Bull. Korean Math. Soc. **37** (2000), No. 4, pp. 755–764

ON A CLASS OF STRONGLY CLOSE-TO-STAR FUNCTIONS

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ABSTRACT. We introduce a new class of functions $H_k(\beta)$ which is related to close-to-star functions and we derive a few geometric properties for the class $H_k(\beta), (2 \leq k \leq 4)$.

I. Introduction

Let $f(z) = \sum_{n=1}^{\infty} a_n z^n (a_1 \neq 0)$ be analytic in $E = \{z : |z| < 1\}$, and let $f(z) \neq 0$ for $z \neq 0$. We say that $f(z)$ is a close-to-star function in E if there exists a univalent star map $s(z)$, starlike with respect to $w = 0$, such that

$$\text{Re}\frac{f(z)}{s(z)} \geq 0 \tag{1.1}$$

holds for $|z| < 1$. Let C^* denote the class of close-to-star functions in E. This class was introduced by Reade[6]. The functions of class C^* are not necessarily univalent. But the close-to-star maps in C^* bear the same relation to the class C of close-to-convex maps. That is,
(1) If $f(z) = \sum_{n=1}^{\infty} a_n z^n (a_1 \neq 0)$ is analytic for $|z| < 1$, then $f(z)$ is in class C^* if and only if $F(z) = \int_0^z \frac{f(\zeta)}{\zeta} d\zeta$ is in C.
(2) If $F(z)$ is analytic for $|z| < 1$, with $F'(z) \neq 0$ for $|z| < 1$, then $F(z)$ is in C if and only if $f(z) = zF'(z)$ is in C^*.

Received June 1, 1999. Revised March 24, 2000.

2000 Mathematics Subject Classification: Primary 30C45.

Key words and phrases: functions of bounded boundary rotation, strongly close-to-star function.

[1]This research was supported in part by the grant of GARC, Seoul National University, 1998-1999.

[2]The second author acknowledges support received from the Ministry of Education, via 1998-99BSRI-98-1424.

M. O. Reade [6] showed that $f(z)$ is close-to-star in E if and only if

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\Big[re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\Big]d\theta > -\pi \tag{1.2}$$

holds for all $\theta_1 < \theta_2$ and for all $0 \le r < 1$. The inequality (1.2) states that the radius vector to the image of $|z| = r < 1$, under close-to-star maps $f(z)$, never turns back by an amount as much as π radians on any arc.

Let $V_k (2 \le k \le 4)$ be the class of all functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in E and let $f'(z) \neq 0$ in E and

$$\limsup_{r\to 1} \int_0^{2\pi} \Big|\mathrm{Re}\Big\{1 + \frac{zf''(z)}{f'(z)}\Big\}\Big| d\theta \le k\pi \quad (z = re^{i\theta}, r < 1). \tag{1.3}$$

V_k is the class of functions with boundary rotation at most $k\pi$. Every function $f \in V_k$ can be given by the Stieltjes integral representation

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z} d\psi(\theta), \tag{1.4}$$

where $\int_0^{2\pi} d\psi(\theta) = 2\pi$ and $\int_0^{2\pi} |d\psi(\theta)| \le k\pi$, $\psi(\theta)$ being a function of bounded variation on $[0, 2\pi]$.

For $2 \le k \le 4$, we define the class W_k as follows ; $f(z) \in V_k$ if and only if $zf'(z) \in W_k$.

DEFINITION. Let $f(z)$ be an analytic function in E with $f(0) = 0, f'(0) = 1$. $f(z)$ belongs to the class $H_k(\beta), (2 \le k \le 4)$ if

$$\Big|\arg \frac{f(z)}{g(z)}\Big| \le \frac{\beta\pi}{2}, \quad (0 \le \beta \le 1,\ z \in E) \tag{1.5}$$

for some $g(z)$ in W_k.

Note that if $k = 2, H_k(\beta)$ reduces to the class of strongly close-to-star functions of order β. If $k = 2, \beta = 0, H_k(\beta)$ reduces to the class of starlike functions. If $k = 2, \beta = 1$, then $H_k(\beta)$ reduces to the class of close-to-star functions which was introduced originally by Reade[6]. If $\beta = 0, 2 \leq k \leq 4, H_k(\beta)$ reduces to the class W_k.

Recently, we have introduced the class $G_k(\beta)$ as follows[3] ; $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to the class $G_k(\beta)$ if $f'(z) \neq 0$ in E and satisfies the condition

$$\left|\arg \frac{f'(z)}{g'(z)}\right| \leq \frac{\beta\pi}{2}, \quad (0 \leq \beta \leq 1, z \in E) \tag{1.6}$$

for some $g(z)$ in $V_k, (2 \leq k \leq 4)$.

From (1.5) and (1.6) it is easy to see that $F(z)$ is in $G_k(\beta)$ if and only if $f(z) = zF'(z)$ is in $H_k(\beta)$.

The purpose of this note is to derive a few geometric properties for the new class $H_k(\beta)$.

II. Main results

THEOREM 2.1. *If $f(z) = \sum_{n=1}^{\infty} a_n z^n$ $(a_1 \neq 0)$ is analytic for $|z| < 1$, with $f(z) \neq 0$ for $z \neq 0$, and $f(z)$ belongs to $H_k(\beta)$ then the inequality*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left[re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right] d\theta > -\frac{k\beta}{2}\pi \quad (0 \leq \beta \leq 1) \tag{2.1}$$

holds for all $\theta_1 < \theta_2$, $0 \leq r < 1$ and $2 \leq k \leq 4$.

Proof. Suppose $f \in H_k(\beta)$ and let g be an associated function in W_k. Then for a suitable choice of arguments

$$|\arg f(z) - \arg g(z)| < \frac{\beta}{2}\pi, \quad 0 \leq \beta \leq 1.$$

Let $F(r, \theta) = \arg f(re^{i\theta})$ and $G(r, \theta) = \arg g(re^{i\theta})$. Since g is a function in W_k, $G(r, \theta)$ is an increasing function of θ. The $H_k(\beta)$ condition takes the form

$$|F(r, \theta) - G(r, \theta)| < \frac{\beta}{2}\pi \leq \frac{k\beta}{4}\pi, \quad 0 \leq \beta \leq 1, \quad 2 \leq k \leq 4.$$

Thus for $\theta_1 < \theta_2$,

$$\begin{aligned}
&F(r,\theta_2) - F(r,\theta_1) \\
&= [F(r,\theta_2) - G(r,\theta_2)] + [G(r,\theta_2) - G(r,\theta_1)] + [G(r,\theta_1) - F(r,\theta_1)] \\
&> -\frac{k\beta}{4}\pi - \frac{k\beta}{4}\pi = -\frac{k\beta}{2}\pi
\end{aligned}$$

i.e. if the function $f(z)$ is in $H_k(\beta)$, then

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}(re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})})d\theta > -\frac{k\beta}{2}\pi,$$

which must hold for all $\theta_1 < \theta_2$ and for all $0 \leq r < 1$. □

REMARK. From Theorem 2.1, we can interprete some geometric meaning for the class $H_k(\beta)$. If we suppose that the image domain is bounded by an analytic curve Γ^*, then from (2.1); it follows that the angle of the radius vector turns back at most $\frac{k\beta}{2}\pi$. This is a necessary condition for a function f to belong to $H_k(\beta)$. It will be interesting to see if this condition is also sufficient.

LEMMA 1. *Let $\phi \in W_k$. Then there are two starlike functions s_1 and s_2 such that for $z \in E$*

$$\phi(z) = \frac{(s_1(z))^{\frac{1}{4}k+\frac{1}{2}}}{(s_2(z))^{\frac{1}{4}k-\frac{1}{2}}}.$$

Proof. We know [1] that if $g \in V_k (2 \leq k \leq 4)$ then there are two starlike functions s_1 and s_2 such that for $z \in E$

$$g'(z) = \frac{(\frac{s_1(z)}{z})^{\frac{k}{4}+\frac{1}{2}}}{(\frac{s_2(z)}{z})^{\frac{k}{4}-\frac{1}{2}}}.$$

Therefore from the definition of W_k, it is clear. □

THEOREM 2.2. *$f(z) \in H_k(\beta),\ 0 \le \beta \le 1$ if and only if*

$$f(z) = \frac{(k_1(z))^{\frac{1}{4}k+\frac{1}{2}}}{(k_2(z))^{\frac{1}{4}k-\frac{1}{2}}},$$

where $k_1(z),\ k_2(z)$ are strongly close-to-star functions of order β.

Proof. From the definition of $H_k(\beta)$, we have $f(z) = \phi(z)h^{\beta}(z)$, $\phi(z) \in W_k$ and $|\arg h(z)| < \frac{\pi}{2}$. Using Lemma 1, we know that there are two starlike functions s_1 and s_2 such that

$$\phi(z) = \frac{(s_1(z))^{\frac{1}{4}k+\frac{1}{2}}}{(s_2(z))^{\frac{1}{4}k-\frac{1}{2}}}, \qquad z \in E.$$

Thus

$$\begin{aligned} f(z) &= \frac{(s_1(z))^{\frac{1}{4}k+\frac{1}{2}}}{(s_2(z))^{\frac{1}{4}k-\frac{1}{2}}} h^{\beta}(z) = \frac{(s_1(z)h^{\beta}(z))^{\frac{1}{4}k+\frac{1}{2}}}{(s_2(z)h^{\beta}(z))^{\frac{1}{4}k-\frac{1}{2}}} \\ &= \frac{(k_1(z))^{\frac{1}{4}k+\frac{1}{2}}}{(k_2(z))^{\frac{1}{4}k-\frac{1}{2}}}, \end{aligned}$$

where k_1 and k_2 are two suitable selected strongly close-to-star fuctions of order β. □

THEOREM 2.3. *If $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belongs to $H_k(\beta)$, then the coefficients satisfy the inequality*

$$\begin{aligned} |b_2| &\le k + 2\beta \\ |b_3| &\le \frac{3k^2}{4} + (2\beta+1)k + (2\beta^2 - 1) \\ |b_4| &\le \frac{k^3}{6} + \frac{3\beta}{2}k^2 + (\frac{4\beta^2}{3} + 2(\beta+1))k + \frac{4\beta^2}{3} + 2(\beta - 1). \end{aligned}$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in $G_k(\beta)$ for $|z| < 1$, $2 \le k \le 4$, $0 \le \beta \le 1$. Then we know [3] that the coefficients satisfy the inequalities

$$|a_2| \le \frac{k}{2} + \beta \tag{2.2}$$

$$|a_3| \le \frac{k^2}{4} + (\frac{2\beta + 1}{3})k + (\frac{2\beta^2 - 1}{3})$$

$$|a_4| \le \frac{k^3}{24} + \frac{3\beta}{8}k^2 + (\frac{\beta^2}{3} + \frac{\beta}{2} + \frac{1}{2})k + (\frac{\beta^2}{3} + \frac{\beta}{2} - \frac{1}{2}).$$

Also, from the definition, $h(z) = z + \sum_{n=2}^{\infty} b_n z^n = zf'(z)$ belongs to $H_k(\beta)$. Then by (2.2) and comparing with the coefficients

$$\begin{aligned}
|b_2| &= 2|a_2| \le 2(\frac{k}{2} + \beta) = k + 2\beta \\
|b_3| &= 3|a_3| \le 3(\frac{k^2}{4} + \frac{2\beta + 1}{3}k + \frac{2\beta^2 - 1}{3}) \\
&= \frac{3k^2}{4} + (2\beta + 1)k + (2\beta^2 - 1) \\
|b_4| &= 4|a_4| \le 4(\frac{k^3}{24} + \frac{3\beta k^2}{8} + (\frac{\beta^2}{3} + \frac{\beta + 1}{2})k + (\frac{\beta^2}{3} + \frac{\beta - 1}{2})) \\
&= \frac{k^3}{6} + \frac{3\beta}{2}k^2 + (\frac{4\beta^2}{3} + 2(\beta + 1))k + \frac{4\beta^2}{3} + 2(\beta - 1). \qquad \square
\end{aligned}$$

LEMMA 2. [7]. *Let $Q(z)$ be analytic for $z \in E$ with $Q(0) = 1$. Then* Re $Q(z) \ge \gamma$ *if and only if*

$$Q(z) = \frac{1 + (1 - 2\gamma)h(z)}{1 - h(z)},$$

where $h(z)$ is analytic, $h(0) = 0$ and $|h(z)| < 1$ for $z \in E$.

THEOREM 2.4. *Let $g(z) \in H_k(\beta)$, $0 \leq \beta \leq 1$, $2 \leq k \leq 4$, then*

$$\frac{r(1-r)^{\frac{1}{2}k-1}}{(1+r)^{\frac{1}{2}k+1}}\{\frac{1-r}{1+r}\}^{\beta} \leq |g(z)| \leq \frac{r(1+r)^{\frac{1}{2}k-1}}{(1-r)^{\frac{1}{2}k+1}}\{\frac{1+r}{1-r}\}^{\beta},$$

Equality holds on the right-hand side for

$$g_1(z) = \frac{z(1+z)^{\frac{1}{2}k-1}}{(1-z)^{\frac{1}{2}k+1}}\{\frac{1+z}{1-z}\}^{\beta},$$

and on the left-hand side for

$$g_2(z) = \frac{z(1-z)^{\frac{1}{2}k-1}}{(1+z)^{\frac{1}{2}k+1}}\{\frac{1-z}{1+z}\}^{\beta}.$$

Proof. Suppose that $\frac{g(z)}{\psi(z)} = Q^{\beta}(z)$, where Re $Q(z) \geq 0$, $0 \leq \beta \leq 1$, $\psi(z) \in W_k$. Then from Lemma 2,

$$\frac{g(z)}{\psi(z)} = [\frac{1+h(z)}{1-h(z)}]^{\beta}, \tag{2.3}$$

where $h(0) = 0$ and $|h(z)| < 1$ for $z \in E$. Since $h(z)$ satisfies the conditions of Schwarz's lemma, (2.3) yields

$$[\frac{1-r}{1+r}]^{\beta} \leq |\frac{g(z)}{\psi(z)}| \leq [\frac{1+r}{1-r}]^{\beta}. \tag{2.4}$$

We know[4] that if $\phi(z)$ belongs to V_k then

$$\frac{(1-r)^{\frac{k}{2}-1}}{(1+r)^{\frac{k}{2}+1}} \leq |\phi'(z)| \leq \frac{(1+r)^{\frac{k}{2}-1}}{(1-r)^{\frac{k}{2}+1}}.$$

From the definition of W_k, $\psi(z) = z\phi'(z)$ is in W_k and

$$\frac{r(1-r)^{\frac{k}{2}-1}}{(1+r)^{\frac{k}{2}+1}} \leq |\psi(z)| \leq \frac{r(1+r)^{\frac{k}{2}-1}}{(1-r)^{\frac{k}{2}+1}}. \tag{2.5}$$

Combining (2.4) and (2.5)

$$\frac{r(1-r)^{\frac{k}{2}-1}}{(1+r)^{\frac{k}{2}+1}}(\frac{1-r}{1+r})^{\beta} \leq |g(z)| \leq \frac{r(1+r)^{\frac{k}{2}-1}}{(1-r)^{\frac{k}{2}+1}}(\frac{1+r}{1-r})^{\beta}.$$

To prove that $g_1(z) \in H_k(\beta)$ and $g_2(z) \in H_k(\beta)$, let

$$g_1(z) = \frac{z(1+z)^{\frac{k}{2}-1}}{(1-z)^{\frac{k}{2}+1}}(\frac{1+z}{1-z})^{\beta},$$

$$g_2(z) = \frac{z(1-z)^{\frac{k}{2}-1}}{(1+z)^{\frac{k}{2}+1}}(\frac{1-z}{1+z})^{\beta}$$

and

$$\psi_1(z) = \frac{z(1+z)^{\frac{k}{2}-1}}{(1-z)^{\frac{k}{2}+1}}, \quad \psi_2(z) = \frac{z(1-z)^{\frac{k}{2}-1}}{(1+z)^{\frac{k}{2}+1}}.$$

Since $\frac{g_1(z)}{\psi_1(z)} = (\frac{1+z}{1-z})^{\beta}$ and $\frac{g_2(z)}{\psi_2(z)} = (\frac{1-z}{1+z})^{\beta}$ have argument which is less than $\frac{\beta\pi}{2}$, it suffices to show that $\psi_1(z), \psi_2(z) \in W_k$. But it was shown[4] that $\phi_1(z) = \int_0^z \frac{(1+t)^{\frac{k}{2}-1}}{(1-t)^{\frac{k}{2}+1}}dt, \phi_2(z) = \int_0^z \frac{(1-t)^{\frac{k}{2}-1}}{(1+t)^{\frac{k}{2}+1}}dt$ belong to V_k. Therefore, it is trivial that $\psi_1(z), \psi_2(z)$ are in W_k. □

THEOREM 2.5. *Let $f(z) \in H_k(\beta)$. Then $f(z)$ maps the disk*

$$|z| < \frac{1}{2}[(k+2\beta) - \sqrt{k^2 + 4\beta(k+\beta) - 4}], \quad 0 \leq \beta \leq 1, \ 2 \leq k \leq 4.$$

onto a starlike-domain.

Proof. By definition of $H_k(\beta)$

$$f(z) = g(z)h^{\beta}(z), \quad g \in W_k, \quad |\arg h(z)| < \frac{\pi}{2}.$$

Thus

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \beta\frac{zh'(z)}{h(z)}$$

and so

$$\mathrm{Re}\frac{zf'(z)}{f(z)} \geq \mathrm{Re}\frac{zg'(z)}{g(z)} - \beta\left|\frac{zh'(z)}{h(z)}\right|.$$

For $k(z) \in V_k$, it is known[2] that for $z = re^{i\theta},\ 0 \leq r < 1$,

$$\mathrm{Re}\frac{(zk'(z))'}{k'(z)} \geq \frac{r^2 - kr + 1}{1 - r^2}.$$

From this and the relation of V_k and W_k, if $g \in W_k$, then

$$\mathrm{Re}\frac{zg'(z)}{g(z)} \geq \frac{r^2 - kr + 1}{1 - r^2}.$$

Hence

$$\mathrm{Re}\frac{zf'(z)}{f(z)} \geq \frac{r^2 - kr + 1}{1 - r^2} - \beta\frac{2r}{1 - r^2} = \frac{r^2 - (k + 2\beta)r + 1}{1 - r^2}.$$

Therefore $f(z)$ maps the disk

$$|z| < \frac{1}{2}[(k + 2\beta) - \sqrt{k^2 + 4\beta(k + \beta) - 4}].$$

onto a starlike domain. □

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