V_3)\mathrm{Sq}^2[(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2] \\ &= [Q_{2,1}\mathrm{Sq}^1(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 + \\ &\quad \mathrm{Sq}^1[(Q_{2,1}X_3 + X_3^3)(\mathrm{Sq}^1(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2}))^2] \\ &\equiv 0 \end{aligned}$$

Suppose that n_0, n_1 are even and n_2 is odd. Putting $n_0 = 2l_0$, $n_1 = 2l_1$, and $n_2 = 2l_2 + 1$, we have the following,

$$\begin{aligned} Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} &= Q_{3,0}^{2l_0}Q_{3,1}^{2l_1}Q_{3,2}^{2l_2+1} \\ &= (Q_{2,1}^2 + \mathrm{Sq}^1(Q_{2,1}X_3 + X_3^3))(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 \text{ (using (2.1.1))} \\ &= (Q_{2,1}Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 + \mathrm{Sq}^1[(Q_{2,1}X_3 + X_3^3))(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2] \\ &\equiv 0 \end{aligned}$$

So Case (1) has been done. □

2.2. Proof of Case (2)

Suppose that $n_0 = 2l_0$, $n_1 = 2l_1 + 1$ and $n_2 = 2l_2 + 1$. We will use the χ-*trick* (1.7) to prove the result. Notice that

(2.2.1)
$$Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} = [\mathrm{Sq}^4(Q_{3,1})](Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2 \equiv Q_{3,1}\chi(\mathrm{Sq}^4)(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})^2.$$

Using the formulae $\chi(\mathrm{Sq}^4) = \mathrm{Sq}^4 + \mathrm{Sq}^2\mathrm{Sq}^2$ and $Q_{3,1} = \mathrm{Sq}^2(Q_{3,2})$, we can easily see that the last term of (2.2.1) can be reduced to

$$\mathrm{Sq}^2(Q_{3,2})[\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 + Q_{3,1}[\mathrm{Sq}^1(\mathrm{Sq}^1(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2}))]^2.$$

Noting that $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$, the last term is zero. Again using the χ-*trick*, the first term is hit if and only if $Q_{3,2}\chi(\mathrm{Sq}^2)[\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2$ is hit. Now

$$\begin{aligned} &Q_{3,2}\chi(\mathrm{Sq}^2)[\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 \\ &= (Q_{2,1}^2 + V_3)[\mathrm{Sq}^1\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 \\ &\equiv V_3[\mathrm{Sq}^1\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2 \\ &= \mathrm{Sq}^1\{(Q_{2,1}X_3 + X_3^3)[\mathrm{Sq}^1\mathrm{Sq}^2(Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2})]^2\}. \end{aligned}$$

Suppose that $n_0 = 2l_0 + 1$, $n_1 = 2l_1$, and $n_2 = 2l_2 + 1$. Then

$$\begin{aligned} Q_{3,0}^{n_0} Q_{3,1}^{n_1} Q_{3,2}^{n_2} &= Q_{3,0}^{2l_0+1} Q_{3,1}^{2l_1} Q_{3,2}^{2l_2+1} \\ &= \mathrm{Sq}^1 [Q_{3,1} Q_{3,2} (Q_{3,0}^{l_0} Q_{3,1}^{l_1} Q_{3,2}^{l_2})^2] \qquad (\text{since } \mathrm{Sq}^1 Q_{3,2} = 0). \end{aligned}$$

Suppose that $n_0 = 2l_0 + 1$, $n_1 = 2l_1 + 1$, and $n_2 = 2l_2$. Then

$$Q_{3,0}^{n_0} Q_{3,1}^{n_1} Q_{3,2}^{n_2} = Q_{3,0}^{2l_0+1} Q_{3,1}^{2l_1+1} Q_{3,2}^{2l_2} = \mathrm{Sq}^2 (Q_{3,2}) [Q_{3,0} (Q_{3,0}^{l_0} Q_{3,1}^{l_1} Q_{3,2}^{l_2})^2]. \tag{2.2.2}$$

Since $\mathrm{Sq}^i (Q_{3,0}) = 0$ for $i = 1, 2$, $\mathrm{Sq}^1 Q_{3,2} = 0$ and $Q_{3,0} = V_3 V_2 V_1$, the last term in (2.2.2) is reduced to

$$\begin{aligned} &Q_{3,2} Q_{3,0} \mathrm{Sq}^2 [(Q_{3,0}^{l_0} Q_{3,1}^{l_1} Q_{3,2}^{l_2})^2] \\ &\quad = \mathrm{Sq}^1 (Q_{2,1} X_3 + X_3^3) V_2 V_1 Q_{2,1}^2 [\mathrm{Sq}^1 (Q_{3,0}^{l_0} Q_{3,1}^{l_1} Q_{3,2}^{l_2})]^2 \\ &\qquad + \mathrm{Sq}^1 (Q_{2,1} X_3 + X_3^3) Q_{3,0} [\mathrm{Sq}^1 (Q_{3,0}^{l_0} Q_{3,1}^{l_1} Q_{3,2}^{l_2})]^2. \end{aligned}$$

The last two terms are hit, since Sq^1 acts on $V_2 V_1$ and $Q_{3,0}$ trivially.

2.3. Proof of Case (3)

Suppose that $n_0 = 4l_0 + 1$, $n_1 = 4l_1 + 1$, and $n_2 = 2l_2 + 1$. Let B be the polynomial $Q_{3,0}^{2l_0} Q_{3,1}^{2l_1} Q_{3,2}^{l_2}$. Then $Q_{3,0}^{n_0} Q_{3,1}^{n_1} Q_{3,2}^{n_2} = Q_{3,0} Q_{3,1} Q_{3,2} B^2$ $= V_3 V_2^3 V_1^7 B^2 + V_3 V_2^5 V_1^3 B^2 + V_3^3 V_2 V_1^7 B^2 + V_3^3 V_2^1 V_1^3 B^2 + V_3^2 V_2^2 V_1^5 B^2 + V_3^2 V_2^4 V_1^1 B^2 + V_3^3 V_2^2 V_1^1 B^2$.

Now we will show that each term after the last equality is hit. Using the following lemma, we will know that the first four terms are hit.

LEMMA 2.3. *For any positive odd integers k_0, k_1, and k_2, the polynomial*

$$V_3^{k_0} V_2^{k_1} V_1^{k_2} B^2$$

is hit.

Proof. Put $k_0 = 2a + 1$, $k_1 = 2b + 1$, and $k_2 = 2c + 1$. Then we have

$$\begin{aligned} & V_3^{k_0} V_2^{k_1} V_1^{k_2} B^2 \\ = \ & V_3^{2a+1} V_2^{2b+1} V_1^{2c+1} B^2 \\ = \ & [\mathrm{Sq}^1 (Q_{2,1} X_3 + X_3^3)] V_3^{2a} V_2^{2b+1} V_1^{2c+1} B^2 \\ = \ & \mathrm{Sq}^1 [(Q_{2,1} X_3 + X_3^3)(V_3^{2a} V_2^{2b+1} V_1^{2c+1} B^2)] \quad (\text{since } \mathrm{Sq}^1 (V_2 V_1) = 0). \ \square \end{aligned}$$

Since $V_3^2V_2^2V_1^5B^2 = V_1(V_3V_2V_1^2B)^2$ and $V_3^2V_2^4V_1^1B^2 = V_1(V_3V_2^2B)^2$, we can verify that these two terms satisfy the condition in Theorem 1.5. So they are hit.

It remains to show that $V_3^3V_2^2V_1^1B^2$ is hit. In fact

$$\begin{aligned}
& V_3^3V_2^2V_1^1B^2 \\
= \; & X_1^3X_2^2X_3^{12}B^2 + X_1X_2^4X_3^{12}B^2 + X_1^7X_2^2X_3^8B^2 + X_1X_2^8X_3^8B^2 \\
& +X_1^8X_2^6X_3^3B^2 + X_1^5X_2^2X_3^{10}B^2 + X_1^4X_2^3X_3^{10}B^2 + X_1^2X_2^5X_3^{10}B^2 \\
& +X_1X_2^6X_3^{10}B^2 + X_1^9X_2^2X_3^6B^2 + X_1^8X_2^3X_3^6B^2 + X_1^2X_2^9X_3^6B^2 \\
& +X_1^4X_2^{10}X_3^3B^2 + X_1X_2^{10}X_3^6B^2 + X_1^9X_2^4X_3^4B^2 + X_1^5X_2^8X_3^4B^2 \\
& +X_1^8X_2^5X_3^4B^2 + X_1^4X_2^9X_3^4B^2 + X_1^7X_2^6X_3^4B^2 + X_1^3X_2^{10}X_3^4B^2 \\
& +X_1^5X_2^3X_3^9B^2 + X_1^3X_2^5X_3^9B^2 + X_1^8X_2^4X_3^5B^2 + X_1^2X_2^{10}X_3^5B^2 \\
& +X_1^9X_2^5X_3^3B^2 + X_1^5X_2^9X_3^3B^2.
\end{aligned}$$

Using Theorem 1.5, we can prove all but the following terms are hit: $X_1^3X_2^5X_3^9B^2$, $X_1^5X_2^3X_3^9B^2$, $X_1^9X_2^3X_3^5B^2$, $X_1^9X_2^5X_3^3B^2$, $X_1^3X_2^9X_3^5B^2$, and $X_1^5X_2^9X_3^3B^2$. Indeed we only need the following proposition to conclude the result.

PROPOSITION 2.4. $X_1^3X_2^5X_3^9B^2 + X_1^5X_2^3X_3^9B^2$, $X_1^9X_2^3X_3^5B^2 + X_1^9X_2^5X_3^3B^2$, and $X_1^3X_2^9X_3^5B^2 + X_1^5X_2^9X_3^3B^2$ are hit.

Proof. Careful observations reveal the following,

(2.3.1)

$$X_1^3X_2^5X_3^9 + X_1^5X_2^3X_3^9 = \mathrm{Sq}^2(X_1^9X_2^3X_3^3) + \mathrm{Sq}^1(X_1^{10}X_2^3X_3^3) + X_1^9X_2^4X_3^4,$$

$$X_1^9X_2^3X_3^5 + X_1^9X_2^5X_3^3 = \mathrm{Sq}^2(X_1^3X_2^9X_3^3) + \mathrm{Sq}^1(X_1^3X_2^{10}X_3^3) + X_1^4X_2^9X_3^4,$$

and

$$X_1^3X_2^9X_3^5 + X_1^5X_2^9X_3^3 = \mathrm{Sq}^2(X_1^3X_2^3X_3^9) + \mathrm{Sq}^1(X_1^3X_2^3X_3^{10}) + X_1^4X_2^4X_3^9.$$

We shall show that $X_1^3X_2^5X_3^9B^2 + X_1^5X_2^3X_3^9B^2$ is hit; the remaining two cases can be proved similarly.

By Theorem 1.5, $X_1^9X_2^4X_3^4B^2 = X_1(X_1^4X_2^2X_3^2B)^2$ in (2.3.1) is hit. So we are left to show that $\mathrm{Sq}^2(X_1^9X_2^3X_3^3)B^2 + \mathrm{Sq}^1(X_1^{10}X_2^3X_3^3)B^2$ is hit. This can be done by using the χ-*trick*. In fact,

$$X_1^3X_2^3X_3^9[\chi(\mathrm{Sq}^2)(B^2)] + X_1^3X_2^3X_3^{10}[\chi(\mathrm{Sq}^1(B^2)] = 0,$$

where we use the facts that $\mathrm{Sq}^2B^2 = 0$ and $\mathrm{Sq}^1B^2 = 0$. □

To finish the proof of Case(3), we are left to show that $Q_{3,0}^{4l_0+3}Q_{3,1}^{4l_1+1}Q_{3,2}^{2l_2+1}$ is hit. The proof can be done as follows,

$$\begin{aligned}
&Q_{3,0}^{4l_0+3}Q_{3,1}^{4l_1+1}Q_{3,2}^{2l_2+1}\\
&= Q_{3,0}^{3}Q_{3,1}Q_{3,2}Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{2l_2}\\
&= \mathrm{Sq}^2[Q_{3,0}Q_{3,1}^{3}Q_{3,2}](Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{2l_2}) + Q_{3,0}Q_{3,1}^{4}(Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{2l_2})\\
&\equiv Q_{3,0}Q_{3,1}^{3}Q_{3,2}\chi(\mathrm{Sq}^2)(Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{2l_2}) + Q_{3,0}^{4l_0+1}Q_{3,1}^{4l_1+4}Q_{3,2}^{2l_2}
\end{aligned}$$

It is easy to see that the first term is zero and that the second term is hit by using the result of Case (1).

2.4. Proof of Case (4)

Put $n_0 = 2l_0 + 1$, $n_1 = 4l_1 + 3$, and $n_2 = 4l_2 + 1$. It is easy to verify that

$$Q_{3,0}Q_{3,1}^{3}Q_{3,2} = \mathrm{Sq}^4[Q_{3,0}Q_{3,1}Q_{3,2}^{3}] + Q_{3,0}^{3}Q_{3,2}^{2} + Q_{3,0}Q_{3,1}Q_{3,2}^{4}.$$

Thus

$$Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} = Q_{3,0}^{2l_0+1}Q_{3,1}^{4l_1+3}Q_{3,2}^{4l_2+1} = Q_{3,0}^{3}Q_{3,1}Q_{3,2}(Q_{3,0}^{2l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2})$$

$$= \mathrm{Sq}^4[Q_{3,0}Q_{3,1}Q_{3,2}^{3}](Q_{3,0}^{2l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}) + Q_{3,0}^{2l_0+3}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2+2} + Q_{3,0}^{2l_0+1}Q_{3,1}^{4l_1+1}Q_{3,2}^{4l_2+4}$$

The last two terms are hit by the results of Cases (1) and (2). Using the χ-*trick* and the result of Case (3), it can be shown that the first term is also hit. We will leave the detail to the reader to verify the conclusion.

2.5. Proof of Case (5)

When $n_0 = 4l_0 + 1$, $n_1 = 4l_1 + 3$, and $n_2 = 4l_2 + 3$,

$$\begin{aligned}
Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} &= Q_{3,0}^{4l_0+1}Q_{3,1}^{4l_1+3}Q_{3,2}^{4l_2+3}\\
&= P_1 Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2},
\end{aligned}$$

where $P_1 = Q_{3,0}Q_{3,1}^{3}Q_{3,2}^{3}$, which is a symmetric polynomial in X_1, X_2, and X_3. Let $X_1^aX_2^bX_3^c$ be a term in the expansion form of P_1. For dimension reason, $a + b + c = 37$. Suppose that a is odd, b and c are even. Then setting $k = 0$, $E = X_1$, and

$$F = X_1^{\frac{a-1}{2}}X_2^{\frac{b}{2}}X_3^{\frac{c}{2}}Q_{3,0}^{2l_0}Q_{3,1}^{2l_1}Q_{3,2}^{2l_2},$$

we have

$$\deg F = \frac{(a-1)+b+c}{2} + (7\cdot 2l_0) + (6\cdot 2l_1) + (4\cdot 2l_2) = 18 + 2(7l_0 + 6l_1 + 4l_2).$$

So $\mu(\deg(F)) \geq 2$. Hence we can use Theorem 1.5 to conclude that

$$X_1^a X_2^b X_3^c Q_{3,0}^{4l_0} Q_{3,1}^{4l_1} Q_{3,2}^{4l_2}$$

is hit. Suppose that a, b, and c are all odd. Then after expanding P_1, we know that up to the permutation, (a, b, c) can only have the following choices,

$$(2.5.1) \qquad \begin{array}{l} (3,13,21),(3,9,25),(5,11,21),(5,13,19), \\ (7,9,21),(7,11,19),(7,13,17),(9,11,17). \end{array}$$

Referring to the above list, we wish to show that $X_1^3 X_2^{13} X_3^{21} Q_{3,0}^{4l_0} Q_{3,1}^{4l_1} Q_{3,2}^{4l_2}$ is hit. Use the condition of Theorem 1.5 as follows. Setting $k = 1$, $E = X_1^3 X_2 X_3$, and $F = X_2^3 X_3^5 Q_{3,0}^{l_0} Q_{3,1}^{l_1} Q_{3,2}^{l_2}$, then

$$\deg E = 5 \text{ and } \deg F = 8 + 7l_0 + 6l_1 + 4l_2.$$

Hence $(2^{k+1}-1)\mu(\deg F) = 3\mu(8+7l_0+6l_1+4l_2)$. Suppose that l_0 is even. Then the condition of Theorem 1.5 is satisfied. Hence we are done. Suppose that l_0 is odd, if the condition of the Peterson Conjecture(Wood's Theorem) is satisfied, namely,

$$3 < \alpha(37 + 4 \cdot 7l_0 + 4 \cdot 6l_1 + 4 \cdot 4l_2 + 3),$$

then we have proved the result. If the condition is not satisfied, then the Minimum Spike exists [6], p.578. So there exist non-negative integers $m \geq r \geq s$, such that $37+4\cdot 7l_0+4\cdot 6l_1+4\cdot 4l_2 = 2^m-1+2^r-1+2^s-1$. This implies that $7l_0 + 6l_1 + 4l_2 = 2^{m-2} + 2^{r-2} - 9$, $s = 2$, and $r \geq 3$. The property for Minimum spike implies that we can assume $m > r$ if $r > s$. So $\mu(8+7l_0+6l_1+4l_2) = \mu(8+2^{m-2}+2^{r-2}-9) \geq 2$. Hence the condition for Theorem 1.5 is satisfied. Therefore $X_1^3 X_2^{13} X_3^{21} Q_{3,0}^{4l_0} Q_{3,1}^{4l_1} Q_{3,2}^{4l_2}$ is hit. Using the same method, we can show that all the polynomials corresponding to the list (2.5.1) are hit except for the case: $(a, b, c) = (7, 11, 19)$ and its permutations.

LEMMA 2.5. *For any non-negative integers l_0, l_1, and l_2, the following statements are true.*

(1) $\mathrm{Sq}^i[Q_{3,0}^{4l_0} Q_{3,1}^{4l_1} Q_{3,2}^{4l_2}] = 0$ *for i not divisible by 4;*

(2) *For any non-negative integer t,* $\mathrm{Sq}^{4t}[Q_{3,0}^{4l_0} Q_{3,1}^{4l_1} Q_{3,2}^{4l_2}] = \sum Q_{3,0}^{4s_0} Q_{3,1}^{4s_1} Q_{3,2}^{4s_2}$ *for some integers s_0, s_1, and s_2;*

(3) $\mathrm{Sq}^4\mathrm{Sq}^4[Q_{3,0}^{4l_0} Q_{3,1}^{4l_1} Q_{3,2}^{4l_2}] = 0$.

Proof. The proposition is a basic consequence of the Cartan formula and Theorem 1.4. We leave the detail to the reader. □

In the following, we will frequently use the above lemma without mentioning each time. Denote by Y the product $Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}$.

LEMMA 2.6.

$$(X_1^7X_2^{11}X_3^{19} + X_1^{11}X_2^7X_3^{19})Y \equiv [X_1^7X_2^{11}X_3^7 + X_1^{11}X_2^7X_3^7]\mathrm{Sq}^8\mathrm{Sq}^4Y.$$

Proof. This is an elementary exercise, using the χ-*trick*, Theorem 1.5 and Lemma 2.5. □

Let P be the set consisting of the index $(1, 2, 3)$ and all its permutations. Then we have the following,

$$\begin{aligned}\sum_P & X_1^7X_2^{11}X_3^{19}Y \\ &= (X_1^7X_2^{11}X_3^{19} + X_1^{11}X_2^7X_3^{19})Y + (X_1^7X_2^{19}X_3^{11} + X_1^{11}X_2^{19}X_3^7)Y \\ &\quad + (X_1^{19}X_2^{11}X_3^7 + X_1^{19}X_2^7X_3^{11})Y \equiv 0 \quad \text{(using Lemma 2.6)}.\end{aligned}$$

2.6. Proof of Case (6)

When $n_0 = 4l_0 + 3$, $n_1 = 4l_1 + 3$, and $n_2 = 4l_2 + 3$,

$$\begin{aligned}Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2} &= Q_{3,0}^{4l_0+3}Q_{3,1}^{4l_1+3}Q_{3,2}^{4l_2+3} \\ &= P_2Y,\end{aligned}$$

where $P_2 = Q_{3,0}^3Q_{3,1}^3Q_{3,2}^3$ and $Y = Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}$. After expanding P_2, by applying Theorem 1.5 and Peterson's Conjecture(Wood's Theorem), we can conclude that all the corresponding terms are hit, except the following terms, up to the permutations,

$$X_1^7X_2^{19}X_3^{25}Y, X_1^7X_2^{11}X_3^{33}Y \quad \text{and} \quad X_1^{11}X_2^{19}X_3^{21}Y. \tag{2.6.1}$$

For example, $X_1^5X_2^{21}X_3^{25}Y$ appears in the expansion of P_2Y. We will show that it is hit. Set $k = 1$, $E = X_1X_2X_3$, and $F = X_1X_2^5X_3^6Q_{3,0}^{l_0}Q_{3,1}^{l_1}Q_{3,2}^{l_2}$. If $\deg F$ is even, then the condition of Theorem 1.5 satisfied. So $EF^4(= X_1^5X_2^{21}X_3^{25}Y)$ is hit. If $\deg F$ is odd, then we can assume that $\alpha(3 + EF^4) < 3$ (otherwise the result is proved using the Peterson Conjecture

(Wood's Theorem)). So the Minimum Spike exists [6], p.578. Hence $\deg(EF^4) = 2^m - 1 + 2^r - 1 + 2^s - 1$ for some integers $m \geq r \geq s$. Therefore we have the following,

$$3 + 48 + 4(7l_0 + 6l_1 + 4l_2) = \deg(EF^4) = 2^m - 1 + 2^r - 1 + 2^s - 1.$$

This implies that $7l_0 + 6l_1 + 4l_2 = 2^{m-2} + 2^{r-2} - 13$, $s = 1$, and $r \geq 2$. Using the property of the Minimum spike, we may assume that $m > r$. So

$$\mu(\deg F) = \mu(12 + 7l_0 + 6l_1 + 4l_2) = \mu(2^{m-2} + 2^{r-2} - 1) \geq 2.$$

Therefore the inequality: $\deg E < (2^2 - 1)\mu(F)$ is true. Hence the condition of Theorem 1.5 is satisfied. Using the same method, we can show that all the terms of P_2Y corresponding to the expansion of P_2 are hit, except the terms listed in (2.6.1).

Recall that Y denotes $Q_{3,0}^{4l_0}Q_{3,1}^{4l_1}Q_{3,2}^{4l_2}$. Using the *$\chi$-trick*, Theorem 1.5 and Lemma 2.5, we can show that

$$(X_1^7X_2^{11}X_3^{33} + X_1^7X_2^{19}X_3^{25})Y \equiv X_1^7X_2^{11}X_3^{25}\mathrm{Sq}^8Y. \tag{2.6.2}$$

It is easy to show that (2.6.2) implies that

$$\sum_P (X_1^7X_2^{11}X_3^{33} + X_1^7X_2^{19}X_3^{25})Y \equiv \sum_P X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^4\mathrm{Sq}^8Y\,, \tag{2.6.3}$$

where P is the set consisting of the index $(1, 2, 3)$ and all its permutations. Again using the *χ-trick*, Theorem 1.5 and Lemma 2.5, we can show the following,

LEMMA 2.7.

$$X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^4\mathrm{Sq}^8Y \equiv (X_1^{11}X_2^{21}X_3^{19} + X_1^{19}X_2^{13}X_3^{19})Y.$$

Combining the above lemma with (2.6.3), we have

$$\begin{aligned}
&\textstyle\sum_P(X_1^7X_2^{11}X_3^{33} + X_1^7X_2^{19}X_3^{25} + X_1^{11}X_2^{21}X_3^{19})Y\\
&\equiv \textstyle\sum_P[X_1^7X_2^{11}X_3^{21}\mathrm{Sq}^4\mathrm{Sq}^8Y + X_1^{11}X_2^{21}X_3^{19}Y]\\
&\equiv \textstyle\sum_P[(X_1^{11}X_2^{21}X_3^{19} + X_1^{19}X_2^{13}X_3^{19})Y + X_1^{11}X_2^{21}X_3^{19}Y]\\
&\equiv 0.
\end{aligned}$$

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