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ON THE SPECTRUM OF THE p-LAPLACIAN ON QUATERNIONIC KAEHLER MANIFOLDS

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ABSTRACT. We study some spectral properties of the p-Laplacian on quaternionic Kaehler manifolds.

1. Introduction

Let (M, g) be a compact manifold of dimension n with metric tensor g. Let $\Delta^p = d\delta + \delta d$ be the Laplace-Beltrami operator acting on the space of smooth p-forms. Then we have the spectrum of Δ^p for each $0 \leq p \leq n$

$$Spec^p(M, g) := \{0 \leq \lambda_{1,p} \leq \lambda_{2,p} \cdots \uparrow +\infty\},$$

where each eigenvalue is repeated according to its multiplicity.

An interesting problem on the spectral geometry is as follows : Let (M, g) and (M', g') be compact Riemannian (resp. Kaehlerian) manifolds with $Spec^p(M, g) = Spec^p(M', g')$ for an arbitrary fixed $p \geq 0$. Then is it true that (M, g) is constant sectional curvature (resp. constant holomorphic sectional curvature) c if and only if (M', g') is of constant sectional curvature (resp. constant holomorphic sectional curvature) c' and $c = c'$?

The answer to the problem is affirmative for any $p \geq 0$ and the particular dimension of M (cf. [4, 5, 6, 7, 8]).

The purpose of the present paper is to study quaternionic analogues for certain results (cf. [2, 5, 6, 7, 8]) of the problem.

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MAIN THEOREM. *Let (M,g) and (M',g') be compact quaternionic Kaehler manifolds with* $\mathrm{Spec}^p(M,g) = \mathrm{Spec}^p(M',g')$ *for an arbitrary fixed $p \geq 0$ (which implies $\dim M = \dim M' = 4m$), $m \geq 2$. If $(m,p) \notin \{(4,2),(4,14)\}$ and $2m(4m-1) - 3p(4m-p) \neq 0$, then M is of constant quaternionic sectional curvature c if and only if M' is of constant quaternionic sectional curvature $c' = c$.*

2. Preliminaries and Proof

Let (M,g) be a real $4m$-dimensional compact quaternionic Kaehler manifold. Then there exists a 3-dimensional vector bundle V of tensors of type $(1,1)$ with local basis of almost Hermitian structures $F = (F_{ij}), G = (G_{ij}), H = (H_{ij})$ such that (i) $FG = -GF = H$, and (ii) for any local cross section ϕ of V, $\nabla_X \phi$ is also a cross section of V, where X is an arbitrary vector field in M and ∇ the Levi-Civita connection on M. It is known that any quaternionic Kaehler manifold (M,g) is an Einstein manifold when $\dim M \geq 8$ (for details, see cf. [1]). By $R = (R^l_{ijk})$, $\rho = (R_{jk}) = (R^l_{ljk})$ and $\sigma = (g^{jk}R_{jk})$ we denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively, and $g = (g_{ij})$ is a Riemannian metric tensor on M, $(g^{ij}) = (g_{ij})^{-1}$. For the tensor field T on M we denote $|T|$ the norm of T with respect to g. We define a *quaternionically projective curvature tensor field* $Q = (Q_{kjih})$[3] defined on M by

$$
\begin{aligned}
Q_{kjih} = R_{kjih} + \frac{1}{4m+8}(&R_{ki}g_{jh} - R_{ji}g_{kh} - 2R_{kl}F^l_jF_{ih} \\
&- R_{kl}F^l_iF_{jh} + R_{jl}F^l_iF_{kh} - 2R_{kl}G^l_jG_{ih} - R_{kl}G^l_iG_{jh} \\
&+ R_{jl}G^l_iG_{kh} - 2R_{kl}H^l_jH_{ih} - R_{kl}H^l_iH_{jh} + R_{jl}H^l_iH_{kh}).
\end{aligned}
$$

Then we obtain

$$
|Q|^2 = |R|^2 - \frac{5m+1}{(m+2)^2}|E|^2 - \frac{5m+1}{4m(m+2)^2}\sigma^2, \tag{2.1}
$$

where we have put $E := (E_{ij} = R_{ij} - \frac{\sigma}{4m}g_{ij})$.

Let $S(X)$ be the so-called *quaternionic section* determined by X, which is a 4-plane spanned by $\{X, FX, GX, HX\}$, where X is a unit vector on the quaternionic Kaehler manifold M. Any 2-plane in a quaternionic section is called a *quaternionic plane*. The sectional curvature of a quaternionic plane π is called the *quaternionic sectional curvature* of π. It is known [3] that if a quaternionic Kaehler manifold M *quaternionically projective flat*, i.e., Q vanishes identically, then it is of constant quaternionic sectional curvature.

Now we introduce the Minakshisundaram-Pleijel-Gaffney asymptotic expansion for $Spec^p(M,g)$, which is given by

$$\sum_{\alpha=0}^{\infty} \exp(-\lambda_{\alpha,p} t) = (4\pi t)^{-\frac{\dim M}{2}} [a_{0,p} + t a_{1,p} + \cdots + t^N a_{N,p}] + o(t^{N-m+\frac{1}{2}}) \quad \text{as} \quad t \downarrow 0, \tag{2.2}$$

where $a_{0,p}, a_{1,p}, a_{2,p}, \cdots$ are numbers which is expressed by (see cf. [4])

$$a_{0,p} = \binom{4m}{p} \int_M dM, \tag{2.3}$$

$$a_{1,p} = \frac{1}{6}\left[\binom{4m}{p} - 6\binom{4m-2}{p-1}\right] \int_M \sigma dM, \tag{2.4}$$

$$\begin{aligned} a_{2,p} = \frac{1}{360} \int_M [& \left\{5\binom{4m}{p} - 60\binom{4m-2}{p-1} + 180\binom{4m-4}{p-2}\right\} \sigma^2 \\ & + \left\{-2\binom{4m}{p} + 180\binom{4m-2}{p-1} - 720\binom{4m-4}{p-2}\right\} |\rho|^2 \\ & + \left\{2\binom{4m}{p} - 30\binom{4m-2}{p-1} + 180\binom{4m-4}{p-2}\right\} |R|^2]\, dM, \end{aligned} \tag{2.5}$$

where dM denotes the volume element of M.

On the other hand, we have for $p \notin \{0, 1, 2, 3, 4m-1, 4m-2\}$,

$$\binom{4m}{p} = \frac{4m(4m-1)(4m-2)(4m-3)}{p(p-1)(4m-p)(4m-p-1)} \binom{4m-4}{p-2}, \tag{2.6}$$

$$\binom{4m-2}{p-1} = \frac{(4m-2)(4m-3)}{(p-1)(4m-p-1)} \binom{4m-4}{p-2}. \tag{2.7}$$

For $p \notin \{0, 1, 2, 3, 4m-1, 4m\}$, substituting (2.1), (2.6), and (2.7) into (2.5) yields

$$(2.8)\quad a_{2,p} = \alpha \int_M \left[4P_1|Q|^2 + \frac{4}{(m+2)^2}P_2|E|^2 + \frac{1}{m(m+2)^2}P_3\sigma^2\right] dM,$$

where

$$\begin{aligned} P_1 := P_1(m,p) = {} & 128m^4 - (480p + 192)m^3 \\ & + (840p^2 - 120p + 88)m^2 \\ & - (360p^3 - 30p^2 + 12)m + 45p^4, \end{aligned}$$

$$\begin{aligned} P_2 := P_2(m,p) = {} & -128m^6 + (2880p + 320)m^5 \\ & - (3600p^2 - 8400p + 664)m^4 \\ & + (1440p^3 - 10020p^2 + 7920p + 676)m^3 \\ & - (180p^4 - 3960p^3 + 12780p^2 + 1560p + 276)m^2 \\ & - (495p^4 - 5400p^3 - 390p^2 - 1440p - 36)m \\ & - 675p^4 - 360p^2, \end{aligned}$$

$$\begin{aligned} P_3 := P_3(m,p) = {} & 1270m^7 - (3840p - 3072)m^6 \\ & + (3840p^2 - 10560p - 1360)m^5 \\ & - (1440p^3 - 11280p^2 + 4944)m^4 \\ & + (180p^4 - 4320p^3 + 3600p^2 + 12720p + 3716)m^3 \\ & + (540p^4 - 1800p^3 - 13980p^2 - 4440p - 756)m^2 \\ & + (225p^4 + 5400p^3 + 1110p^2 + 1440p + 36)m \\ & - 675p^4 - 360p^2, \end{aligned}$$

$$\alpha := \frac{\binom{4m-4}{p-2}}{360p(p-1)(4m-p)(4m-p-1)}.$$

For $p \in \{0,1,2,3,4m-1,4m\}$, the formula (2.5) is of the form ;

$$(2.9) \quad a_{2,p} = \beta \int_M \Big[4Q_1|Q|^2 + \frac{8}{m+2}Q_2|E|^2 + \frac{4}{m(m+1)}Q_3\sigma^2\Big]\, dM,$$

where for $i = 1,2,3$ and $m > 1$

(i) if $p = 0$, then

$$\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,0)}{4m(4m-1)(4m-2)(4m-3)},$$

(ii) if $p = 1$, then

$$\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,1)}{(4m-1)(4m-2)(4m-3)},$$

(iii) if $p = 2$, then

$$\beta = \frac{1}{2 \times 360}, \quad Q_i = Q_i(m) = \frac{P_i(m,2)}{(4m-2)(4m-3)},$$

(iv) if $p = 3$, then

$$\beta = \frac{1}{6 \times 360}, \quad Q_i = Q_i(m) = \frac{P_i(m,3)}{4m-3},$$

(v) if $p = 4m-1$, then

$$\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,4m-1)}{(4m-1)(4m-2)(4m-3)},$$

(vi) if $p = 4m$, then

$$\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,4m)}{4m(4m-1)(4m-2)(4m-3)}.$$

REMARK 1. The sign of the coefficients of $|Q|^2$, $|E|^2$, and σ^2 in the formula (2.9) are respectively determined by the polynomials P_1, P_2, and P_3.

From now on we shall write (2.8) and (2.9) in the following form ;

(2.10)

$$a_{2,p} = \gamma \int_M \left[4R_1|Q|^2 + \frac{4}{(m+2)^2}R_2|E|^2 + \frac{1}{m(m+2)^2}R_3\sigma^2\right] dM,$$

where γ is either α or β, and R_i is either P_i or Q_i (i=1,2,3).

REMARK 2. The equation $\binom{4m}{p} - 6\binom{4m-2}{p-1} = 0$ if and only if $2m(4m-1) - 3p(4m-p) = 0$ if and only if $u^2 - 12v^2 = 1$, where $m = \frac{u-1}{4}$, $p = \frac{u-1}{2} \pm v$. The least solutions are $(u,v) = (7,2), (97,28), (1351,390), \cdots$, which give $(m,p) = (24,20), (24,76), \cdots$.

REMARK 3. The polynomial R_1 has the only solutions $(m,p) = (4,2)$, $(4,14)$ (for the proof, see Theorem 3.1 in [6]).

REMARK 4. Assume that $\mathrm{Spec}^p(M,g) = \mathrm{Spec}^p(M',g')$. Then $\dim M = \dim M'$ is derived from (2.2).

Proof of Main Theorem. Since M and M' ($\dim M = \dim M' = 4m \geq 8$) are Einstein manifolds, $E = 0 = E'$ and σ, σ' are constants. From (2.3) and (2.4), we have $\sigma = \sigma'$. And (2.10) with $\mathrm{Spec}^p(M,g) = \mathrm{Spec}^p(M',g')$ yields

$$\int_M 4R_1|Q|^2 dM = \int_{M'} 4R_1|Q'|^2 dM'.$$

But for $(m,p) \notin \{(4,2),(4,14)\}, R_1 \neq 0$ (Remark 3). Hence $Q = 0$ if and only if $Q' = 0$. □

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