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Decompositions of transformation semigroups

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ABSTRACT

We introduce the concepts of *TL*-finite state machines, *TL*-transformation semigroups and coverings, and several decompositions of transformation semigroups and investigate some of their algebraic structures.

1. Introduction

Since Wee[8] in 1967 introduced the concept of fuzzy automata following Zadeh [9], fuzzy automata theory has been developed by many researchers. Recently Malik *et al.* [4-6] introduced the concepts of fuzzy state machines and fuzzy transformation semigroups based on Wee's concept of fuzzy automata and related concepts and applied algebraic technique. In [2,3] Cho *et al.* introduced the notion of *T*-fuzzy state machine and *T*-fuzzy transformation semigroup that are extensions of fuzzy state machine and fuzzy transformation semigroup, respectively. In this paper, we introduce the concepts of *TL*-finite state machines and *TL*-transformation semigroups, coverings, restricted direct products and full direct products of *TL*-finite state machines and *TL*-transformation semigroups that are generalizations of crisp concepts in algebraic automata theory and investigate their algebraic structures.

For the terminology in (crisp) algebraic automata theory, we refer to [1].

2. *TL*-finite state machines and *TL*-transformation semigroups

We let L denote a complete lattice that contains at least two distinct elements. The meet, join, and partial ordering will be written as $\wedge$, $\vee$, and $\leq$, respectively. We also write 1 and 0 for the greatest element and least element of L, respectively.

Definition 2.1 A triple $\mathcal{M} = (Q, X, \tau)$ where Q and X are finite nonempty sets and τ is an L-subset of $Q\times X\times Q$, i.e., τ is a function from $Q\times X\times Q$ to L, is called an L-finite state machine.

Let $\mathcal{M} = (Q, X, \tau)$ be an L-finite state machine. Then Q is called the set of states and X is called the set of input symbols. Let X^+ denote the set of all words of elements of X of finite length with the empty word λ.

Definition 2.2 [7] A binary operation T on L is called a t-norm if

(1) $T(a, 1) = a$,
(2) $T(a, b) \leq T(a, c)$ whenever $b \leq c$,
(3) $T(a, b) = T(b, a)$,
(4) $T(a, T(b, c)) = T(T(a, b), c)$

for all $a, b, c \in L$.

From this definition one gets immediately $T(0, a) = 0$ and $T(a, b) \leq a\wedge b$ for all $a, b\in L$. A t-norm T on L is said to be $\vee$-distributive if $T(a, b\vee c) = T(a, b) \vee T(a, c)$ for all $a, b, c\in L$. And T is said to be positive-definite if $T(a, b)>0$ for all $a, b\in L\backslash\{0\}$.

Throughout this paper, T shall mean a positive-definite and $\vee$-distributive t-norm on L unless otherwise specified.

We will denote $T(a_1, T(a_2, \cdots, T(a_{n-2}, T(a_{n-1}, a_n))\cdots))$ by $T(a_1,\cdots, a_n)$ where $a_1, \cdots, a_n\in L$.

Example 2.3 Let $L=[0, 1]\times[0, 1]$. Define a partial order $\leq$ on L by for $a = (a_1, a_2)$, $b = (b_1, b_2)\in L$, $a\leq b$ if $a_1\leq b_1$ and $a_2\leq b_2$. Define $T(a, b) = (a_1b_1, a_2b_2)$ where $a = (a_1, a_2)$, $b = (b_1, b_2)\in L$. Then T is a positive-definite and $\vee$-distributive t-norm on L.

Definition 2.4 Let $\mathcal{M}=(Q, X, \tau)$ be an L-finite state machine. Define $\tau^+ : Q\times X^+\times Q \to L$ by

$$\tau^+(q, \lambda, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

$$\tau^+(p, a_1, \cdots, a_n, q) = \vee\{T(\tau(p, a_1, r_1), \tau(r_1, a_2, r_2), \cdots, \tau(r_{n-2}, a_{n-1}, r_{n-1}), \tau(r_{n-1}, a_n, q)) \mid r_i\in Q\}$$

where $p, q\in Q$ and $a_1, \cdots, a_n\in X$. When T is applied to $\mathcal{M}$ as above, $\mathcal{M}$ is called a TL-finite state machine(briefly, a TL-fsm).

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Remark In Definition 2.4 if we let $T=\wedge$ and $L=[0, 1]$, then the concept of a TL-fms is the concept of [5].

Proposition 2.5 Let (Q, X, τ) be a TL-fsm. Then

$$\tau^+(p, xy, q) = \vee\{T(\tau^+(p, x, r), \tau^+(r, y, q)) \mid r\in Q\}$$
for all $p, q\in Q$ and $x, y\in X^+$.

Proof Let $p, q\in Q$. Let $x = a_1\cdots a_n$ and $y = b_1\cdots b_m$ with $a_1, \cdots, a_n, b_1, \cdots, b_m\in X$. Then

$$\begin{aligned}
&\vee\{T(\tau^+(p, x, r), \tau^+(r, y, q)) \mid r\in Q\}\\
&= \vee\{T(\tau^+(p, a_1\cdots a_n, r), \tau^+(r, b_1\cdots b_m, q)) \mid r\in Q\}\\
&= \vee\{T(\vee\{T(\tau(p, a_1, q_1), \cdots, \tau(q_{n-1}, a_n, r)) \mid q_1, \cdots, q_{n-1}\in Q\},\\
&\quad \vee\{T(\tau(r, b_1, q_n), \cdots, \tau(q_{n+m-1}, b_m, q)) \mid q_n, \cdots q_{n+m-1}\in Q\}) \mid r\in Q\} \quad \text{by Definition 2.4}\\
&= \vee\{T(\tau(p, a_1, q_1), \cdots, \tau(q_{n-1}, a_n, r), \tau(r, b_1, q_n), \cdots, \tau(r, b_1, q_n), \cdots, \tau(q_{n+m-1}, b_m, q)) \mid q_1, \cdots q_{n+m-1}, r\in Q\}\\
&= \tau^+(p, a_1\cdots a_n b_1\cdots b_m, q) \quad \text{by Definition 2.4}\\
&= \tau^+(p, xy, q).
\end{aligned}$$

For a TL-fsm, let $\equiv$ be a relation on X^+ defined by $x\equiv y$ if $\tau^+(p, x, q) = \tau^+(p, y, q)$ for all $p, q\in Q$.

Lemma 2.6 Let (Q, X, τ) be a TL-fsm. Then $\equiv$ is a congruence relation on X^+.

Proof Clearly $\equiv$ is an equivalence relation on X^+. Let $z\in X^+$ and $x\equiv y$. Then for all $p, q\in Q$,

$$\begin{aligned}
\tau^+(p, xz, q) &= \vee\{T(\tau^+(p, x, r), \tau^+(r, z, q)) \mid r\in Q\}\\
&= \vee\{T(\tau^+(p, y, r), \tau^+(r, z, q)) \mid r\in Q\}\\
&= \tau^+(p, yz, q)
\end{aligned}$$

by Proposition 2.5. So $xz\equiv yz$. Similarly $zx\equiv zy$. Thus $\equiv$ is a congruence relation on X^+.

Given a TL-fsm $\mathcal{M}=(Q, X, \tau)$, we will write $\{y\in X^+ \mid x\equiv y\}$ by $[x]$ where $x\in X^+$ and $X^+/\equiv\ =\{[x] \mid x\in X^+\}$ by $S(\mathcal{M})$.

Theorem 2.7 Let $\mathcal{M}=(Q, X, \tau)$ be a TL-fsm. Then $S(\mathcal{M})$ is a semigroup, where the binary operation on $S(\mathcal{M})$ is defined by $[x][y]=[xy]$.

Proof Clearly the operation is well-defined because $\equiv$ is a congruence relation by Lemma 2.6, and is associative. So $S(\mathcal{M})$ is a semigroup.

Remark In general $S(\mathcal{M})$ is not finite in Theorem 2.7. But if we let $T=\wedge$ and $L=[0, 1]$, then $S(\mathcal{M})$ is always finite.

Definition 2.8 A TL-fsm (Q, S, ρ) is called a TL-transformation semigroup if S is a semigroup and if it satisfies the following:

(i) $\rho(p, uv, q)=\vee\{T(\rho(p, u, r), \rho(r, v, q)) \mid r\in Q\}$ for all $p, q\in Q$ and $u, v\in S$.

(ii) For $u, v\in S$, if $\rho(p,u,q)=\rho(p,v,q)$ for all $p, q\in Q$, then $u = v$.

When a TL-transformation semigroup $\mathcal{S}=(Q, S, \rho)$ is regarded as a TL-fsm (Q, S, τ_ρ) by taking $\tau_\rho = \tau_\rho^+ = \rho$, we will write it by $SM(\mathcal{S})$.

Proposition 2.5 and Theorem 2.7 seem to suggest that a TL-fsm $\mathcal{M}=(Q, X, \tau)$ naturally induces a TL-transformation semigroup $(Q, S(\mathcal{M}), \rho_\tau)$ where ρ_τ is defined by $\rho_\tau(p, [x], q) = \tau^+(p, x, q)$. We call $(Q, S(\mathcal{M}), \rho_\tau)$ by the TL-transformation semigroup induced by $\mathcal{M}$ and denote it by $TS(\mathcal{M})$.

3. Coverings

Definition 3.1 Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 =(Q_2, X_2, \tau_2)$ be TL-finite state machines. If $\xi: X_1\to X_2$ is a function and $\eta: Q\to Q_1$ is a surjective partial function such that $\tau_1^+(\eta(p), x, \eta(q))\le \tau_2^+(p, \xi(x), q)$ for all p, q in the domain of η and $x\in X_1^+$, then we say that (η, ξ) is a covering of $\mathcal{M}_1$ by $\mathcal{M}_2$ and that $\mathcal{M}_2$ covers $\mathcal{M}_1$ and denote by $\mathcal{M}_1\le\mathcal{M}_2$. Moreover, if the inequality always turns out equality, then we say that (η, ξ) is a complete covering of $\mathcal{M}_1$ by $\mathcal{M}_2$ and that $\mathcal{M}_2$ completely covers $\mathcal{M}_1$ and denote by $\mathcal{M}_1\le_c\mathcal{M}_2$.

We will write the natural semigroup homomorphism from X_1^+ to X_2^+ induced by ξ by ξ for convenience sake.

Example 3.2 Let $\mathcal{M}=(Q, X, \tau)$ be a TL-fsm. Define an equivalence relation $\sim$ on X by $a\sim b$ if and only if $\tau(p, a, q) = \tau(p, b, q)$ for all $p, q\in Q$. Construct a TL-fsm $\mathcal{M}_1 = (Q, X/\sim, \tau^\sim)$ by defining $\tau^\sim(p, [a], q) = \tau(p, a, q)$. Now define $\xi: X\to X/\sim$ by $\xi(a) = [a]$ and $\eta = 1_Q$. Then (η, ξ) is a complete covering of $\mathcal{M}$ by $\mathcal{M}_1$ clearly.

Definition 3.3 Let $\mathcal{S}_1=(Q_1, S_1, \rho_1)$ and $\mathcal{S}_2=(Q_2, S_2, \rho_2)$ be TL-transformation semigroups. If $\eta: Q_2\le Q_1$ is a surjective partial function and for each $s\in S_1$ there exists $t_s\in S_2$ such that $\rho_1(\eta(p), s, \eta(q))\le\rho_2(p, t_s, q)$ for all p, q in the domain of η, then we say that η is a covering of $\mathcal{S}_1$ by $\mathcal{S}_2$ and that $\mathcal{S}_2$ covers $\mathcal{S}_1$ and denote by $\mathcal{S}_1\le\mathcal{S}_2$. Moerover, if the inequality always turns out equality then we say that η is a complete covering of $\mathcal{S}_1$ by $\mathcal{S}_2$ and that $\mathcal{S}_2$ completely covers $\mathcal{S}_1$ and denote by $\mathcal{S}_1\le_c\mathcal{S}_2$.

Proposition 3.4 (1) Let $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be TL-

finite state machines. If $\mathcal{M}_1 \leq \mathcal{M}_2$ [resp. $\mathcal{M}_1 \leq_c \mathcal{M}_2$] and $\mathcal{M}_2 \leq \mathcal{M}_3$ [resp. $\mathcal{M}_2 \leq \mathcal{M}_3$], then $\mathcal{M}_1 \leq \mathcal{M}_3$ [resp. $\mathcal{M}_1 \leq \mathcal{M}_3$].

(2) Let $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ be TL-transformation semigroups. If $\mathcal{S}_1 \leq \mathcal{S}_2$ [resp. $\mathcal{S}_1 \leq_c \mathcal{S}_2$] and $\mathcal{S}_2 \leq \mathcal{S}_3$ [resp. $\mathcal{S}_2 \leq_c \mathcal{S}_3$], then $\mathcal{S}_1 \leq \mathcal{S}_3$[resp. $\mathcal{S}_1 \leq_c \mathcal{S}_3$].

Proof It is trivial.

Theorem 3.5 Let $\mathcal{M}_1=(Q_1, X_1, \tau_1)$ and $\mathcal{M}_2=(Q_2, X_2, \tau_2)$ be TL-finite state machines such that $\mathcal{M}_1 \leq \mathcal{M}_2$ with covering (η, ξ). Then $TS(\mathcal{M}_1) \leq TS(\mathcal{M}_2)$. Moreover, if $\mathcal{M}_1 \leq_c \mathcal{M}_2$ and η is a function, then $TS(\mathcal{M}_1) \leq_c TS(\mathcal{M}_2)$.

Proof Let $a_1, \cdots, a_n \in X_1$. Then we have

$$\begin{aligned}
&\rho_{\tau_1}(\eta(p), [a_1, \cdots, a_n], \eta(p)) \\
&= \tau_1^+(\eta(p), a_1, \cdots, a_n, \eta(p)) \\
&= \vee\{T(\tau_1(\eta(p), a_1, r_1'), \tau_1(r_1', a_2, r_2'), \cdots, \tau_1(r'_{n-1}, a, \eta(p))) \mid r_i' \in Q_1\} \\
&= \vee\{T(\tau_1(\eta(p), a_1, \eta(r_1)), \tau_1(\eta(r_1), a_2, \eta(r_2)), \cdots, \tau_1(\eta(r_{n-1}), a_n, \eta(q)) \mid r_i \in Q\}
\end{aligned}$$

because η is surjective (Q denotes the domain of η)}

$$\begin{aligned}
&\leq \vee\{T(\tau_2(p, \xi(a_1), r_1), \tau_2(r_1, \xi(a_2), r_2), \cdots, \tau_2(r_{n-1}, \xi(a_n), q)) \mid r_i \in Q\} \\
&\leq \vee\{T(\tau_2(p, \xi(a_1), r_1), \tau_2(r_1, \xi(a_2), r_2), \cdots, \tau_2(r_{n-1}, \xi(a_n), q)) \mid r_i \in Q_2\} \\
&= \tau_2^+(p, \xi(a_1) \cdots \xi(a_n), q) \\
&= \tau_2^+(p, \xi(a_1 \cdots a_n), q) \\
&= \rho_{\tau_2}(p, [\xi(a_1 \cdots a_n)], q)
\end{aligned}$$

for all p, q in the domain of η. Hence η is a covering of $TS(\mathcal{M}_1)$ by $TS(\mathcal{M}_2)$. Now let $\mathcal{M}_1 \leq_c \mathcal{M}_2$ and η a function. Then the first inequality in the first part of the proof turns out equality because $\mathcal{M}_1 \leq_c \mathcal{M}_2$. And the second inequality in the first part of the proof turns out equality because the domain of η is Q_2. This completes the proof.

4. Products

In this section, we consider restricted direct products and full directed products of TL-finite state machines and TL-transformation semigroups.

Definition 4.1 Let $\mathcal{M}_1 = (Q_1, X, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X, \tau_2)$ be TL-finite state machines. The restricted direct product $\mathcal{M}_1 \wedge_T \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the TL-fsm $(Q_1 \times Q_2$, X, $\tau_1 \wedge_T \tau_2)$ with

$$(\tau_1 \wedge_T \tau_2)((p_1, p_2), a, (q_1, q_2)) = T(\tau_1(p_1, a, q_1), \tau_2(p_2, a, q_2)).$$

Clearly $(Q_1 \times Q_2, X, \tau_1 \wedge_T \tau_2)$ is a TL-fsm.

Lemma 4.2 Let $\mathcal{M}_1 = (Q_1, X, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X, \tau_2)$ be TL-finite state machines. Then for $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$ and $x \in X^+$,

$$(\tau_1 \wedge_T \tau_2)^+((p_1, p_2), x, (q_1, q_2)) = T(\tau_1^+(p_1, x, q_1), \tau_2^+(p_2, x, q_2))$$

Proof Let $x = a_1 a_2 \cdots a_n$ where $a_1, a_2, \cdots, a_n \in X$. Then we have

$$\begin{aligned}
&(\tau_1 \wedge_T \tau_2)^+((p_1, p_2), x, (q_1, q_2)) \\
&= (\tau_1 \wedge_T \tau_2)^+((p_1, p_2), a_1 \cdots a_n, (q_1, q_2)) \\
&= \vee\{T((\tau_1 \wedge_T \tau_2)((p_1, p_2), a_1, (r_{11}, r_{12})), (\tau_1 \wedge_T \tau_2)((r_{11}, r_{12}), a_2, (r_{21}, r_{22})), \cdots, (\tau_1 \wedge_T \tau_2)((r_{(n-1)1}, r_{(n-1)2}), a_n, (q_1, q_2)) \mid (r_{i1}, r_{i2}) \in Q_1 \times Q_2\} \\
&= \vee\{T(\tau_1(p_1, a_1, r_{11}), \tau_2(p_2, a_1, r_{12})), T(\tau_1(r_{11}, a_2, r_{21}), \tau_2(r_{12}, a_2, r_{22})), \cdots, T(\tau_1(r_{(n-1)1}, a_n, q_1), \tau_2(r_{(n-1)2}, a_n, q_2)) \mid r_{i1} \in Q_1, r_{i2} \in Q_2\} \\
&= T(\vee\{T(\tau_1(p_1, a_1, r_{11}), \tau_1(r_{11}, a_2, r_{21}), \cdots, \tau_1(r_{(n-1)1}, a_n, q_1)) \mid r_{i1} \in Q_1\}, \vee\{T(\tau_2(p_2, a_1, r_{12}), \tau_1(r_{12}, a_2, r_{22}), \cdots, \tau_2(r_{(n-1)2}, a_n, q_2)) \mid r_{i2} \in Q_1) \\
&= T(\tau_1^+(p_1, a_1 \cdots a_n, q_1), \tau_2^+(p_2, a_1 \cdots a_n, q_2)) \\
&= T(\tau_1^+(p_1, x, q_1), \tau_2^+(p_2, x, q_2))
\end{aligned}$$

for all $p_1, q_1 \in Q_1$ and $p_2, q_2 \in Q_2$.

Definition 4.3 Let $\mathcal{S}_1=(Q_1, S_1, \rho_1)$ and $\mathcal{S}_2=(Q_2, S_2, \rho_2)$ be TL-transformation semigroups such that there exists a free semigroup F with epimorphisms $\theta_1 : F \to S_1$ and $\theta_2 : F \to S_2$. The restricted direct product $\mathcal{S}_1 \wedge_T \mathcal{S}_2$ of $\mathcal{S}_1$ and $\mathcal{S}_2$(with respect to θ_1 and θ_2) is the TL-transformation semigroup $(Q_1 \times Q_2, S, \rho_1 \wedge_T \rho_2)$ with $S = F/(R_1 \cap R_2)$ where R_1 and R_2 are the equivalence relations on F defined by θ_1 and θ_2, respectively, and $(\rho_1 \wedge_T \rho_2)((p_1, p_2), [x], (q_1, q_2)) = T(\rho_1(p_1, [x]_{R_1}, q_1), \rho_2(p_2, [x]_{R_2}, q_2))$.

Theorem 4.4 Let $\mathcal{M}_1=(Q_1, X, \tau_1)$ and $\mathcal{M}_2=(Q_2, X, \tau_2)$ be TL-finite state machines. Then $TS(\mathcal{M}_1 \wedge_T \mathcal{M}_2) = TS(\mathcal{M}_1) \wedge_T TS(\mathcal{M}_2)$.

Proof Let $TS(\mathcal{M}_1)=(Q_1, X^+/R_1, \rho_1)$, $TS(\mathcal{M}_2)=(Q_2, X^+/R_2, \rho_2)$ and $TS(\mathcal{M}_1 \wedge_T \mathcal{M}_2)=(Q_1 \times Q_2, X^+/R, \rho)$. Let $a_1, \cdots, a_n \in X_1$. Then we have

$$\begin{aligned}
&\rho((p_1, p_2), [a_1 \cdots a_n]_R, (q_1, q_2)) \\
&= (\tau_1 \wedge_T \tau_2)^+((p_1, p_2), a_1 \cdots a_n, (q_1, q_2)) \\
&= T(\tau_1^+(p_1, a_1 \cdots a_n, q_1), \tau_2^+(p_2, a_1 \cdots a_n, q_2)) \quad \text{by Lemma 4.2} \\
&= T(\rho_1(p_1, [a_1 \cdots a_n]_{R_1}, q_1), \rho_2(p_2, [a_1 \cdots a_n]_{R_2}, q_2)) \\
&= (\rho_1 \wedge_T \rho_2)((p_1, p_2), [a_1 \cdots a_n]_R, (q_1, q_2))
\end{aligned}$$

for all $p_1, q_1 \in Q_1$ and $p_2, q_2 \in Q_2$.

Definition 4.5 Let $\mathcal{M}_1=(Q_1, X, \tau_1)$ and $\mathcal{M}_2=(Q_2, X, \tau_2)$ be TL-finite state machines. The full direct product $\mathcal{M}_1 \times_T \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the TL-fsm $(Q_1 \times Q_2, X_1 \times X_2, \tau_1 \times_T \tau_2)$ with $(\tau_1 \times_T \tau_2)((p_1, p_2), (a, b), (q_1, q_2)) = T(\tau_1(p_1, a, q_1), \tau_2(p_2, b, q_2))$.

Clearly $(Q_1 \times Q_2, X_1 \times X_2, \tau_1 \times_T \tau_2)$ is a TL-fsm.

Lemma 4.6 Let $\mathcal{M}_1=(Q_1, X_1, \tau_1)$ and $\mathcal{M}_2=(Q_2, X_2, \tau_2)$ be TL-finite state machines. Then $(\tau_1 \times_T \tau_2)^+((p_1, p_2), (a_1 \cdots a_n, b_1 \cdots b_n), (q_1, q_2)) = T(\tau_1^+(p_1, a_1 \cdots a_n, q_1), \tau_2^+(p_2, b_1 \cdots b_n, q_2))$ for all $a_1, \cdots, a_n \in X_1$, $b_1, \cdots b_n \in X_2$, $p_1, q_1 \in Q_1$ and $p_2, q_2 \in Q_2$.

Proof Let $a_1, \cdots, a_n \in X_1$ and $b_1, \cdots b_n \in X_2$. Then we have

$$\begin{aligned}
&(\tau_1 \times_T \tau_2)^+((p_1, p_2),(a_1 \cdots a_n, b_1 \cdots b_n), (q_1, q_2)) \\
&= \vee \{T((\tau_1 \times_T \tau_2)((p_1, p_2), (a_1, b_1), (r_{11}, r_{12})), (\tau_1 \times_T \tau_2) \\
&\quad ((r_{11}, r_{12}), (a_2, b_2), (r_{21}, r_{22})), \cdots, (\tau_1 \times_T \tau_2)((r_{(n-1)1}, r_{(n-1)2}), \\
&\quad (a_n, b_n), (q_1, q_2))) \mid (r_{i1}, r_{i2}) \in Q_1 \times Q_2\} \\
&= \vee \{T(T(\tau_1(p_1, a_1, r_{11}), \tau_2(p_2, b_1, r_{12})), T(\tau_1(r_{11}, a_2, r_{22}), \\
&\quad \tau_2(r_{12}, b_2, r_{22})), \cdots, T(\tau_1(r_{(n-1)1}, a_n, q_1), \tau_2((r_{(n-1)2}, b_n, \\
&\quad q_2)) \mid r_{i1} \in Q_1, r_{i2} \in Q_2\} \\
&= T(\vee \{T(\tau_1(p_1, a_1, r_{11}), \tau_1(r_{11}, a_2, r_{21}), \cdots, \tau_1(r_{(n-1)1}, a_n, \\
&\quad q_1)) \mid r_{i1} \in Q_1\}, \vee \{T(\tau_2(p_2, b_1, r_{12}), \tau_2(r_{12}, b_2, r_{22}), \cdots, \\
&\quad \tau_2(r_{(n-1)2}, b_n, q_2)) \mid r_{i2} \in Q_2\} \\
&= T(\tau_1^+(p_1, a_1 \cdots a_n, q_1), \tau_2^+(p_2, b_1 \cdots b_n, q_2))
\end{aligned}$$

for all $p_1, q_1 \in Q_1$ and $p_2, q_2 \in Q_2$.

Definition 4.7 Let $\mathcal{S}_1=(Q_1, S_1, \rho_1)$ and $\mathcal{S}_2=(Q_2, S_2, \rho_2)$ be TL-transformation semigroups. The full direct product $\mathcal{S}_1 \times_T \mathcal{S}_2$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ is the TL-transformation semigroup $(Q_1 \times Q_2, S_1 \times S_2, \tau_1 \times_T r_2)$ with $(\rho_1 \times_T \rho_2)((p_1, p_2), (u, v), (q_1, q_2))=T(\rho_1(p_1, u, q_1), \rho_2(p_2, v, q_2))$.

Theroem 4.8 Let $\mathcal{M}_1=(Q_1, X_1, \tau_1)$ and $\mathcal{M}_2=(Q_2, X_2, \tau_2)$ be TL-finite state machines. Then $TS(\mathcal{M}_1 \times_T \mathcal{M}_2) \leq_c TS(\mathcal{M}_1) \times_T TS(\mathcal{M}_2)$.

Proof Let $TS(\mathcal{M}_1)=(Q_1, X_1^+/R_1, \rho_1)$, $TS(\mathcal{M}_2)=(Q_2, X_2^+/R_2, \rho_2)$ and $TS(\mathcal{M}_1 \times_T \mathcal{M}_2)=(Q_1 \times Q_2, (X_1 \times_T X_2)+/R_3, \rho_3)$. Let $a_1, \cdots, a_n \in X_1$ and $b_1 \cdots, b_n \in X_2$. Then we have

$$\begin{aligned}
&\rho_3((p_1, p_2), [(a_1 \cdots a_n, b_1 \cdots b_n)]_{R_3}, (q_1, q_2)) \\
&= (\tau_1 \times_T \tau_2)^+((p_1, p_2),(a_1 \cdots a_n, b_1 \cdots b_n), (q_1, q_2)) \\
&= T(\tau_1^+(p_1, a_1 \cdots a_n, q_1), \tau_2^+(p_2, b_1 \cdots b_n, q_2)) \\
&\qquad \text{by Lemma 4.6} \\
&= T(\rho_1(p_1, [a_1 \cdots a_n]_{R1}, q_1), \rho_2(p_2,[b_1 \cdots b_n]_{R2}, q_2)) \\
&= (\rho_1 \times_T \rho_2)((p_1, p_2), ([a_1 \cdots a_n]_{R_1}, [b_1 \cdots b_n]_{R2}), (q_1, q_2))
\end{aligned}$$

for all $p_1, q_1 \in Q_1$ and $p_2, q_2 \in Q_2$.

Proposition 4.9 Let $\mathcal{M}_1=(Q_1, X, \tau_1)$ and $\mathcal{M}_2=(Q_2, X, \tau_2)$ be T-fuzzy state machines. Then the following hold:

(1) $\mathcal{M}_1 \wedge_T \mathcal{M}_2 \leq_c \mathcal{M}_1 \times_T \mathcal{M}_2$.

(2) $TS(\mathcal{M}_1 \wedge_T \mathcal{M}_2)_c\ TS(\mathcal{M}_1 \times_T \mathcal{M}_2)$.

Proof (1) Let $\eta = 1_{Q_1 \times Q_1}$ and define $\xi : X \to X \times X$ by $\xi(a)=(a, a)$. Then (η, ξ) is a complete covering of $\mathcal{M}_1 \wedge_T \mathcal{M}_2$ by $\mathcal{M}_1 \times_T \mathcal{M}_2$ clearly.

(2) It is clear from (1) and Theorem 3.5.

The following propositions are direct consequences of the associativity of t-norm of T.

Proposition 4.10 Let $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be TL-finite state machines.

(1) $(\mathcal{M}_1 \wedge_T \mathcal{M}_2) \wedge_T \mathcal{M}_3 = \mathcal{M}_1 \wedge_T (\mathcal{M}_2 \wedge_T \mathcal{M}_3)$.

(2) $(\mathcal{M}_1 \times_T \mathcal{M}_2) \times_T \mathcal{M}_3 = \mathcal{M}_1 \times_T (\mathcal{M}_2 \times_T \mathcal{M}_3)$.

Proposition 4.11 Let $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ be TL-transformation semigroups. Then the following hold:

(1) $(\mathcal{S}_1 \wedge_T \mathcal{S}_2) \wedge_T \mathcal{S}_3 = \mathcal{S}_1 \wedge_T (\mathcal{S}_2 \wedge_T \mathcal{S}_3)$.

(2) $(\mathcal{S}_1 \times_T \mathcal{S}_2) \times_T \mathcal{S}_3 = \mathcal{S}_1 \times_T (\mathcal{S}_2 \times_T \mathcal{S}_3)$.

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