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CONTINUITY OF ONE-SIDED BEST SIMULTANEOUS APPROXIMATIONS

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Abstract. In the space $C_1(X)$ of real-valued continuous functions with L_1-norm, every bounded set has a relative Chebyshev center in a finite-dimensional subspace S. Moreover, the set function $F \to Z_S(F)$ corresponding to F the set of its relative Chebyshev centers, is continuous on the space $B[C_1(X)]$ of nonempty bounded subsets of $C_1(X)$ with the Hausdorff metric. In particular, every bounded set has a relative Chebyshev center in the closed convex set $S(F)$ of S and the set function $F \to Z_{S(F)}(F)$ is continuous on $B[C_1(X)]$ with a condition that the sets $S(\cdot)$ are equal.

1. Introduction

Let $C_1(X)$ be the normed linear space $C(X)$ with the $L_1(X,\mu)$-norm, where μ is an admissible measure on X that is a compact Hausdorff space, and $C(X)$ denote the set of all real-valued continuous functions on X. But $C_1(X)$ is not a Banach space, however, a dense linear subspace of $L_1(X,\mu)$.

Denote by $B[C_1(X)]$ the space of nonempty bounded subsets in the space $C_1(X)$, $C[C_1(X)]$ the family of nonempty compact subsets in the space $C_1(X)$, and let H be the Hausdorff metric on $B[C_1(X)]$:

$$H(A,B) = \max\left\{ \sup_{a\in A}\inf_{b\in B} d(a,b),\ \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}.$$

Suppose that S is a finite-dimensional subspace of $C_1(X)$ which contains a strictly positive function throughout this article.

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For any $f \in C_1(X)$ and $r > 0$, let $B(f,r) = \{g \in C_1(X) : ||f - g|| \leq r\}$ be the closed $r-$ball around f. For any nonempty bounded subset F and $f \in C_1(X)$, we denote

$$r(f,F) = \inf\{r > 0 : F \subset B(f,r)\}$$

and

$$r(F) = \inf_{f \in C_1(X)} r(f,F)$$

is the Chebyshev radius of F and

$$Z(F) = \{f \in C_1(X) : r(f,F) = r(F)\}$$

is the set of Chebyshev centers of F.

For any subset $M \subset C_1(X)$, we can consider the relative Chebyshev radius of F in M

$$r_M(F) = \inf_{m \in M} r(m,F)$$

and the set of relative Chebyshev centers of F in M

$$Z_M(F) = \{m \in M : r(m,F) = r_M(F)\}.$$

In section 2, we will study the relative Chebyshev center in a finite-dimensional subspace S and a set function $Z_S(\cdot)$ on $B[C_1(X)]$ with the Hausdorff metric. J. D. Ward [5] proved that in the space $C(T,E)$ of E-valued bounded continuous functions on a topological space T, with the norm $||f|| = \sup_{t \in T} ||f(t)||$, every bounded set in $C(T,E)$ has Chebyshev centers in each of the following two cases: (1) E is a finite-dimensional strictly convex (hence uniformly convex) normed space and T is paracompact. (2) E is a Hilbert space and T is normal. And then D. Amir [1] proved that if E is a uniformly convex Banach space and T is any topological space, then every bounded set in $C(T,E)$ with the norm $||f|| = \sup_{t \in T} ||f(t)||$ has Chebyshev centers, moreover every bounded set has a relative Chebyshev center in a closed subspace. Amir asked whether a normed space Y is uniformly convex iff every selection for the set-valued map $Z : F \to Z(F)$ is uniformly continuous on $B[Y]$. Our first result shows that the above cited results are in the affirmative in $C_1(X)$. We note that $C_1(X)$ is the normed linear space $C(X)$ with the $L_1(X,\mu)$-norm, and $C_1(X)$ is not a Banach space, however, a dense linear subspace of $L_1(X,\mu)$. A. Pinkus [4] proved in the case that the bounded sets are singleton in $C_1(X)$, then $Z_S(F)$ is the set of best approximations of f in a finite dimensional subspace S, that is, $Z_S(F) = P_S(f)$.

In section 3, we will consider the relative Chebyshev center of a bounded set F in the closed convex set $S(F)$ and the continuity of the parameter map $F \to Z_{S(F)}(F)$ on bounded sets $B[C_1(X)]$ with the Hausdorff metric. A. Pinkus [4] proved in the case that the bounded sets are singleton in $C_1(X)$, then $Z_{S(F)}(F)$ is the set of one-sided best approximations of f in the closed convex subset $S(F)$ of a finite dimensional subspace S, that is, $Z_{S(F)}(F) = P_{S(f)}(f)$. S. G. Mabizela [2] considered the relative Chebyshev center of x in the closed convex subset $K(p)$ of X for each $(p,x) \in (P,X)$ and the continuity of the parameter map $P : (p,x) \to P_{K(p)}(x)$ on a normed linear space X and (P,d) a metric space.

2. Continuity of best simultaneous approximations

REMARK 1. Suppose that S is a finite-dimensional subspace in $C_1(X)$. Then S is a complete subspace in $C_1(X)$.

THEOREM 2. *Let $\varepsilon > 0$ and a bounded set F in $C_1(X)$ be fixed. If $s_0 \in S$ with $r(s_0, F) \leq (1+\varepsilon) r_S(F)$, then there exists $s^* \in S$ such that $s^* \in Z_S(F)$ with $\|s_0 - s^*\| \leq \varepsilon \cdot \mu(X)$.*

Proof. We may assume, without loss of generality, that $r_S(F) = 1$ and $\mu(X) = 1$. We can choose $s_0 \in S$ such that $r(s_0, F) \leq (1+\varepsilon)$. Take any $v \in S$ with $r(v, F) \leq (1 + \frac{\varepsilon}{2})$ and define

$$\beta(t) = \begin{cases} 1 & \text{if } \quad |v(t) - s_0(t)| \leq \frac{\varepsilon}{2} \\ \varepsilon/2 \cdot |v(t) - s_0(t)| & \text{if } \quad |v(t) - s_0(t)| > \frac{\varepsilon}{2} \end{cases}$$

and

$$s_1(t) = s_0(t) + \beta(t)(v(t) - s_0(t)).$$

Then $s_1 \in S$ and

$$\begin{aligned}||s_1 - s_0|| &= \int_X \beta(t)|v(t) - s_0(t)|\, d\mu \\ &\leq \frac{\varepsilon}{2} \cdot \mu\Big\{t : |v(t) - s_0(t)| \leq \frac{\varepsilon}{2}\Big\} + \frac{\varepsilon}{2} \cdot \mu\Big\{t : |v(t) - s_0(t)| > \frac{\varepsilon}{2}\Big\} \\ &= \frac{\varepsilon}{2}.\end{aligned}$$

By [1], for any $f \in F, ||s_1 - f||_\infty \leq 1 + \frac{\varepsilon}{2}$. Then

$$\begin{aligned}||s_1 - f|| &= \int_X |s_1(t) - f(t)|\, d\mu \\ &\leq 1 + \frac{\varepsilon}{2}.\end{aligned}$$

Inductively, we find s_{n+1} with

$$||s_{n+1} - s_n|| \leq \frac{\varepsilon}{2^{n+1}}$$

and

$$r(s_{n+1}, F) \leq 1 + \frac{\varepsilon}{2^{n+1}}.$$

The Cauthy sequence (s_n) converges to some $s^* \in S$ with $||s^* - s_0|| \leq \varepsilon$. Then

$$1 \leq r(s^*, F) \leq \lim r(s_n, F) \leq 1.$$

Hence, $r(s^*, F) = 1$. □

In Theorem 2, if $r_S(F) = 0$, that is, $F = \{f\} \subset S$, then it is trivial. In Theorem 3, in case of $r_S(F) = 0$ and $r_S(G) = 0$, it is easy. So we assume that $r_S(F) \neq 0$ and $r_S(G) \neq 0$.

THEOREM 3. *Suppose that S is a finite-dimensional subspace of $C_1(X)$. Then the mapping $F \to Z_S(F)$ is continuous in $B[C_1(X)]$.*

Proof. Case 1. Given any $\varepsilon > 0$, let

$$0 < \delta \leq \frac{\varepsilon}{2} \cdot \min\{r_S(F), r_S(G)\},$$

where F, G in $B[C_1(X)]$. Assume that $\mu(X) = 1$. If $H(F,G) < \delta$, then

$$|r(x,F) - r(x,G)| \leq \delta$$

for any $x \in C_1(X)$. For any $x \in S$,

$$r(x,F) \leq r(x,G) + \delta$$

so

$$\inf_{x \in S} r(x,F) \leq \inf_{x \in S} r(x,G) + \delta,$$

that is,

$$r_S(F) \leq r_S(G) + \delta.$$

Interchanging the roles of F and G, we obtain

$$r_S(G) \leq r_S(F) + \delta.$$

Thus

$$|r_S(F) - r_S(G)| \leq \delta.$$

For any $z \in Z_S(F)$,

$$\begin{aligned} r(z,G) &\leq r(z,F) + \delta \\ &= r_S(F) + \delta \\ &\leq r_S(G) + 2\delta \\ &\leq r_S(G) + \varepsilon \cdot r_S(G) \\ &= (1+\varepsilon) r_S(G). \end{aligned}$$

By Theorem 2, there exists $w \in S$ such that $w \in Z_S(G)$ and $||z - w|| \leq \varepsilon$. So,

$$\sup_{z \in Z_S(F)} \inf_{w \in Z_S(G)} ||z - w|| \leq \varepsilon.$$

Similarly,

$$\sup_{z\in Z_S(G)} \inf_{w\in Z_S(F)} ||z-w|| \leq \varepsilon,$$

i.e., $H(Z_S(F), Z_S(G)) \leq \varepsilon$. Hence $Z_S : F \to Z_S(F)$ is continuous on $B[C_1(X)]$.

Case 2. If $r_S(F) = 0, r_S(G) \neq 0$, then let

$$0 < \delta \leq \frac{\varepsilon}{2} \cdot \min\{1, r_S(G)\}.$$

Similarly, we can prove it. □

THEOREM 4 (E. MICHAEL). *Let X be a paracompact space and let Z be a set-valued map from X to a* Banach *space Y whose values are closed, convex and nonempty sets. If Z is lower semicontinuous, then Z admits a continuous selection.*

Since $(B[C_1(X)], H)$ is a metric space and hence is paracompact. Every finite-dimensional subspace S of $C_1(X)$ is complete by Remark 1, so S is a Banach space. By E. Michael, we have the following result.

COROLLARY 5. *The mapping $F \longmapsto Z_S(F)$ in $B[C_1(X)]$ has a continuous selection.*

If F is singleton, then the following results are added.

DEFINITION 6 [4]. The metric selection s from $C_1(X)$ onto S is said to be an L_1-continuous selection if s is continuous with respect to L_1-convergence. That is, for any f, f_n in $C_1(X)$,

$$\lim_{n\to\infty} ||f - f_n||_1 = 0 \Longrightarrow \lim_{n\to\infty} ||s(f) - s(f_n)||_1 = 0.$$

The metric selection s from $C_1(X)$ onto S is said to be an L_∞-continuous selection if s is continuous with respect to L_∞-convergence. That is, for any f, f_n in $C_1(X)$,

$$\lim_{n\to\infty} ||f - f_n||_\infty = 0 \Longrightarrow \lim_{n\to\infty} ||s(f) - s(f_n)||_1 = 0.$$

REMARK 7. If s is an L_1-continuous selection, then it is also L_∞-continuous selection.

COROLLARY 8. *The map* $F \longmapsto Z_S(F)$ *in* $B[C_1(X)]$ *has a* L_∞-*continuous selection.*

Proof. By Corollary 5, the map $F \longmapsto Z_S(F)$ in $B[C_1(X)]$ has a continuous selection, that is an L_1-continuous selection. Also it is a L_∞-continuous selection by Remark 7. □

THEOREM 9 [4]. *Let S be a finite-dimensional subspace of $C_1(X)$. There exists an L_1-continuous selection onto S if and only if S is a unicity space for $C_1(X)$. Furthermore, if X is connected and S is not a unicity space, then there exists no L_∞-continuous selection.*

THEOREM 10. *A finite-dimensional subspace S is a unicity space for $C_1(X)$.*

Proof. It follows from Corollary 5 and Theorem 9. □

3. Continuity of one-sided best simultaneous approximations

For each bounded set F, set

$$S(F) := \bigcap_{f\in F} S(f) := \bigcap_{f\in F} \{\tilde{s} \in S | \tilde{s} \le f\}.$$

If there exists a function $s^* \in S(F)$ such that

$$\max_{f\in F} \|f - s^*\|_1 = r(s^*, F) = r_{S(F)}(F) = d(F, S(F)),$$

then s^* is called a one-sided best simultaneous L_1-approximation for F. That is, s^* is a relative Chebyshev center of F in $S(F)$.

Firstly, if for any bounded set F, $S(F)$ is nonempty, then there exists a one-sided best simultaneous L_1-approximation on $S(F)$, that is, there exists a relative Chebyshev center of F in $S(F)$ by the following theorem. If S contains a strictly positive function, then $S(F)$ is nonempty, and so each compact set $F \in C[C_1(X)]$ has a best simultaneous L_1-approximation from $S(F)$ where S is a finite-dimensional subspace [3].

THEOREM 11. *Suppose that F is a bounded set in $C_1(X)$ and that ε is a positive real number. If $r(s_0, F) \leq (1+\varepsilon) r_{S(F)}(F)$ and $s_0 \in S(F)$, then there exists $s^* \in S(F)$ such that it is a relative Chebyshev center for F, that is, $s^* \in Z_{S(F)}(F)$ with $||s_0 - s^*|| \leq 2\varepsilon$.*

Proof. Suppose that $r_{S(F)}(F) = 1$. Assume that for any $\varepsilon > 0$, there exist $s_0, v_1 \in S(F)$ such that $r(s_0, F) \leq 1+\varepsilon$ and $r(v_1, F) \leq 1 + \frac{\varepsilon}{2}$. Let $s_1 = s_0$ if $v_1 \leq s_0$, and $s_1 = v_1$ if not. Then $f - s_1 \geq 0$ for any $f \in F$, that is, $s_1 \in S(F)$. Thus

$$\begin{aligned} ||f - s_1|| &= \int_X |f(x) - s_1(x)| d\mu \\ &\leq \int_X |f(x) - v_1(x)| d\mu \\ &\leq 1 + \frac{\varepsilon}{2}, \end{aligned}$$

that is, $r(s_1, F) \leq 1 + \frac{\varepsilon}{2}$ and $||s_1 - s_0|| \leq \varepsilon$. In fact, if $||s_0 - s_1|| > \varepsilon$, then there exists $f \in F$ such that

$$\begin{aligned} 1 + \varepsilon &\geq ||f - s_0|| \\ &= ||f - s_1|| + ||s_0 - s_1|| \\ &\geq 1 + ||s_0 - s_1|| \\ &> 1 + \varepsilon. \end{aligned}$$

There exists $v_2 \in S(F)$ such that $r(v_2, F) \le 1 + \frac{\varepsilon}{4}$. Let $s_2 = s_1$ if $v_2 \le s_1$, and $s_2 = v_2$ if not. Then $s_2 \in S(F)$ and $r(s_2, F) \le 1 + \frac{\varepsilon}{4}$ and $||s_1 - s_2|| \le \frac{\varepsilon}{2}$. Inductively, we find s_{n+1} in $S(F)$ with

$$||s_{n+1} - s_n|| \le \frac{\varepsilon}{2^n}$$

and

$$r(s_n, F) \le 1 + \frac{\varepsilon}{2^n}.$$

The Cauchy sequence (s_n) converges to some $s^* \in S(F)$ with $||s^* - s_0|| \le 2\varepsilon$. Then

$$\begin{aligned}
1 \le r(s^*, F) &= r(\lim_{n\to\infty} s_n, F) \\
&= \sup_{f\in F} ||\lim_{n\to\infty} s_n - f|| \\
&= \sup_{f\in F} \lim_{n\to\infty} ||s_n - f|| \\
&\le \lim_{n\to\infty} \sup_{f\in F} ||s_n - f|| \\
&= \lim_{n\to\infty} r(s_n, F) \\
&\le 1.
\end{aligned}$$

Hence $r(s^*, F) = 1$ and s^* is a relative Chebyshev center for F in $S(F)$. □

THEOREM 12. *Suppose that S is a finite-dimensional subspace of $C_1(X)$. For any F, G in $B[C_1(X)]$ with $S(F) = S(G)$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$H(F, G) < \delta \quad \textit{implies} \quad H(Z_{S(F)}(F), Z_{S(G)}(G)) \le 2\varepsilon.$$

Proof. For any $\varepsilon > 0$, let

$$0 < \delta \le \frac{\varepsilon}{2} \cdot \min\{r_{S(F)}(F), r_{S(G)}(G)\}$$

where F, G in $B[C_1(X)]$ with $S(F) = S(G)$. Assume that $H(F,G) < \delta$. Then for any $x \in C_1(X)$

$$|r(x,F) - r(x,G)| \leq \delta.$$

In fact, for any $u \in F$, there exists $v \in G$ such that $||u - v|| < \delta$. Then

$$||u - x|| - ||x - v|| \leq ||u - v|| < \delta,$$

so

$$||u - x|| \leq ||x - v|| + \delta.$$

Thus

$$r(x,F) \leq r(x,G) + \delta.$$

Similarly, $r(x,G) \leq r(x,F) + \delta$. For any $x \in Z_{S(G)}(G)$,

$$r(x,F) \leq r(x,G) + \delta,$$

so

$$r(x,F) \leq r_{S(G)}(G) + \delta.$$

Thus

$$r_{S(F)}(F) \leq r_{S(G)}(G) + \delta.$$

Hence

$$|r_{S(F)}(F) - r_{S(G)}(G)| \leq \delta.$$

For any $z \in Z_{S(F)}(F)$,

$$\begin{aligned} r(z,G) &\leq r(z,F) + \delta \\ &= r_{S(F)}(F) + \delta \\ &\leq r_{S(G)}(G) + 2\delta \\ &\leq r_{S(G)}(G)(1 + \varepsilon). \end{aligned}$$

There exists $w \in Z_{S(G)}(G)$ with $||z - w|| \leq 2\varepsilon$. So,

$$\sup_{z \in Z_{S(F)}(F)} \inf_{w \in Z_{S(G)}(G)} ||z - w|| \leq 2\varepsilon.$$

Hence, $(Z_{S(F)}(F), Z_{S(G)}(G)) \leq 2\varepsilon$. □

REMARK 13. (1) In the Theorems 11 and 12, even if $r_{S(F)}(F) = 0$ or $r_{S(G)}(G) = 0$, the results are proved similarly.

(2) In the Theorem 12, we wish to drop the condition that $S(F) = S(G)$. If we can drop it, then by the same proof, we have the following:

Suppose that S is a finite-dimensional subspace of $C_1(X)$. Then the mapping $F \to Z_{S(F)}(F)$ is uniformly continuous in $B[C_1(X)]$.

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