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ON SUFFICIENT CONDITIONS FOR MULTIVALENT STARLIKENESS

DINGGONG YANG

ABSTRACT. Let $S_n(p,\alpha)(p,n \in N = \{1,2,3,\cdots\}, 0 \leq \alpha < 1)$ denote the class of functions $f(z) = z^p + a_{p+n}z^{p+n} + \ldots$ which are p-valently starlike of order α in the unit disk. Some criteria for a function $f(z)$ to be in the class $S_n(p,\alpha)$ are given.

1. Introduction

Let $A_n(p)\big(p, n \in N = \{1,2,3,\cdots\}\big)$ be the class of functions of the form

$$f(z) = z^p + \sum_{m=n}^{\infty} a_{p+m} z^{p+m}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. A function $f(z) \in A_n(p)$ is called p-valently starlike of order α in $E, 0 \leq \alpha < 1$, if it satisfies

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > p\alpha \quad (z \in E).$$

We denote by $S_n(p,\alpha)$ the subclass of $A_n(p)$ consisting of functions $f(z)$ which are p-valently starlike of order α in E. Clearly $S_n(p,\alpha) \subset S_n(p,0)$ for $0 \leq \alpha < 1$. Also, we write

$$A_1(p) = A(p), \quad S_n(p,0) = S_n(p) \quad \text{and} \quad S_1(p) = S(p).$$

For the starlikeness of functions in $A(p)$, the following results are proved.

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THEOREM A([6]). *If* $f(z) \in A(1)$ *satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right\} > -\frac{1}{2} \quad (z \in E),$$

then $f(z) \in S(1)$.

THEOREM B. *If* $f(z) \in A(p)$ *satisfies* $f(z) \neq 0 (0 < |z| < 1)$ *and*

$$\left|\arg\left\{\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)-\left(1+\frac{1}{4p}\right)\right\}\right| > 0 \quad (z \in E),$$

then $f(z) \in S(p)$ *and*

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p \quad (z \in E).$$

THEOREM C. *If* $f(z) \in A(p)$ *satisfies* $f(z) \neq 0 \quad (0 < |z| < 1)$ *and*

$$\left|\arg\left\{\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)-\left(1+\frac{1}{2p}\right)\right\}\right| > 0 \quad (z \in E),$$

then $f(z) \in S(p)$.

Theorem B is the main result of Owa, Nunokawa and Fukui [4] and Theorem C was obtained by Owa, Nunokawa and Saitoh [5].

The object of the present paper is to derive some criteria for a function $f(z)$ to be in the class $S_n(p, \alpha)$. In particular, we improve or extend the above theorems.

2. Lemmas

Let $g(z)$ and $h(z)$ be analytic in E. Then the function $g(z)$ is said to be subordinate to $h(z)$, written $g(z) \prec h(z)$, if $h(z)$ is univalent in $E, g(0) = h(0)$ and $g(E) \subset h(E)$.

To derive our results, we need the following lemmas.

LEMMA 1. *Let* $g(z) = 1 + g_n z^n + \cdots (n \in N)$ *be analytic in* E *and let* $q(z) = 1 + q_1 z + \cdots$ *be analytic and univalent on* $\overline{E}$. *If* $g(z)$ *is not* subordinate to $q(z)$, then there exist points $z_0 \in E$ and $t_0 \in \partial E$, and a *real number* $\lambda \geq n$, *such that*

$$g(|z| < |z_0|) \subset q(E), \quad g(z_0) = q(t_0) \ \text{and} \ z_0 g'(z_0) = \lambda t_0 q'(t_0).$$

This lemma is due to Eenigenburg et al [1].

Applying Lemma 1, we derive

LEMMA 2. *Let* $g(z) = 1 + g_n z^n + \cdots (n \in N)$ *be analytic in* E *and let* $h(z)$ *be analytic and starlike (with respect to the origin) univalent in* E *with* $h(0) = 0$. *If*

$$zg'(z) \prec h(z), \tag{1}$$

then

$$g(z) \prec 1 + \frac{1}{n}\int_0^z \frac{h(u)}{u}du.$$

Proof. Let $g_\rho(z) = g(\rho z), h_\rho(z) = h(\rho z)$ and $q_\rho(z) = q(\rho z)$, where $0 < \rho < 1$ and

$$q(z) = 1 + \frac{1}{n}\int_0^z \frac{h(u)}{u}du.$$

Then $g_\rho(z) = 1 + g_n \rho^n z^n + \cdots$ is analytic on $\overline{E}, h_\rho(z)$ is analytic and starlike univalent on $\overline{E}$, and $q_\rho(z) = 1 + q_1 z + \cdots$ is analytic and univalent on $\overline{E}$. From (1) we have

$$zg'_\rho(z) \prec h_\rho(z). \tag{2}$$

We want to show that $g_\rho(z) \prec q_\rho(z)$. For otherwise, by Lemma 1, there exist $z_0 \in E$ and $t_0 \in \partial E$ such that $g_\rho(z_0) = q_\rho(t_0)$ and

$$z_0 g'_\rho(z_0) = \lambda t_0 q'_\rho(t_0) \quad (\lambda \geq n).$$

Since $t_0 q'_\rho(t_0) = h_\rho(t_0)/n$ and $h_\rho(E)$ is a starlike domain, it follows that

$$z_0 g'_\rho(z_0) = \frac{\lambda}{n} h_\rho(t_0) \notin h_\rho(E),$$

which contradicts (2). Hence $g(\rho z) \prec q(\rho z)$, and by letting $\rho \to 1$ we have $g(z) \prec q(z)$. □

LEMMA 3. *Let $g(z) = a + g_n z^n + g_{n+1} z^{n+1} + \cdots (n \in N)$ be analytic in E with $g(z) \not\equiv a$. If $0 < |z_0| < 1$ and $\mathrm{Re}g(z_0) = \min_{|z| \leq |z_0|} \mathrm{Re}g(z)$, then*

$$z_0 g'(z_0) \leq -\frac{n|a - g(z_0)|^2}{2\mathrm{Re}(a - g(z_0))}.$$

We owe Lemma 3 to Miller and Mocanu [2, Theorem 4(i)].

3. Main results

THEOREM 1. *If $f(z) \in A_n(p)$ satisfies $f(z) \neq 0 \quad (0 < |z| < 1)$ and*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right\} > -\frac{np(1-\alpha)}{2\log 2} \quad (z \in E), \tag{3}$$

where $0 \leq \alpha < 1$, then $f(z) \in S_n(p, \alpha)$ and the order α is sharp.

Proof. Let

$$g(z) = \frac{zf'(z)}{pf(z)}.$$

Then $g(z) = 1 + g_n z^n + \cdots$ is analytic in E and

$$zg'(z) = \frac{zf'(z)}{pf(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right).$$

Hence (3) can be written as

$$zg'(z) \prec \frac{n(1-\alpha)}{\log 2}\frac{z}{1-z}.$$

Now an application of Lemma 2 yields

$$g(z) \prec 1 - \frac{1-\alpha}{\log 2}\log(1-z),$$

which implies that $\mathrm{Re}g(z) > \alpha (z \in E)$. This shows that $f(z) \in S_n(p, \alpha)$.

If we take

$$f(z) = z^p \exp\left\{-\frac{p(1-\alpha)}{\log 2}\int_0^z \frac{\log(1-t^n)}{t}dt\right\},$$

then it is easy to check that $f(z) \in A_n(p)$ satisfies the condition (3). Note that

$$\mathrm{Re}\frac{zf'(z)}{pf(z)} \to \alpha \quad \text{as} \quad z \to e^{i\pi/n}.$$

Thus the order α cannot be increased and the theorem is proved. □

REMARK 1. For $p = n = 1$ and $\alpha = 0$, our theorem improves Theorem A by Owa and Obradovic.

THEOREM 2. *Let* $g(z) = 1 + g_n z^n + g_{n+1} z^{n+1} + \cdots (n \in N)$ *be analytic in* E *with* $g(z) \neq 0$ *for* $z \in E$. *If* $g(z)$ *satisfies*

$$1 + \frac{\alpha}{p}\frac{zg'(z)}{g^2(z)} \neq \beta \quad (z \in E) \tag{4}$$

for all real $\beta \geq M$, *where* $\alpha > 0$ *and*

$$1 < M \leq 1 + \frac{n\alpha}{2p}, \tag{5}$$

then

$$\mathrm{Re}\frac{1}{g(z)} > 1 - \frac{2p(M-1)}{n\alpha} \quad (z \in E). \tag{6}$$

The bound in (6) *is best possible.*

Proof. Since the univalent function $w = -z/(1-z)^2$ maps E onto the complex plane minus the halfline $\mathrm{Re}w \geq 1/4, Imw = 0$, from (4) and (5) we have

$$\frac{\alpha}{p}\frac{zg'(z)}{g^2(z)} \prec -\frac{4(M-1)z}{(1-z)^2}$$

or

$$z\left(\frac{1}{g(z)}\right)' \prec \frac{4p(M-1)z}{\alpha(1-z)^2}.$$

Therefore, using Lemma 2, we get

$$\frac{1}{g(z)} \prec 1 + \frac{4p(M-1)z}{n\alpha(1-z)} \equiv h(z). \tag{7}$$

Since $h(z)$ is (convex) univalent in E and

$$\mathrm{Re}h(z) > 1 - \frac{2p(M-1)}{n\alpha} \quad (z \in E),$$

it follows from (7) that the inequality (6) holds.

To show that the bound in (6) cannot be increased, we consider

$$g(z) = \left(1 + \frac{4p(M-1)z^n}{n\alpha(1-z^n)}\right)^{-1} \quad (z \in E),$$

where M satisfies (5). It is easily verified that the function $g(z)$ is analytic in E,

$$g(z) = 1 - \frac{4p(M-1)}{n\alpha} z^n + \cdots \neq 0 \quad (z \in E)$$

and satisfies (4). On the other hand we have

$$\mathrm{Re}\frac{1}{g(z)} \to 1 - \frac{2p(M-1)}{n\alpha} \quad \text{as} \quad z \to e^{i\pi/n}.$$

The proof of the theorem is now complete. □

Corollary 1. *Let $f(z) \in A_n(p)$ with $f(z)f'(z) \neq 0 \quad (0 < |z| < 1)$. If*

$$\frac{f(z)}{zf'(z)}\left\{\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{zf'(z)}{f(z)}\right\} \neq \beta \quad (z \in E)$$

for all real $\beta \geq M$, where $\alpha > 0$ and $1 < M \leq 1 + \frac{n\alpha}{2p}$, then

$$\mathrm{Re}\frac{f(z)}{zf'(z)} > \frac{1}{p} - \frac{2(M-1)}{n\alpha} \quad (z \in E). \tag{8}$$

The bound in (8) is best possible.

Proof. Putting $g(z) = \frac{zf'(z)}{pf(z)}$ in Theorem 2, the corollary follows at once. □

Remark 2. Setting $n = \alpha = 1$ and $M = 1 + \frac{1}{4p}$ in the corollary, we easily have Theorem B. For $n = \alpha = 1$ and $M = 1 + \frac{1}{2p}$, Corollary 1 implies Theorem C. Furthermore we see that both Theorem B and Theorem C are sharp.

Corollary 2. *Let $p \geq 3$. If $f(z) \in A_n(p)$ satisfies $f^{(p)}(z) \neq 0$ for $z \in E$ and*

$$\left|\arg\left\{\frac{zf^{(p+1)}(z)}{(f^{(p)}(z))^2} - \frac{n}{2(p!)}\right\}\right| > 0 \quad (z \in E), \tag{9}$$

then $f(z) \in S_n(p)$.

Proof. The condition (9) implies that

$$1+\frac{\alpha(p!)}{p}\frac{zf^{(p+1)}(z)}{(f^{(p)}(z))^2}\neq\beta \quad (z\in E)$$

for all real $\beta\geq 1+\frac{n\alpha}{2p}$. Therefore, taking

$$g(z)=\frac{f^{(p)}(z)}{p!},\quad M=1+\frac{n\alpha}{2p} \quad \text{and} \quad p\geq 3$$

in Theorem 2, we get

$$\text{Re}f^{(p)}(z)>0 \quad (z\in E). \tag{10}$$

Making use of the main theorem of Nunokawa et al [3], it follows from (10) that $f(z)\in S_n(p)$ for $p\geq 3$. □

THEOREM 3. *If $f(z)\in A_n(p)$ satisfies $f(z)\neq 0$ $(0<|z|<1)$ and*

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-p\alpha\right)\right\}\right|<\frac{\pi\beta}{2} \quad (z\in E), \tag{11}$$

where $0<\alpha<1$ and

$$\beta=1+\frac{2}{\pi}\arctan\left\{\frac{[n(2p(1-\alpha)+n)]^{1/2}}{p\alpha}\right\}, \tag{12}$$

then $f(z)\in S_n(p,\alpha)$.The bound in (11) is best possible.

Proof. Define the function $g(z)$ by

$$g(z)=\frac{1}{1-\alpha}\left(\frac{zf'(z)}{pf(z)}-\alpha\right). \tag{13}$$

Then $g(z)=1+g_nz^n+\cdots$ is analytic in E and

(14)

$$\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-p\alpha\right)=p(1-\alpha)[p\alpha g(z)+p(1-\alpha)g^2(z)+zg'(z)].$$

Suppose that there exists a point $z_0\in E$ such that

$$\text{Re}g(z)>0 \quad (|z|<|z_0|), \quad g(z_0)=ib, \tag{15}$$

where $b\neq 0$ is a real number. Then, applying Lemma 3, we get

$$-z_0g'(z_0)\geq\frac{n}{2}(1+b^2). \tag{16}$$

If $b>0$, then it follows from (14) and (16) that

$$\begin{aligned}
&\arg\left\{\frac{z_0f'(z_0)}{f(z_0)}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}-p\alpha\right)\right\}\\
&=\arg\{ip\alpha b-p(1-\alpha)b^2+z_0g'(z_0)\}\\
&=\frac{\pi}{2}+\arctan\left\{\frac{p(1-\alpha)b^2-z_0g'(z_0)}{p\alpha b}\right\}\\
&\geq\frac{\pi}{2}+\arctan\left\{\frac{(2p(1-\alpha)+n)b^2+n}{2p\alpha b}\right\}\\
&\geq\frac{\pi\beta}{2},
\end{aligned}$$

where $0<\alpha<1$ and β is given by (12). This is a contradiction to (11).

Similarly, if $b<0$, then we have

$$\arg\left\{\frac{z_0f'(z_0)}{f(z_0)}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}-p\alpha\right)\right\}\leq-\frac{\pi\beta}{2},$$

which also contradicts (11). Thus $\mathrm{Re}g(z)>0\ \ (z\in E)$, that is, $f(z)\in S_n(p,\alpha)$.

Next, we consider the function

$$f(z)=\frac{z^p}{(1-z^n)^{2p(1-\alpha)/n}}\in S_n(p,\alpha). \tag{17}$$

Then for $z=e^{i\theta/n}, 0<\theta<\pi$, we have

$$\begin{aligned}
&\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-p\alpha\right)\\
&=p(1-\alpha)\frac{1+z^n}{1-z^n}\left(p\alpha+p(1-\alpha)\frac{1+z^n}{1-z^n}+\frac{2nz^n}{1-z^{2n}}\right)\\
&=\frac{ip(1-\alpha)}{\tan\frac{\theta}{2}}\left\{p\alpha+i\left(\frac{p(1-\alpha)}{\tan\frac{\theta}{2}}+\frac{n}{\sin\theta}\right)\right\},
\end{aligned}$$

and so

$$\begin{aligned}
&\arg\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-p\alpha\right)\right\}\\
&=\frac{\pi}{2}+\arctan\left\{\frac{2p(1-\alpha)+n+n\tan^2\frac{\theta}{2}}{2p\alpha\tan\frac{\theta}{2}}\right\},
\end{aligned}$$

which attains the minimum value $\frac{\pi\beta}{2}$ when

$$\theta = 2\arctan\left(1+\frac{2p(1-\alpha)}{n}\right)^{1/2}$$

Hence the bound in (11) cannot be increased and the proof of the theorem is complete. □

THEOREM 4. *If $f(z) \in A_n(p)$ satisfies $f(z) \neq 0 \quad (0<|z|<1)$ and*

(18)
$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)+\frac{np(1-\alpha)}{2}\right\}\right| < \pi \quad (z \in E),$$

where $0 \leq \alpha < 1$, then $f(z) \in S_n(p,\alpha)$ and the order α is sharp.

Proof. The function $g(z)$ defined by (13) is analytic in E and

$$\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) = p(1-\alpha)zg'(z). \tag{19}$$

If there exists a point $z_0 \in E$ such that $g(z)$ satisfies (15), then it follows from (19) and (16) that

$$\frac{z_0f'(z_0)}{f(z_0)}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0f'(z_0)}{f(z_0)}\right)+\frac{np(1-\alpha)}{2} < 0,$$

which contradicts (18). Consequently, $f(z) \in S_n(p,\alpha)$.

It is easy to verify that the function $f(z)$ defined by (17) satisfies (18). On the other hand, we have

$$\mathrm{Re}\frac{zf'(z)}{pf(z)} \to \alpha \quad as \quad z \to e^{i\pi/n},$$

which shows that the order α is exact. □

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DEPARTMENT OF MATHEMATICS, SUZHOU UNIVERSITY, SUZHOU, JIANGSU 215006, P. R. CHINA