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Initial fuzzy quasi-proximities

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ABSTRACT

We prove the existence of initial fuzzy quasi-proximity spaces. We study the relationship between the initial structure of the fuzzy closure spaces and that of fuzzy quasi-proximity spaces. We give some examples of those.

1. Introduction

A.S. Mashhour and M.H. Ghanim [14] introduced fuzzy closure spaces as a generalization of closure spaces. In [10], it was introduceda fuzzy quasi-proximity space in a sense of [4]. It is weaker than the definitions of A.K. Katsaras and C.G. Petalas [8] and others [5,7,9]. In [8], it was proved the existence of initial fuzzy proximity spaces in a view of topogenous order. We reprove it with respect to the relation of fuzzy quasi-proximity in our sense. Furthermore, M. Khare [9] prove the existence of the coarsest proximity on X for which each proximity is finer than every given proximities on X. The existence of our sense is more general than it of [9]. In particular, we study the relationship between the initial structure of the fuzzy closure spaces and that of fuzzy quasi-proximity spaces. We give some examples of those.

2. Preliminaries

Throughout this paper, I denote the unit interval. A member μ of I^X is called a fuzzy set. $\tilde{0}$ and $\tilde{1}$ denote constant fuzzy sets taking the values 0 and 1 on X, respectively. For $\lambda, \mu \in I^X$, the fuzzy set λ is *quasi-coincident* with μ, denoted by $\lambda\ q\ \mu$, if there exists $x \in X$ such that $\lambda(x)+\mu(x)>1$. If λ is not quasi-coincident with μ, we denote $\lambda\ \bar{q}\ \mu$. Let $f: X \to Y$ be a function, $\mu \in I^X$ and $\nu \in I^Y$. We define:

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) \mid x \in f^{-1}(\{y\})\}, & \text{if } f^{-1}(\{y\}) \neq 0, \\ 0, & \text{if } f^{-1}(\{y\}) = 0. \end{cases}$$

and $f^{1}(\nu)(x)=\nu(f(x))$.

Lemma 2.1 [4,11] For $\lambda, \mu, \mu_i \in I^X$, we have the following properties.

(1) If $\lambda\ q\ \mu$, then $\lambda \wedge \mu \neq 0$.
(2) $\lambda\ \bar{q}\ \mu$ iff $\lambda \leq \tilde{1} - \mu$.
(3) $x_t\, q \bigvee_{i\in\Gamma} \mu_i$ iff there exists $i_0 \in \Gamma$ such that $x_t\ q\ \mu_{i_0}$.
(4) If $f: X \to Y$ is a function and $\lambda\ q\ \mu$, then $f(\lambda)\ q\ f(\mu)$.

Lemma 2.2 [11,12] If $f: X \to Y$, then we have the following properties for direct and inverse image of fuzzy sets under mappings: for $\nu, \nu_i \in I^Y$ and $\mu, \mu_i \in I^X$,

(1) $\nu \geq f(f^{-1}(\nu))$ with equality if f is surjective,
(2) $\mu \leq f^{-1}(f(\mu))$ with equality if f is injective,
(3) $f^{-1}(\tilde{1} - \nu) = \tilde{1} - f^{-1}(\nu)$,
(4) $f(\tilde{1} - \mu) = \tilde{1} - f(\mu)$ if f is bijective,
(5) $f^{-1}(\bigvee_{i\in\Gamma} \nu_i) = \bigvee_{i\in\Gamma} f^{-1}(\nu_i)$,
(6) $f^{-1}(\bigwedge_{i\in\Gamma} \nu_i) = \bigwedge_{i\in\Gamma} f^{-1}(\nu_i)$,
(7) $f(\bigvee_{i\in\Gamma} \mu_i) = \bigvee_{i\in\Gamma} f(\mu_i)$,
(8) $f(\bigwedge_{i\in\Gamma} \mu_i) \leq \bigwedge_{i\in\Gamma} f(\mu_i)$ with equality if f is injective.

Definition 2.3 [2] A subset $\mathcal{T}$ of I^X is called a *fuzzy topology* on X if it satisfies the following conditions:

(O1) $\tilde{0}, \tilde{1} \in \mathcal{T}$.
(O2) If $\mu_1, \mu_2 \in \mathcal{T}$, then $\mu_1 \wedge \mu_2 \in \mathcal{T}$.
(O3) If $\mu_i \in \mathcal{T}$ for each $i \in \Gamma$, then $\bigvee_{i\in\Gamma} \mu_i \in \mathcal{T}$

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space*.

Let $(X, \mathcal{T}_1)$, $(Y, \mathcal{T}_2)$ be fuzzy topological spaces. A function $f: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called *fuzzy continuous* if $f^{-1}(\mu) \in \mathcal{T}_1$, for all $\mu \in \mathcal{T}_2$.

Definition 2.4 [14] A function $C: I^X \to I^X$ is called a *fuzzy closure operator* on X if it satisfies the following conditions:

(C1) $C(\tilde{0}) = \tilde{0}$.
(C2) $C(\lambda) \geq \lambda$, for all $\lambda \in I^X$.
(C3) $C(\lambda \vee \mu) = C(\lambda) \vee C(\mu)$, for all $\lambda, \mu \in I^X$.

The pair (X, C) is called *a fuzzy closure space*.

A fuzzy closure space (X, C) is called *topological* provided that

(C4) $C(C(\lambda)) = C(\lambda)$, for all $\lambda \in I^X$.

Let (X, C_1) and (Y, C_2) be fuzzy closure spaces. A function $f: (X, C_1) \to (Y, C_2)$ is calleda *fuzzy closure* *map* (for short C-map) if for all $\lambda \in I^X$, $f(C_1(\lambda)) \leq C_2(f(\lambda))$.

Theorem 2.5 [11,12] Let $(X, \mathcal{T})$ be a fuzzy topological space. We define an operator $C_{\mathcal{T}}: I^X \rightarrow I^X$ as follows: for each $\lambda \in I^{X'}$

$C_{\mathcal{T}}(\lambda) = \wedge \{\mu \mid \mu \geq \lambda, \tilde{1} - \mu \in \mathcal{T}\}$.

Then $(X, C_{\mathcal{T}})$ is a topological fuzzy closure space.

Theorem 2.6 [11,12] Let (X, C) be a fuzzy closure space. Define $\mathcal{T}_C$ on X by

$\mathcal{T}_C = \{\tilde{1} - \lambda \mid C(\lambda) = \lambda\}$.

Then:
(1) $\mathcal{T}_C$ is a fuzzy topology on X.
(2) $C = C_{\mathcal{T}_C}$ iff (X, C) is topological.

Theorem 2.7 [11] Let $(X, \mathcal{T})$ be a fuzzy topological space. Then $T_{C_{\mathcal{T}}} = \mathcal{T}$

Definition 2.8 [4,10] A binary relation δ on I^X is said to be a *fuzzy quasi-proximity* on X which satisfies the following conditions: for $\lambda, \mu, \rho \in I^X$,
(FQP1) $(\tilde{1}, \tilde{0}) \notin \delta$ and $(\tilde{1}, \tilde{0}) \notin \delta$.
(FQP2) $(\lambda \vee \rho, \mu) \in \delta$ iff $(\lambda, \mu) \in \delta$ or $(\rho, \mu) \in \delta$ and $(\mu, \lambda \vee \rho) \in \delta$ iff $(\mu, \lambda) \in \delta$ or $(\mu, \rho) \in \delta$.
(FQP3) If $(\lambda, \mu) \notin \delta$, then $\lambda \bar{q} \mu$.
The pair (X, δ) is called a *fuzzy quasi-proximity space*.

A fuzzy quasi-proximity space (X, δ) is called a *fuzzy proximity space* if it satisfies:
(FP) If $(\lambda, \mu) \in \delta$ for $\lambda, \mu \in I^X$, then $(\mu, \lambda) \in \delta$.
Let δ_1 and δ_2 be fuzzy quasi-proximities on X. We say δ_2 is *finer* than δ_1 (δ_1 is *coarser* than δ_2) if $(\lambda, \mu) \in \delta_2$ implies $(\lambda, \mu) \in \delta_1$.

Remark 1. Let (X, δ) be a fuzzy quasi-proximity space.
(1) If $(\lambda, \nu) \notin \delta$ and $\rho \leq \lambda$, then, by (FQP2), we have $(\rho, \nu) \notin \delta$.
(2) We define a binary relation δ^{-1} on I^X if for any $\lambda, \mu \in I^X$,

$(\lambda, \mu) \in \delta^{-1}$ iff $(\mu, \lambda) \in \delta$.

Then (X, δ^{-1}) be a fuzzy quasi-proximity space.

Theorem 2.9 [4] Let (X, δ) be a fuzzy quasi-proximity space. For each $\lambda \in I^X$, we define operator $C_\delta: I^X \rightarrow I^X$ as follows:

$C_\delta(\lambda) = \wedge \{\tilde{1} - \rho \mid (\rho, \lambda) \notin \delta\}$.

Then (X, C_δ) is a fuzzy closure space.

Theorem 2.10 [10] Let (X, C) be a fuzzy closure space. We define a binary relation δ_C on I^X as follows: for $\lambda, \mu \in I^X$

$(\lambda, \mu) \notin \delta_C$ iff $\lambda \bar{q} C(\mu)$.

Then:
(1) $C_{\delta_C} = C$.
(2) For any fuzzy quasi-proximity δ on X, δ_{C_δ} is finer than δ.

Corollary 2.11 Let $(X, \mathcal{T})$ be a fuzzy topological space. Then $\mathcal{T}_{\delta_{C_{\mathcal{T}}}} = \mathcal{T}$.

Let (X, δ_1) and (Y, δ_2) be fuzzy quasi-proximity spaces. A function $f: (X, \delta_1) \rightarrow (Y, \delta_2)$ is a *fuzzy quasi-proximity map* (for short, P-map) if for each $(\mu, \nu) \in \delta_1$, we have $(f(\mu), f(\nu)) \in \delta_2$, equivalently, $(\rho, \eta) \notin \delta_2$, we have $(f^{-1}(\rho), f^{-1}(\eta)) \notin \delta_1$.

Theorem 2.12 [4,10] Let (X, δ_1) and (Y, δ_2) be fuzzy quasi-proximity spaces.
If $f: (X, \delta_1) \rightarrow (Y, \delta_2)$ is a P-map, then:
(1) $f: (X, C_{\delta_1}) \rightarrow (Y, C_{\delta_2})$ is a C-map.
(2) $C_{\delta_1}(f^{-1}(\mu)) \leq f^{-1}(C_{\delta_2}(\mu))$, for each $\mu \in I^Y$.
(3) $f: (X, \mathcal{T}_{C_{\delta_1}}) \rightarrow (Y, \mathcal{T}_{C_{\delta_2}})$ is a fuzzy continuous map.

3. Initial fuzzy quasi-proximities

Definition 3.1 Let $(X_i, \delta_i)_{i \in \Gamma}$ be a family of fuzzy quasi-proximity spaces. Let X be a set and, for each $i \in \Gamma$, $f_i: X \rightarrow X_i$ a function. The initial structure δ is the coarsest fuzzy quasi-proximity on X for which each f_i is a P-map.

Theorem 3.2 (Existence of initial structure) Let $\{(X_i, \delta_i) \mid i \in \Gamma\}$ be a family of fuzzy quasi-proximity spaces. Let X be a set and, for each $i \in \Gamma$, $f_i: X \rightarrow X_i$ a mapping. Define a binary relation $\delta \subset I^X \times I^X$ on X by $(\lambda, \mu) \notin \delta$ iff there exist finite families $\{\lambda_j \mid \lambda = \vee_{j=1}^{p} \lambda_j\}$ and $\{\mu_k \mid \mu = \vee_{k=1}^{q} \mu_k\}$ satisfying the following condition: for any λ_j, μ_k, there exists $i_{jk} \in \Gamma$ such that $(f_{i_{jk}}(\lambda_j), f_{i_{jk}}(\mu_k)) \notin \delta_{i_{jk}}$. Then:
(1) δ is the initial structure on X.
(2) A map $f: (Y, \delta') \rightarrow (X, \delta)$ is a P-map iff $f_i \circ f \rightarrow (Y, \delta') \rightarrow (X_i, \delta_i)$ is a P-map for each $i \in \Gamma$.

Proof. (1) First, we will show that δ is a fuzzy quasi-proximity on X.
(FQP1) For families $\{\tilde{0}\}$ and $\{\tilde{1}\}$, we have $(f_i(\tilde{0}), f_i(\tilde{1})) \notin \delta_i$. Hence $(\tilde{0}, \tilde{1}) \notin \delta$. Similarly, $(\tilde{1}, \tilde{0}) \notin \delta$.
(FQP2) For any $\lambda, \mu, \nu \in I^X$, we will show that $(\lambda \vee \rho, \mu) \notin \delta$ iff $(\lambda, \mu) \notin \delta$ and $(\rho, \mu) \notin \delta$.

If $(\lambda, \mu) \notin \delta$ and $(\rho, \mu) \notin \delta$. there are finite families

$\{\lambda_j \mid \lambda=\vee \lambda_j\}$, $\{\mu_k \mid \mu=\vee \mu_k\}$
$\{\rho_l \mid \rho=\vee \rho_l\}$ and $\{\mu'_m \mid \mu=\vee \mu'_m\}$

satisfying the following conditions: for each j, k, there exists $i_{jk}\in\Gamma$ such that

$(f_{i_{jk}}(\lambda_j), f_{i_{jk}}(\mu_k)))\notin \delta_{i_{jk}}$

and for each l, m, there exists $i_{lm}\in\Gamma$ such that

$(f_{i_{lm}}(\rho_l), f_{i_{lm}}(\mu'_m)))\notin \delta_{i_{lm}}$.

So, there exist finite families

$\{\lambda_j, \rho_l \mid \lambda\vee\rho=(\vee\lambda_j)\vee(\vee\rho_l)\}$,
$\{\mu_k\wedge\mu'_m \mid \mu=\vee_{k,m}(\mu_k\wedge \mu'_m)\}$.

For each λ_j and $\mu_k\wedge\mu'_m$, there exists $i_{jk}\in\Gamma$ such that, by Remark 1 (1),

$(f_{i_{jk}}(\lambda_j), f_{i_{jk}}(\mu_k)))\notin \delta_{i_{jk}} \Rightarrow (f_{i_{jk}}(\lambda_j), f_{i_{jk}}(\mu_k\wedge \mu'_m))\notin \delta_{i_{jk}}$.

For each ρ_l and $\mu_k\wedge\mu'_m$, there exists $i_{lm}\in\Gamma$ such that

$(f_{i_{lm}}(\rho_l), f_{i_{lm}}(\mu'_m))\notin \delta_{i_{lm}} \Rightarrow (f_{i_{lm}}(\rho_l), f_{i_{lm}}(\mu_k\wedge\mu'_m))\notin \delta_{i_{ml}}$.

Hence we have $(\lambda\vee\rho, \mu)\notin\delta$.

If $(\lambda\vee\rho, \mu)\notin\delta$, then there exist finite families $\{\omega_j \mid \lambda\vee\rho=\vee_{j=1}^{p}\omega_j\}$ and $\{\mu_k \mid \mu=\vee_{k=1}^{q}\mu_k\}$ satisfying the following condition: for any ω_j, μ_k, there exists $i_{jk}\in\Gamma$ such that $(f_{i_{jk}}(\omega_j), f_{i_{jk}}(\mu_k))\notin\delta_{i_{jk}}$.

Since $\lambda=(\lambda\vee\rho)\wedge\lambda$, we have

$(\lambda\vee\rho)\wedge\lambda=\vee_{j=1}^{p}(\omega_j\vee\lambda)$.

Since $f_{i_{jk}}(\omega_j\wedge\lambda)\le f_{i_{jk}}(\omega_j)$, by Remark 1 (1),

$(f_{i_{jk}}(\omega_j), f_{i_{jk}}(\mu_k))\notin\delta_{i_{jk}} \Rightarrow (f_{i_{jk}}(\omega_j\wedge\lambda), f_{i_{jk}}(\mu_k))\notin\delta_{i_{jk}}$.

Thus, there exist finite families $\{\omega_j\wedge\lambda \mid \lambda=\vee_{j=1}^{p}(\omega_j\vee\lambda)$ and $\{\mu_k \mid \mu=\vee_{k=1}^{q}\mu_k\}$ satisfying the following condition: for any $\omega_j\wedge\lambda$, μ_k, there exists $i_{jk}\in\Gamma$ such that $(f_{i_{jk}}(\omega_j\wedge\lambda), f_{i_{jk}}(\mu_k))\notin\delta_{i_{jk}}$. Hence $(\lambda, \mu)\notin\delta$. Similarly, $(\rho, \mu)\notin\delta$.

By a similar method, we can prove

$(\mu, \lambda\vee\rho)\notin\delta$ iff $(\mu, \lambda)\notin\delta$ and $(\mu, \rho)\notin\delta$.

(FQP3) We will show that if $\lambda\not\le\tilde{1}-\mu$, then $(\lambda, \mu)\in\delta$.

If $\lambda\not\le\tilde{1}-\mu$, then, for every finite families $\{\lambda_j \mid \lambda=\vee\lambda_j\}$ and $\{\mu_k \mid \mu=\vee\mu_k\}$, there exist j_0, k_0, x_0 such that

$\lambda_{j0}(x_0)+\mu_{k0}(x_0)>1$.

It follows that, for all $i\in\Gamma$,

$f_i(\lambda_{j0})\ (f_i(x_0))+f_i(\mu_{k0})\ (f_i(x_0))\ge\lambda_{j0}(x_0)+\mu_{k0}(x_0)>1$.

Since for each $i\in\Gamma$, δ_i is a fuzzy quasi-proximity on X_i, by (FQP3), we have $(f_i(\lambda_{j0}), f_i(\mu_{k0}))\in\delta_i$. Hence $(\lambda, \mu)\in\delta$.

Second, if $(f_i(\lambda), f_i(\mu))\notin\delta_i$ for each $i\in\Gamma$, for families $\{\lambda\}$ and $\{\mu\}$, we have $(\lambda, \mu)\notin\delta$. Hence for each $i\in\Gamma$, $f_i:(X, \delta)\to(X_i, \delta_i)$ is a P-map.

Finally, if $f_i:(X, \delta')\to(X_i, \delta_i)$ is a P-map, then we will show that for all λ, $\mu\in I^X$,

$(\lambda, \mu)\notin\delta \Rightarrow (\lambda, \mu)\notin\delta'$.

If for any λ, $\mu\in I^X$, $(\lambda, \mu)\notin\delta$, then there are finite families $\{\lambda_j \mid \lambda=\vee_{j=1}^{p}\lambda_j\}$ and $\{\mu_k \mid \mu=\vee_{k=1}^{q}\mu_k\}$ satisfying the following condition: for each j, k, there exists $i_{jk}\in\Gamma$ such that

$(f_{i_{jk}}(\lambda_j), f_{i_{jk}}(\mu_k)))\notin\delta_{i_{jk}}$.

Since $f_i:(X, \delta')\to(X_i, \delta_i)$ is a P-map for every $i\in\Gamma$, then $(\lambda_j, \mu_k)\notin\delta'$. For fixed k, by (FQP2), $(\lambda, \mu_k)\notin\delta'$. Again, by (FQP2), $(\lambda, \mu)\notin\delta'$.

(2) Necessity of the composition condition is clear since the composition of P-maps is a P-map.

Conversely, if $(f(\lambda), f(\mu))\notin\delta$, then there are finite families $\{\rho_j \mid f(\lambda)=\vee_{j=1}^{p}\rho_j\}$ and $\{\eta_k \mid f(\mu)=\vee_{k=1}^{q}\eta_k\}$ satisfying the followings: for any j, k, there exists an $i_{jk}\in\Gamma$ such that

$(f_{i_{jk}}(\rho_j), f_{i_{jk}}(\eta_k))\notin\delta_{i_{jk}}$.

On the other hand, since $f_i\circ f$ is a P-map,

$((f_{i_{jk}}\circ f)^{-1}(f_{i_{jk}}(\rho_j)), (f_{i_{jk}}\circ f)^{-1}(f_{i_{jk}}(\eta_k)))\notin\delta'$.

For any j, k, by Lemma 2.2(2), since

$f^{-1}(\rho_j)\le f^{-1}((f_{i_{jk}}^{-1}(f_{i_{jk}}(\rho_j))),$
$f^{-1}(\eta_k)\le f^{-1}((f_{i_{jk}}^{-1}(f_{i_{jk}}(\eta_k))),$

we have, by Remark 1(1),

$(f^{-1}(\rho_j), f^{-1}(\eta_k))\notin\delta'$.

For fixed k, by (FQP2) and Lemma 2.2 (5),

$(\vee_{j=1}^{p}f^{-1}(\rho_j), f^{-1}(\eta_k))=(f^{-1}(\vee_{j=1}^{p}\rho_j), f^{-1}(\eta_k))\notin\delta'$.

Since $\lambda\le f^{-1}(f(\lambda))=\vee_{j=1}^{p}f^{-1}(\rho_j)$, by Remark 1(1), we have

$(\lambda, f^{1}(\eta_k))\notin\delta'$.

Again, by (FQP2), $(\lambda, \mu)\notin\delta'$. □

Theorem 3.3 [11] (Existence of initial fuzzy closure structure) Let $\{(X_i, C_i) \mid i\in\Gamma\}$ be a family of fuzzy closure spaces. Let X be a set and, for each $i\in\Gamma$, $f_i: X\to X_i$ a function. The structure C on X is defined by

$C(\lambda)=\wedge\{\vee_j^p(\wedge_{i\in\Gamma}f_i^{-1}(C_i(f_i(\lambda_j))))\}$,

where the first $\wedge$ is taken for every finite families $\{\lambda_j \mid \lambda=\vee_{j=1}^{p}\lambda_j\}$.

Then:

(1) C is the initial structure on X which for each $i\in\Gamma$, f_i is a C-map.

(2) A map $f:(Y, C')\to(X, C)$ is a C-map iff $f_i\circ f:(Y,$

$C') \to (X_i, C_i)$ is a C-map for each $i \in \Gamma$.

Theorem 3.4 Let $\{(X_i, \delta_i) \mid i \in \Gamma\}$ be a family of fuzzy quasi-proximity spaces. Let $f_i : X \to X_i$ be a function for each $i \in \Gamma$. Let δ is the initial fuzzy quasi-proximity on X for which each f_i is a P-map. Then:

(1) $C_\delta = C$ where C is the initial fuzzy closure operator on X for which each $f_i : (X, C) \to (X, C_{\delta_i})$ is a C-map.

(2) $\mathcal{T}_{C_\delta} = \mathcal{T}_C$ where $\mathcal{T}_C$ is the initial fuzzy topology on X for which each $f_i : (X, \mathcal{T}_C) \to (X, \mathcal{T}_{C_{\delta_i}})$ is fuzzy continuous.

Proof. (1) Since $f_i : (X, \delta) \to (X_i, \delta_i)$ is a P-map for $i \in \Gamma$, by Theorem 2.12, then $f_i : (X, C_\delta) \to (X_i, C_{\delta_i})$ is a C-map. Therefore, by Theorem 3.3, the identity function $id_X : (X, C_\delta) \to (X, C)$ is a C-map. Hence we have $C_\delta(\lambda) \le C(\lambda)$ for $\lambda \in I^X$.

We will show that $C(\lambda) \le C_\delta(\lambda)$ for $\lambda \in I^X$.

By C_δ of Theorem 2.9, we have

$$C_\delta(\lambda) = \wedge \{\tilde{1} - \rho \mid (\rho, \lambda) \notin \delta\}.$$

For $(\rho, \lambda) \notin \delta$, there are finite families $\{\rho_k \mid \rho = \vee_{k=1}^{q} \rho_k\}$ and $\{\lambda_j \mid \lambda = \vee_{j=1}^{p} \lambda_j\}$ satisfying the following conditions: for all k, j, there exists $i_{kj} \in \Gamma$ such that

$$(f_{i_{kj}}(\rho_k), f_{i_{kj}}(\lambda_j)) \notin \delta_{i_{kj}}.$$

Hence

$$C_{\delta_{i_{kj}}}(f_{i_{kj}}(\lambda_j)) \le \tilde{1} - f_{i_{kj}}(\rho_k). \quad (1)$$

For fixed j, we have

$$\begin{aligned} \wedge_{i \in \Gamma} f_i^{-1}(C_i(f_i(\lambda_j))) &\le \wedge_{k=1}^{q} f^{-1}_{i_{kj}}(C_{\delta_{i_{kj}}}(f_{i_{kj}}(\lambda_j))) \\ &\le \wedge_{k=1}^{q} f^{-1}_{i_{kj}}(\tilde{1} - f_{i_{kj}}(\rho_k)) \text{ (by (1))} \\ &= \wedge_{k=1}^{q} \tilde{1} - f^{-1}_{i_{kj}}(f_{i_{kj}}(\rho_k)) \\ &\qquad \text{(by Lemma 2.2(3))} \\ &\le \wedge_{k=1}^{q} (\tilde{1} - \rho_k) \\ &\qquad \text{(by Lemma 2.2(2))} \\ &= \tilde{1} - \rho. \end{aligned}$$

It follows, by the definition of C from Theorem 3.3,

$$\begin{aligned} C(\lambda) &= \wedge \{\vee_j^p (\wedge_{i \in \Gamma} f_i^{-1}(C_i(f_i(\lambda_j))))\} \\ &\le \wedge \{\vee_j^p (\tilde{1} - \rho)\} \\ &= \tilde{1} - \rho \end{aligned}$$

where the first $\wedge$ is taken for every finite families $\{\lambda_j \mid j = 1, \cdots, p\}$ such that $\lambda = \vee_{j=1}^{p} \lambda_j$. Hence $C(\lambda) \le C_\delta(\lambda)$.

(2) It is easily proved by (1), Theorem 2.12(3) and Theorem 2.6.

Theorem 3.5 Let δ be a fuzzy quasi-proximity on X. Let δ^* be the initial structure on X for which two identity maps $id_X : (X, \delta^*) \to (X, \delta)$ and $id_X : (X, \delta^*) \to (X, \delta^{-1})$ are P-maps. Then δ^* is a fuzzy proximity on X.

Proof. From Theorem 3.2, we only show that $(\lambda, \mu) \notin \delta^*$ iff $(\mu, \lambda) \notin \delta^*$.

Since $(\lambda, \mu) \notin \delta^*$, there exist finite families $\{\lambda_j \mid \lambda = \vee_{j=1}^{p} \lambda_j\}$ and $\{\mu_k \mid \mu = \vee_{k=1}^{m} \mu_k\}$ satisfying the following condition: for any λ_j, μ_k, there exists $\delta_{jk} \in \{\delta, \delta^{-1}\}$ such that $(\lambda_j, \mu_k) \notin \delta_{jk}$. Since $(\lambda_j, \mu_k) \notin \delta_{jk}$ iff $(\mu_k, \lambda_j) \notin \delta_{jk}^{-1}$ and $\delta_{jk}^{-1} \in \{\delta, \delta^{-1}\}$, we have $(\mu, \lambda) \notin \delta^*$.

Conversely, we similarly prove it. □

Example 1. Let μ_1 and μ_2 be nonempty fuzzy sets of I^X. Define δ_1 and δ_2 on X as follows:

$$(\lambda, \mu) \notin \delta_1 \begin{cases} \text{if } \lambda = \tilde{0} \text{ or } \mu = \tilde{0}, \\ \text{if } \lambda \le \tilde{1} - \mu_1,\ \mu \le \mu_1. \end{cases}$$

and

$$(\lambda, \mu) \notin \delta_2 \begin{cases} \text{if } \lambda = \tilde{0} \text{ or } \mu = \tilde{0}, \\ \text{if } \lambda \le \tilde{1} - \mu_2,\ \mu \le \mu_2. \end{cases}$$

Then δ_1 and δ_2 are fuzzy quasi-proximities on X (ref.[10]).

From Theorem 3.2, we can obtain the coarsest fuzzy quasi-proximity δ finer than δ_1 and δ_2 as follows:

(Case 1) If $\lambda \le (\tilde{1} - \mu_1) \vee (\tilde{1} - \mu_2)$, $\mu \le \mu_1 \wedge \mu_2$, then there exist families $\{\lambda \wedge (\tilde{1} - \mu_1), \lambda \wedge (\tilde{1} - \mu_2) \mid \lambda = (\lambda \wedge (\tilde{1} - \mu_1)) \vee (\lambda \wedge (\tilde{1} - \mu_2))\}$ and $\{\mu\}$ such that

$$(\lambda \wedge (\tilde{1} - \mu_1), \mu) \notin \delta_1 \text{ and } (\lambda \wedge (\tilde{1} - \mu_2), \mu) \notin \delta_2.$$

Hence $(\lambda, \mu) \notin \delta$.

(Case 2) If $\lambda \not\le (\tilde{1} - \mu_1) \vee (\tilde{1} - \mu_2)$, $0 \ne \mu \le \mu_1 \wedge \mu_2$, for every finite families $\{\lambda_j \mid \lambda = \vee_{j=1}^{p} \lambda_j\}$ and $\{\mu_k \mid \mu = \vee_{k=1}^{q} \mu_k\}$, since $\vee_{j=1}^{p} \lambda_j \not\le (\tilde{1} - \mu_1 \vee (\tilde{1} - \mu_2)$, there exist λ_j and $x \in X$ such that

$$\lambda_j(x) > (\tilde{1} - \mu_1)(x) \vee (\tilde{1} - \mu_2)(x).$$

For λ_j and $\mu_k \ne \tilde{0}$, since $\lambda_j \not\le (\tilde{1} - \mu_1)$ and $\lambda_j \not\le (\tilde{1} - \mu_2)$, we have $(\lambda_j, \mu_k) \in \delta_i$, for $i = 1, 2$. Hence $(\lambda, \mu) \in \delta$.

(Case 3) If $\lambda \le (\tilde{1} - \mu_1) \vee (\tilde{1} - \mu_2)$, $\lambda \not\le (\tilde{1} - \mu_1)$, $\lambda \not\le (\tilde{1} - \mu_2)$ and $\mu \not\le \mu_1 \wedge \mu_2$, for every finite families $\{\lambda_j \mid \lambda = \vee_{j=1}^{p} \lambda_j\}$ and $\{\mu \mid \mu = \vee_{k=1}^{q} \mu_k\}$, since $\vee_{k=1}^{q} \mu_k \not\le \mu_1 \wedge \mu_2$, there exist μ_k and $x \in X$ such that

$$\mu_k(x) > \mu_1(x) \wedge \mu_2(x).$$

Without loss of generality, we may assume that $\mu_k(x) > \mu_1(x)$. Then, since $\vee_{j=1}^{p} \lambda_j \not\le \tilde{1} - \mu_2$, there exist λ_j and $x_1 \in X$ such that

$$\lambda_j(x_1) > (\tilde{1} - \mu_2)(x_1).$$

For λ_j and μ_k, since $\lambda_j \not\le (\tilde{1} - \mu_2)$ and $\mu_k \not\le \mu_1$, we have $(\lambda_j, \mu_k) \in \delta_1$ and $(\lambda_j, \mu_k) \in \delta_2$, respectively. Hence $(\lambda, \mu) \in \delta$.

(Case 4) If $\lambda \leq (\tilde{1} - \mu_1)$, $\lambda \nleq (\tilde{1} - \mu_2)$ and $\mu \leq \mu_2$, $\mu \nleq \mu_1$, for every finite families $\{\lambda_j \mid \lambda = \vee_{j=1}^{p} \lambda_j\}$ and $\{\mu \mid \mu = \vee_{k=1}^{q} \mu_k\}$, since $\vee_{j=1}^{p} \lambda_j \nleq (\tilde{1} - \mu_2)$, there exists λ_j such that $\lambda_j \nleq (\tilde{1} - \mu_2)$. Since $\vee_{k=1}^{q} \mu_k \nleq \mu_1$, there exists μ_k such that $\mu_k \nleq \mu_1$.

For λ_j and μ_k, we have $(\lambda_j, \mu_k) \in \delta_i$ for $i=1, 2$. Hence$(\lambda, \mu) \in \delta$.

By a similar way, we obtain

$$(\lambda, \mu) \notin \delta \begin{cases} \text{if } \lambda = \tilde{0} \text{ or } \mu = \tilde{0}, \\ \text{if } \tilde{0} \neq \lambda \leq \tilde{1} - (\mu_1 \wedge \mu_2),\ \tilde{0} \neq \mu \leq \mu_1 \wedge \mu_2 \\ \text{if } \lambda \leq \tilde{1} - \mu_1,\ \mu \leq \mu_1, \\ \text{if } \lambda \leq \tilde{1} - \mu_2,\ \mu \leq \mu_2, \\ \text{if } \lambda \leq \tilde{1} - (\mu_1 \vee \mu_2),\ \mu \leq \mu_1 \vee \mu_2. \end{cases}$$

We obtain fuzzy closure operators C_{δ_1} and C_{δ_1} from Theorem 2.9 as follows:

$$C_{\delta_1}(\lambda) = \begin{cases} \tilde{0}, \text{ if } \lambda = \tilde{0}, \\ \mu_1, \text{ if } \tilde{0} \neq \lambda \leq \mu_1, \\ \tilde{1} \quad \text{otherwise.} \end{cases}$$

and

$$C_{\delta_2}(\lambda) = \begin{cases} \tilde{0}, \text{ if } \lambda = \tilde{0}, \\ \mu_2, \text{ if } \tilde{0} \neq \lambda \leq \mu_2, \\ \tilde{1} \quad \text{otherwise.} \end{cases}$$

Furthermore, we obtain fuzzy topological spaces from Theorem 2.6 as follows:

$\mathcal{T}_{C\delta_1} = \{ \tilde{0}, \tilde{1}, \tilde{1} - \mu_1 \}$, $\mathcal{T}_{C\delta_2} = \{ \tilde{0}, \tilde{1}, \tilde{1} - \mu_2 \}$.

We can obtain the coarsest C finer than C_{δ_1} and C_{δ_2} from Theorem 3.3 (ref.[10]) as follows:

$$C(\lambda) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0}, \\ \mu_1 \wedge \mu_2, & \text{if } \tilde{0} \neq \lambda \leq \mu_1 \wedge \mu_2, \\ \mu_1, & \text{if } \lambda \leq \mu_1,\ \lambda \nleq \mu_2, \\ \mu_2, & \text{if } \lambda \leq \mu_2,\ \lambda \nleq \mu_1, \\ \mu_1 \vee \mu_2, & \text{if } \lambda \leq \mu_1 \vee \mu_2,\ \lambda \nleq \mu_2,\ \lambda \nleq \mu_2, \\ \tilde{1} & \text{otherwise.} \end{cases}$$

Hence we easily show $C_\delta = C$. Furthermore, since $\mathcal{T}_C = \{ \tilde{0}, \tilde{1}, \tilde{1} - \mu_1, \tilde{1} - \mu_2, \tilde{1} - (\mu_1 \vee \mu_2), \tilde{1} - (\mu_1 \wedge \mu_2) \}$,

we have $\mathcal{T}_{C\delta} = \mathcal{T}_C$.

Example 2. Define a fuzzy quasi-proximity δ on X as follows:

$$(\lambda, \mu) \notin \delta \begin{cases} \text{if } \lambda = \tilde{0} \text{ or } \mu = \tilde{0}, \\ \text{if } \lambda \leq \tilde{1} - \mu_1,\ \mu \leq \mu_1. \end{cases}$$

Then we obtain a fuzzy quasi-proximity δ^{-1} on X as follows:

$$(\lambda, \mu) \notin \delta^{-1} \begin{cases} \text{if } \lambda = \tilde{0} \text{ or } \mu = \tilde{0}, \\ \text{if } \lambda \leq \mu_1,\ \mu \leq \tilde{1} - \mu_1. \end{cases}$$

We can obtain the coarsest fuzzy proximity δ^* finer than δ and δ^{-1} from Example 1 on putting $\mu_2 = \tilde{1} - \mu_1$ as follows:

$$(\lambda, \mu) \notin \delta \begin{cases} \text{if } \lambda = \tilde{0} \text{ or } \mu = \tilde{0}, \\ \text{if } \lambda \leq (\tilde{1} - \mu_1) \vee \mu_1,\ \mu \leq \mu_1 \wedge (\tilde{1} - \mu_1) \\ \text{if } \lambda \leq \tilde{1} - \mu_1,\ \mu \leq \mu_1, \\ \text{if } \lambda \leq \mu_1,\ \mu \leq \tilde{1} - \mu_1, \\ \text{if } \lambda \leq (\tilde{1} - \mu_1) \wedge \mu_1,\ \mu \leq (\tilde{1} - \mu_1) \vee \mu_1. \end{cases}$$

Similarly, we obtain C, $\mathcal{T}_C$, $C_{\delta^*} = C$ and $\mathcal{T}_{C\delta^*} = \mathcal{T}_C$.

Theorem 3.6 Let (X, δ_1) and (Y, δ_2) be fuzzy quasi-proximity spaces. If $f: (X, \delta_1) \to (Y, \delta_2)$ is a P-map, then:

(1) $f: (X, \delta_1^{-1}) \to (Y, \delta_2^{-1})$ is a P-map.

(2) $f: (X, \delta_1^*) \to (Y, \delta_2^*)$ is a P-map where each $i = 1, 2$, δ_i^* is defined as Theorem 3.5.

Proof. (1) It is proved from the followings:

$$\begin{aligned} \forall (\lambda, \mu) \in \delta_1^{-1} &\Rightarrow (\mu, \lambda) \in \delta_1 \\ &\Rightarrow (f(\mu), f(\lambda)) \in \delta_2 \\ &\qquad \text{(since } f \text{ is a P-map)} \\ &\Rightarrow (f(\lambda)), f(\mu)) \in \delta_2^{-1}. \end{aligned}$$

(2) If $(\lambda, \mu) \notin \delta_2^*$, there exist finite families $\{\lambda_j \mid \lambda = \vee_{j=1}^{p} \lambda_j\}$ and $\{\mu_k \mid \mu = \vee_{k=1}^{q} \mu_k\}$ satisfying the following condition: for any λ_j, μ_k, there exists $\delta_{jk} \in \{\delta_2, \delta_2^{-1}\}$ such that $(\lambda_j, \mu_k) \notin \delta_{jk}$.

Without loss of generality, we may assume that $(\lambda_j, \mu_k) \notin \delta_2^{-1}$ for a fixed j, k. Since $f: (X, \delta_1^{-1}) \to (Y, \delta_2^{-1})$ is a P-map, we have

$(f^{-1}(\lambda_j), f^{-1}(\mu_k)) \notin \delta_1^{-1}$.

Hence there exist finite families

$\{f^{-1}(\lambda_j) \mid f^{-1}(\lambda) = \vee_{j=1}^{p} f^{-1}(\lambda_j)\}$

and

$\{f^{-1}(\mu_k) \mid f^{-1}(\mu) = \vee_{k=1}^{q} f^{-1}(\mu_k)\}$

satisfying the following condition: for any $f^{-1}(\lambda_j)$, $f^{-1}(\mu_k)$, there exists $\delta_{jk} \in \{\delta_1, \delta_1^{-1}\}$ such that $(f^{-1}(\lambda_j), f^{-1}(\mu_k)) \notin \delta_{jk}$. Therefore

$$(f^{-1}(\lambda), f^{-1}(\mu)) \notin \delta_1^{*}. \qquad \square$$

From Theorem 3.2, we can define subspaces and products in the obvious way.

Definition 3.7 Let (X, δ) be a fuzzy quasi-proximityand A be a subset of X. The pair (A, δ_A) is said to be a *subspace* of (X, δ) if it is endowed with the initial fuzzy quasi-proximity structure with respect to the inclusion map.

Definition 3.8 Let X be the product $\Pi_{i \in \Gamma} X_i$ of the family $\{(X_i, \delta_i) \mid i \in \Gamma\}$ of fuzzy quasi-proximity spaces. An initial fuzzy quasi-proximity structure $\delta = \otimes \delta_i$ on X with respect to all the projections $\pi_i : X \to X_i$ is called the *product fuzzy quasi-proximity structure* of $\{\delta_i \mid i \in \Gamma\}$, and $(X, \otimes \delta_i)$ is called the *product fuzzy quasi-proximity space*.

Let $\{\delta_i \mid i \in \Gamma\}$ be a family of fuzzy quasi-proximities on X. From Theorem 3.2, there exists an initial fuzzy quasi-proximity structure δ on X with respect to all identity function $id_X : X \to (X, \delta_i)$. We obtain the following corollary to coincide with Theorem 4.1.3 of [9].

Corollary 3.9 Let $\{\delta_i \mid i \in \Gamma\}$ be a family of fuzzy quasi-proximities on X. Define a binary relation $\delta \subset I^X \times I^X$ on X by $(\lambda, \mu) \notin \delta$ iff there exist finite families $\{\lambda_j \mid \lambda = \vee_{j=1}^{p} \lambda_j\}$ and $\{\mu_k \mid \mu = \vee_{k=1}^{q} \mu_k\}$ satisfying the following condition: for any λ_j, μ_k, there exists $i_{jk} \in \Gamma$ such that $(\lambda_j, \mu_k) \notin \delta_{i_{jk}}$.

Then δ is the coarsest quasi-proximity on X for which each quasi-proximity is finer than δ_i.

Using Theorem 3.2 and Theorem 3.4, we have the following corollary.

Corollary 3.10 Let $\{(X_i, \delta_i) \mid i \in \Gamma\}$ be a family of fuzzy quasi-proximity spaces. Let $(X, \otimes \delta_i)$ bea product fuzzy quasi-proximity space. Then $C_{\otimes \delta_i} = \otimes C_{\delta_i}$.

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