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Default Bayes Factors for Testing the Equality of Poisson Population Means

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Abstract

Default Bayes factors are computed to test the equality of one Poisson population mean and the equality of two independent Possion population means. As default priors are assumed Jeffreys priors, noninformative improper priors, and default Bayes factors such as three intrinsic Bayes factors of Berger and Pericchi(1996, 1998), the arithmetic, the median, and the geometric intrinsic Bayes factor, and the fractional Bayes factor of O'Hagan(1995) are computed. The testing results by each default Bayes factor are compared with those by the classical method in the simulation study.

Keywords : Noninformative improper prior, Default Bayes factors, Testing on the equality of independent Poisson population means

1. Introduction

In quality control the Poisson(μ) distribution

$$f(x \mid \mu) = \frac{\mu^x e^{-\mu}}{x!}, \qquad x=0,1,2,\cdots$$

is typically used as a probability model of the number of defects or nonconformaties occurred per unit of a product, where μ is a positive parameter called the mean occurrence rate of defects.

A test on the equality of one Poisson population mean is to test the hypothesis

$$H_0 : \mu = \mu_0 \qquad \text{versus} \qquad H_1 : \mu \neq \mu_0, \tag{1.1}$$

where μ_0 is a fixed positive constant. Let $X_1, X_2, \cdots, X_n$ be a random sample from a

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Poisson(μ) distribution to test the hypothesis (1.1). Then $Y = \sum_{i=1}^{n} X_i$ is a sufficient statistic for μ with Poisson($n\mu$). The classical method to test H_0 versus H_1 has two decision rules according to the size of μ_0. When $\mu_0 < 5$, the test rule with the significance level α is to

$$\text{reject } H_0 \text{ if } \sum_{x=0}^{y} f(x \mid n\mu_0) \le \frac{\alpha}{2} \quad \text{or} \quad \sum_{x=y}^{\infty} f(x \mid n\mu_0) \le \frac{\alpha}{2}.$$

The p-value of the computed value of test statistic is $2 \cdot \sum_{x=0}^{y} f(x \mid n\mu_0)$ if $\sum_{x=0}^{y} f(x \mid n\mu_0) < 0.5$ and $2 \cdot \sum_{x=y}^{\infty} f(x \mid n\mu_0)$ otherwise. When $\mu_0 \geqq 5$, the test rule by normal approximation is to

$$\text{reject } H_0 \text{ if } |Z_0| = \left| \frac{Y - n\mu_0}{\sqrt{n\mu_0}} \right| \le z_{\alpha/2},$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution. The p-value of z_0, the computed value of Z_0, is $2[1-\Phi(|z_0|)]$, where $\Phi(\cdot)$ is the standard normal cumulative distribution function.

A test on the equality of two Poisson population means is to test the hypothesis

$$H_0 : \mu_1 = \mu_2 = \mu \quad \text{versus} \quad H_1 : \mu_1 \neq \mu_2, \tag{1.2}$$

where μ_i is a mean parameter of the Poisson population i and μ is an unknown common mean parameter. To test the hypothesis (1.2) a random sample $\{X_{ij}\}$, $j = 1, 2, \ldots, n_i$, is sampled from each Poisson (μ_i), $i = 1, 2$, and the chi-square test statistic

$$\chi_0^2 = \frac{\sum_{i=1}^{2} \left(\sum_{j=1}^{n_i} X_{ij} - \overline{X} \right)^2}{\overline{X}},$$

where $\overline{X} = (1/2) \sum_{i=1}^{2} \sum_{j=1}^{n_i} X_{ij}$, is used. The test rule with the significance level α is to

$$\text{reject } H_0 \text{ if } \chi_0^2 > \chi_{1,\alpha}^2,$$

where $\chi_{1,\alpha}^2$ is the upper α percentage point of the chi-square distribution with 1 degree of freedom. The p-value of χ_0^2 is $1 - F_1(\chi_0^2)$, where $F_1(\cdot)$ is the chi-square cumulative distribution function with 1 degrees of freedom.

By now, we review the classical approach to test the equality of Poisson population means. In this paper we are interested in Bayesian solution on the same subject. We wish to test the equality of Poisson population means with only the least prior information, i.e. noninformative improper prior for the parameters. So we use as Bayesian test tools the default Bayes factors such as the intrinsic Bayes factor of Berger and Pericchi(1996, 1998) and the fractional Bayes factor of O'Hagan(1995).

In section 2, default Bayes factors are introduced. In section 3 and section 4 they are computed to test the equality of one Poisson population mean and the equality of two Poisson population means, respectively. In the last section, the results of the classical test summarized in section 1 and Bayesian test obtained in section 3 and section 4 are applied to simulated data.

2. Default Bayes Factors

Suppose that we wish to test the following hypothesis

$$H_0 : \mathbf{X}=\{ X_1, X_2, \ldots, X_n\} \sim f(x \mid \theta_0), \quad \theta_0 \in \Theta_0,$$
$$H_1 : \mathbf{X}=\{ X_1, X_2, \ldots, X_n\} \sim f(x \mid \theta_1), \quad \theta_1 \in \Theta_1.$$

There are the Bayes Factor and the posterior probability of hypothesis or model as tools for Bayesian testing or Bayesian model selection.

The Bayes factor B_{10} to test the hypothesis H_0 versus H_1 is defined by $B_{10}(\mathbf{x} \mid 1)$, where

$$B_{10}(\mathbf{x} \mid b) = \frac{m_1(\mathbf{x} \mid b)}{m_0(\mathbf{x} \mid b)}, \tag{2.1}$$

and

$$m_i(\mathbf{x} \mid b) = \int_{\Theta_i} \pi_i(\theta_i)\, l^b(\theta_i \mid \mathbf{x})\, d\theta_i$$

with a likelihood function $l(\theta_i \mid \mathbf{x}) = \Pi_{j=1}^{n} f(x_j \mid \theta_i)$, a fraction b, $b < 0 < 1$, of likelihood function and a prior distribution $\pi(\theta_i)$ of parameter θ_i under the hypothesis H_i, $(i=0,1)$. Here $m_i(\mathbf{x} \mid 1)$, $i=0,1$, is called a marginal or a predictive density of the hypothesis H_i, $i=0,1$.

Under the assumption of equal prior probability of each hypothesis being true the posterior probability of H_1 being true is

$$P(H_1|\mathbf{x}) = \frac{B_{10}}{1+B_{10}}, \tag{2.2}$$

where B_{10} is a Bayes factor, $B_{10}(x|1)$, defined in (2.1).

Bayesian decision rule for testing is to

$$\text{reject} \quad H_0 \quad \text{if} \quad B_{10}(\mathbf{x}|1) > 1 \quad \text{or} \quad P(H_1|\mathbf{x}) > 0.5.$$

There are a group of Bayesians who use default priors, most of which are typically improper. Noninformative improper priors are objective priors that need not any subjective consideration. But there is an inevitable obstacle in computing the Bayes factor using the noninformative improper prior $\pi_i^N(\theta_i)$ since $\pi_i^N(\theta_i)$ is defined only up to arbitrary constant c_i. Hence

$$B_{10}^N(\mathbf{x}|1) = \frac{m_1^N(\mathbf{x}|1)}{m_0^N(\mathbf{x}|1)} = \frac{\int_{\Theta_1} \pi_1^N(\theta_1) l_1(\theta_1|\mathbf{x})\, d\theta_1}{\int_{\Theta_0} \pi_0^N(\theta_0) l_0(\theta_0|\mathbf{x})\, d\theta_0}, \tag{2.3}$$

is defined only up to arbitrary constant c_1/c_0.

The intrinsic Bayes factor (IBF) of Berger and Pericchi(1996) and the fractional Bayes factor (FBF) of O'Hagan(1995) are objective and automatic priors. The idea of intrinsic Bayes factor is to use the minimal training sample $\mathbf{x}(l)$, the part of full sample, to convert the improper prior $\pi_i^N(\theta_i)$ to the proper posterior density. A training sample, $\mathbf{x}(l)$, is called a minimal training sample if it has the minimal sample size to guarantee $0 < m_i^N(\mathbf{x}|1) < \infty$ for all H_i. The result is

$$B_{10}(l) = B_{10}^N(\mathbf{x}|1) \cdot B_{01}^N(\mathbf{x}(l)|1),$$

where

$$B_{01}^N(\mathbf{x}(l)|1) = \frac{m_0^N(\mathbf{x}(l)|1)}{m_1^N(\mathbf{x}(l)|1)}, \tag{2.4}$$

and

$$m_i^N(\mathbf{x}(l)|1) = \int \pi_i^N(\theta_i)\, l_i(\theta_i|\mathbf{x}(l))\, d\theta_i.$$

Clearly, c_1/c_0 in $B_{10}^N(\mathbf{x}|1)$ and c_1/c_0 in $B_{01}^N(\mathbf{x}(l)|1)$ are cancelled by the multiplication.

Now the intrinsic Bayes factor B_{10} is defined by

$$B^{I}_{10} = E_l[\, B_{10}(l)] = B^{N}_{10}(\, \boldsymbol{x}\,|1) \cdot \; E_l[\, B^{N}_{01}(\, \boldsymbol{x}(l)|1)].$$

But there is practically a difficulty in obtaining the expectation of $B^{N}_{01}(\boldsymbol{x}(l)|1)$ over l. Hence instead of its expectation can be used a sample mean, a sample median, or a sample geometric mean. Thus an arithmetic IBF(AIBF), B^{AI}_{10}, a median IBF(MIBF), B^{MI}_{10}, and a geometric IBF(GIBF), B^{GI}_{10}, of Berger and Pericchi (1996, 1998) are defined as follows

$$B^{AI}_{10} = B^{N}_{10}(\, \boldsymbol{x}\,|1) \cdot \frac{1}{L} \sum_{l=1}^{L} B^{N}_{01}(\, \boldsymbol{x}(l)|1), \tag{2.5}$$

$$B^{MI}_{10} = B^{N}_{10}(\, \boldsymbol{x}\,|1) \cdot \underset{1 \le l \le L}{Median} \left\{ B^{N}_{01}(\, \boldsymbol{x}(l)|1) \right\}, \tag{2.6}$$

$$B^{GI}_{10} = B^{N}_{10}(\, \boldsymbol{x}\,|1) \cdot \{ \Pi^{L}_{l=1} B^{N}_{01}(\, \boldsymbol{x}(l)|1) \}^{1/L} \;, \tag{2.7}$$

where N implies the use of noninformative improper prior, $\boldsymbol{x}(l)$ is the $l-th$ minimal training sample, and L is the frequency of minimal training sample possible in the sample.

Finally the FBF of O'Hagan (1995) is defined by

$$B^{F}_{10} = B^{N}_{10}(\, \boldsymbol{x}\,|1) \cdot B^{N}_{01}(\, \boldsymbol{x}\,|\,b) \;,$$

where the fraction b of likelihood is usually used as $b = m/n$ with the size m of a minimal training sample. Similarily in B^{F}_{10}, c_1/c_0 in $B^{N}_{10}(\boldsymbol{x}|1)$ and c_0/c_1 in $B^{N}_{01}(\boldsymbol{x}|b)$ are cancelled by the multiplication

3. Default Bayes Factors for Testing the Equality of One Poisson population Mean.

Consider the following hypothesis on the equality of one Poisson population mean,

$$H_0 : \mu = \mu_0 \qquad \text{versus} \qquad H_1 : \mu \neq \mu_0$$

The prior $\pi_0(\mu) = \mathbf{1}_{A_0}(\mu)$ for H_0 and Jeffreys prior, the noninformative improper prior, $\pi_1(\mu) = \mu^{-1/2}\, \mathbf{1}_{A_1}(\mu)$ for H_1 are assumed, where $\mathbf{1}(\cdot)$ is an indicator function, $A_0 = \{\mu | \mu = \mu_0,\ \mu_0$ is a fixed positive number $\}$, an $A_1 = \{\mu | \mu \neq \mu_0,\ \mu > 0\}$.

We compute the following functions,

$$m_0^N(\boldsymbol{x}\mid b) = \pi_0(\mu_0)\, l_0^b(\mu_0\mid \boldsymbol{x}) = \frac{\mu_0^{b\sum_{j=1}^n x_j}\, e^{-bn\mu_0}}{(\Pi_{j=1}^n x_j!)^b} \tag{3.1}$$

and

$$m_1^N(\boldsymbol{x}\mid b) = \int_0^\infty \pi_1(\mu)\, l_1^b(\mu\mid \boldsymbol{x})\, d\mu$$

$$= \frac{\Gamma(b\sum_{j=1}^n x_j + 0.5)}{(bn)^{(b\sum_{j=1}^n x_j+0.5)}\,(\Pi_{j=1}^n x_j!)^b}. \tag{3.2}$$

The size of a minimal training sample is 1 of being equal to a minimum sample size required to guarantee a finite marginal density. Replacing $\boldsymbol{x}$ by x_l, n by 1, b by 1, and x_j by x_l in (3.1) and (3.2), the marginal density of a minimal training sample under each hypothesis is obtained by for $l = 1, 2, \ldots, n$,

$$m_0^N(x_l\mid 1) = (\mu_0^{x_l} e^{-\mu_0})/\, x_l!\,, \tag{3.3}$$

and

$$m_1^N(x_l\mid 1) = \Gamma(x_l + 0.5)/\, x_l!\,. \tag{3.4}$$

Finally, after (2.1), (2.3)-(2.7) are filled with (3.1)-(3.4) the AIBF, B_{10}^{AI}, the MIBF B_{10}^{MI}, the GIBF, B_{10}^{GI}, and the FBF, B_{10}^{F}, are respectively obtained by

$$B_{10}^{AI} = Y(1, n\mid 1)\cdot \frac{1}{n}\sum_{l=1}^n Y^{-1}(l, l\mid 1)\,, \tag{3.5}$$

$$B_{10}^{MI} = Y(1, n\mid 1)\cdot \underset{1\le l\le n}{Median}\, \{Y^{-1}(l, l\mid 1)\}\,, \tag{3.6}$$

$$B_{10}^{GI} = Y(1, n\mid 1)\cdot \{\Pi_{l=1}^n Y^{-1}(l, l\mid 1)\}^{1/n}\,, \tag{3.7}$$

and

$$B_{10}^{F} = Y(1, n\mid 1)/\, Y\left(1, n\mid \frac{1}{n}\right), \tag{3.8}$$

where

$$Y(c, d\mid b) = \frac{e^{b(d-c+1)\mu_0}\, \Gamma\left\{b\sum_{j=c}^d x_j + 0.5\right\}}{\{b(d-c+1)\}^{b\sum_{j=c}^d x_j + 0.5}\ \mu_0^{b\sum_{j=c}^d x_j}}.$$

4. Default Bayes Factors for Testing the Equality of Two Poisson population Means

Consider the following hypothesis on the equality of two Poisson population means

$$H_0: \mu_1 = \mu_2 = \mu \qquad \text{versus} \qquad H_1: \mu_1 \neq \mu_2 .$$

Jeffreys prior $\pi_0(\mu)$ for H_0 and Jeffreys prior $\pi_i(\mu_1, \mu_2)$ for H_1 are assumed as follows,

$$\pi_0(\mu) = \mu^{-1/2}\, 1_{A_0}(\mu), \quad \text{where } A_0 = \{\mu \mid \mu > 0\},$$

$$\pi_1(\mu_1, \mu_2) = \mu_1^{-1/2} \mu_2^{-1/2}\, 1_{A_1}(\mu_1, \mu_2)\ , \text{ where } A_1 = \{(\mu_1, \mu_2) \mid \mu_1 \neq \mu_2,\ \mu_1, \mu_2 > 0\}.$$

We compute the following functions,

$$m_0^N(\boldsymbol{x}_1, \boldsymbol{x}_2 \mid b) = \int_0^\infty \pi_0^N l_0^b(\mu \mid \boldsymbol{x}_1, \boldsymbol{x}_2)\, d\mu$$

$$= \frac{\Gamma\left\{b\left(\sum_{j=1}^{n_1} x_{1j} + \sum_{j=1}^{n_2} x_{2j}\right) + 0.5\right\}}{\left(\Pi_{j=1}^{n_1} x_{1j}!\right)^b \left(\Pi_{j=1}^{n_2} x_{2j}!\right)^b \{b(n_1 + n_2)\}^{b\left\{\sum_{j=1}^{n_1} x_{1j} + \sum_{j=1}^{n_2} x_{2j}\right\} + 0.5}} \tag{4.1}$$

and

$$m_1^N(\boldsymbol{x}_1, \boldsymbol{x}_2 \mid b) = \frac{\Gamma\left(b\sum_{j=1}^{n_1} x_{1j} + 0.5\right)\Gamma\left(b\sum_{j=1}^{n_2} x_{2j} + 0.5\right)}{\left(\Pi_{j=1}^{n_1} x_{1j}!\right)^b \left(\Pi_{j=1}^{n_2} x_{2j}!\right)^b (bn_1)^{b\sum_{j=1}^{n_1} x_{1j} + 0.5} (bn_2)^{b\sum_{j=1}^{n_2} x_{2j} + 0.5}}. \tag{4.2}$$

The size of a minimal training sample is 2. Each of two is from each population. Replacing $\boldsymbol{x}$ by $\boldsymbol{xy}(k, l)$, n_1 by 1, n_2 by 1, x_{11} by x_{1k}, and x_{21} by x_{2l} in (4.1) and (4.2) the marginal density of a minimal training sample under each hypothesis is given by for $k = 1, 2, \ldots, n_1$ and $l = 1, 2, \ldots, n_2$

$$m_0^N(\boldsymbol{xy}(k, l) \mid 1) = \frac{\Gamma(x_{1k} + x_{2l} + 0.5)}{x_{1k}!\, x_{2l}!\, 2^{x_{1k} + x_{2l} + 0.5}} \tag{4.3}$$

and

$$m_1^N(\mathbf{xy}(k,l)|1)=\frac{\Gamma(x_{1k}+0.5)\Gamma(x_{2l}+0.5)}{x_{1k}!\,x_{2l}!} \tag{4.4}$$

Finally, after (2.1), (2.3)-(2.7) are filled with (4.1)-(4.4) the AIBF, B_{10}^{AI}, the MIBF, B_{10}^{MI} the GIBF, B_{10}^{GI}, and the FBF, B_{10}^{F}, are respectively obtained by

$$B_{10}^{AI}=Z(1,n_1,1,n_2|1)=\frac{1}{n_1n_2}\sum_{l=1}^{n_2}\sum_{k=1}^{n_1}Z^{-1}(k,k,l,l|1)\ , \tag{4.5}$$

$$B_{10}^{MI}=Z(1,n_1,1,n_2|1)\cdot \underset{\substack{1\le k\le n_1\\ 1\le l\le n_2}}{Median}\{Z^{-1}(k,k,l,l|1)\}, \tag{4.6}$$

$$B_{10}^{GI}=Z(1,n_1,1,n_2|1)\{\Pi_{l=1}^{n_2}\Pi_{k=1}^{n_1}Z^{-1}(k,k,l,l|1)\}^{1/(n_1\cdot n_2)}, \tag{4.7}$$

and

$$B_{10}^{F}=Z(1,n_1,1,n_2|1)/Z\left(1,n_1,1,n_2\,|\,\frac{2}{n_1+n_2}\right), \tag{4.8}$$

where

$$Z(c,d,e,f|b)=\frac{(d-c+f-e+2)^{b\left(\sum_{k=c}^{d}x_{1k}+\sum_{l=e}^{f}x_{2l}\right)+0.5}}{b^{\frac{1}{2}}(d-c+1)^{b\sum_{k=c}^{d}x_{1k}+0.5}(f-e+1)^{b\sum_{l=e}^{f}x_{2l}+0.5}}$$

$$\cdot\frac{\Gamma\left(b\sum_{k=c}^{d}x_{1k}+0.5\right)\Gamma\left(b\sum_{l=e}^{f}x_{2l}+0.5\right)}{\Gamma\left\{b\left(\sum_{k=c}^{d}x_{1k}+\sum_{l=e}^{f}x_{2l}\right)+0.5\right\}}$$

5. A Simulation Study

To see the performance of tests by default Bayes factors in testing the equality of Poisson population means Poisson data of size 30 with 100 replications are simulated from Poisson(μ) distributions.

Results of tests on the equality of one Poisson population mean are shown in Table 5.1 and Table 5.2 and those on the equality of two independent Poisson population means in Table 5.3 and Table 5.4.

Mean values of B_{10} and $P(H_1|\boldsymbol{x}) = 1.0 - P(H_0|\boldsymbol{x})$ under the assumption of equal prior probability for each hypothesis and powers by B_{10} in 100 replications are larger as Poisson data are farther from the population of H_0. These results meet our theoretical expectations. Also results in cells shadowed of Tables explain that the data are sampled from the population of H_0. Powers by B_{10} when H_0 is true are about 2%~7%, which are compared with powers, 1%, 5%, and 10%, by the classical tests.

Though the differences in powers are small among four default Bayes factors, generally the GIBF's give smaller powers and the FBF's give larger powers.

The conflicts between the p-value, the observed significance level, and $P(H_0|\boldsymbol{x})$, the posterior probability of H_0, are also shown in our results as Berger and Sellke(1987).

6. Concluding Remarks

In a simulation study we can see that default Bayes factors under the least prior information perform coincidently with the logic of test.

In addition to experiments with the sample size $n=30$ and $(n_1, n_2)=(30,30)$, experiments were carried for simulated data with the small sample size $n=8, 15$ in case of one sample and $(n_1, n_2)=(8,8)$, $(15,15)$ in case of two samples, though their tables are not presented here because of limited space. We can see that their results for small samples are similar to those of the sample size $n=30$ and $(n_1, n_2)=(30,30)$ and they more agree with our theoretical expectation for testing as the sample sizes are larger.

It seems to be complicated to extend to the test on the equality of k independent Poisson population means since the likelihood function of $H_1: \mu_i \neq \mu_j$ for some i, j $(i, j=1, 2, \ldots, k)$ must be written for any k.

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Table 5.1 : Testing results of $H_0: \mu = 1.0$ versus $H_1: \mu \neq 1.0$ for Poisson data of size $n = 30$ simulated from Poisson(μ) distribution.

μ		AIBF	MIBF	GIBF	FBF	p-value
0.1	Mean of B_{10}	0.160D+11	0.160D+11	0.160D+11	0.161D+11	0.000
	(s.d. of B_{10})	(0.101D+12)	(0.101D+12)	(0.101D+12)	(0.101D+12)	(0.000)
	Mean of $P(H_0\|\boldsymbol{x})$	0.000	0.000	0.000	0.000	
	(s.d. of $P(H_0\|\boldsymbol{x})$)	(0.000)	(0.000)	(0.000)	(0.000)	
	power by B_{10}	1.000	1.000	1.000	1.000	
0.2	Mean of B_{10}	0.722D+10	0.721D+10	0.722D+10	0.722D+10	0.000
	(s.d. of B_{10})	(0.718D+11)	(0.718D+11)	(0.718D+11)	(0.718D+11)	(0.000)
	Mean of $P(H_0\|\boldsymbol{x})$	0.000	0.000	0.000	0.000	
	(s.d. of $P(H_0\|\boldsymbol{x})$)	(0.001)	(0.001)	(0.001)	(0.001)	
	power by B_{10}	1.000	1.000	1.000	1.000	
0.5	Mean of B_{10}	0.264D+03	0.212D+03	0.252D+03	0.327D+03	0.032
	(s.d. of B_{10})	(0.114D+04)	(0.908D+03)	(0.109D+04)	(0.134D+04)	(0.100)
	Mean of $P(H_0\|\boldsymbol{x})$	0.161	0.184	0.167	0.136	
	(s.d. of $P(H_0\|\boldsymbol{x})$)	(0.229)	(0.243)	(0.235)	(0.204)	
	power by B_{10}	0.850	0.850	0.850	0.900	
0.7	Mean of B_{10}	0.183D+02	0.145D+02	0.174D+02	0.234D+02	0.178
	(s.d. of B_{10})	(0.105D+03)	(0.825D+02)	(0.997D+02)	(0.130D+03)	(0.241)
	Mean of $P(H_0\|\boldsymbol{x})$	0.494	0.525	0.505	0.440	
	(s.d. of $P(H_0\|\boldsymbol{x})$)	(0.285)	(0.282)	(0.288)	(0.273)	
	power by B_{10}	0.450	0.420	0.450	0.560	
1.0	Mean of B_{10}	0.297D+00	0.277D+00	0.268D+00	0.415D+00	0.512
	(s.d. of B_{10})	(0.420D+00)	(0.380D+00)	(0.376D+00)	(0.537D+00)	(0.294)
	Mean of $P(H_0\|\boldsymbol{x})$	0.805	0.814	0.819	0.749	
	(s.d. of $P(H_0\|\boldsymbol{x})$)	(0.119)	(0.117)	(0.115)	(0.127)	
	power by B_{10}	0.030	0.030	0.030	0.050	
1.5	Mean of B_{10}	0.296D+03	0.312D+03	0.206D+03	0.353D+03	0.072
	(s.d. of B_{10})	(0.121D+04)	(0.128D+04)	(0.818D+03)	(0.142D+04)	(0.121)
	Mean of $P(H_0\|\boldsymbol{x})$	0.360	0.364	0.384	0.311	
	(s.d. of $P(H_0\|\boldsymbol{x})$)	(0.286)	(0.290)	(0.294)	(0.261)	
	power by B_{10}	0.640	0.650	0.590	0.760	
2.0	Mean of B_{10}	0.140D+09	0.151D+09	0.760D+08	0.121D+09	0.002
	(s.d. of B_{10})	(0.119D+10)	(0.125D+10)	(0.634D+09)	(0.103D+10)	(0.007)
	Mean of $P(H_0\|\boldsymbol{x})$	0.028	0.029	0.032	0.022	
	(s.d. of $P(H_0\|\boldsymbol{x})$)	(0.087)	(0.090)	(0.096)	(0.071)	
	power by B_{10}	1.000	1.000	1.000	1.000	
3.0	Mean of B_{10}	0.849D+27	0.231D+27	0.184D+27	0.287D+27	0.000
	(s.d. of B_{10})	(0.844D+28)	(0.227D+28)	(0.183D+28)	(0.286D+28)	(0.000)
	Mean of $P(H_0\|\boldsymbol{x})$	0.000	0.000	0.000	0.000	
	(s.d. of $P(H_0\|\boldsymbol{x})$)	(0.000)	(0.000)	(0.000)	(0.000)	
	power by B_{10}	1.000	1.000	1.000	1.000	

Table 5.2 : Testing results of $H_0: \mu = 10.0$ versus $H_1: \mu \neq 10.0$ for Poisson data of size $n=30$ simulated from Poisson(μ) distribution.

μ		AIBF	MIBF	GIBF	FBF	p-value
7.0	Mean of B_{10}	0.549D+11	0.526D+11	0.357D+11	0.488D+11	0.000
	(s.d. of B_{10})	(0.467D+12)	(0.455D+12)	(0.301D+12)	(0.406D+12)	(0.000)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.001	0.001	0.001	0.001	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.003)	(0.003)	(0.004)	(0.003)	
	power by B_{10}	1.000	1.000	1.000	1.000	
8.0	Mean of B_{10}	0.846D+06	0.640D+06	0.498D+06	0.823D+06	0.011
	(s.d. of B_{10})	(0.821D+07)	(0.618D+07)	(0.482D+07)	(0.793D+07)	(0.034)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.096	0.090	0.106	0.080	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.189)	(0.181)	(0.203)	(0.164)	
	power by B_{10}	0.930	0.940	0.910	0.950	
9.0	Mean of B_{10}	0.115D+03	0.147D+03	0.896D+03	0.127D+03	0.178
	(s.d. of B_{10})	(0.108D+04)	(0.140D+04)	(0.840D+04)	(0.119D+04)	(0.228)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.542	0.522	0.570	0.488	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.272)	(0.270)	(0.273)	(0.263)	
	power by B_{10}	0.400	0.420	0.390	0.480	
10.0	Mean of B_{10}	0.529D+00	0.590D+00	0.463D+00	0.666D+00	0.470
	(s.d. of B_{10})	(0.262D+01)	(0.293D+01)	(0.238D+01)	(0.300D+01)	(0.279)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.805	0.790	0.826	0.754	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.132)	(0.136)	(0.127)	(0.136)	
	power by B_{10}	0.030	0.050	0.020	0.060	
11.0	Mean of B_{10}	0.833D+01	0.922D+01	0.625D+01	0.103D+02	0.193
	(s.d. of B_{10})	(0.352D+02)	(0.386D+02)	(0.248D+02)	(0.431D+02)	(0.224)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.583	0.566	0.611	0.531	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.278)	(0.277)	(0.277)	(0.269)	
	power by B_{10}	0.310	0.330	0.260	0.360	
12.0	Mean of B_{10}	0.331D+04	0.359D+04	0.238D+04	0.357D+04	0.019
	(s.d. of B_{10})	(0.136D+05)	(0.145D+05)	(0.950D+04)	(0.143D+05)	(0.050)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.146	0.139	0.161	0.125	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.234)	(0.227)	(0.245)	(0.210)	
	power by B_{10}	0.870	0.870	0.860	0.890	
13.0	Mean of B_{10}	0.144D+10	0.136D+10	0.793D+09	0.123D+10	0.001
	(s.d. of B_{10})	(0.107D+11)	(0.107D+11)	(0.585D+10)	(0.909D+10)	(0.009)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.012	0.011	0.014	0.010	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.069)	(0.065)	(0.075)	(0.061)	
	power by B_{10}	0.990	0.990	0.990	0.990	
14.0	Mean of B_{10}	0.394D+16	0.267D+16	0.190D+16	0.283D+16	0.000
	(s.d. of B_{10})	(0.375D+17)	(0.250D+17)	(0.182D+17)	(0.270D+17)	(0.000)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.000	0.000	0.000	0.000	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.001)	(0.001)	(0.001)	(0.001)	
	power by B_{10}	1.000	1.000	1.000	1.000	

Table 5.3 : Testing results of $H_0: \mu_1 = \mu_2 = \mu$ versus $H_1: \mu_1 \neq \mu_2$ for two independent Poisson data of size $n_1 = n_2 = 30$ simulated from Poisson(1.0) and Poisson(μ) distribution.

μ		AIBF	MIBF	GIBF	FBF	p-value
0.1	Mean of B_{10}	0.480D+10	0.326D+10	0.359D+10	0.381D+10	0.000
	(s.d. of B_{10})	(0.385D+11)	(0.240D+11)	(0.285D+11)	(0.306D+11)	(0.000)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.002	0.002	0.002	0.002	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.006)	(0.008)	(0.007)	(0.006)	
	power by B_{10}	1.000	1.000	1.000	1.000	
0.2	Mean of B_{10}	0.431D+06	0.409D+06	0.353D+06	0.389D+06	0.009
	(s.d. of B_{10})	(0.230D+07)	(0.223D+07)	(0.186D+07)	(0.205D+07)	(0.083)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.027	0.030	0.029	0.025	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.093)	(0.092)	(0.095)	(0.091)	
	power by B_{10}	0.990	0.990	0.990	0.990	
0.5	Mean of B_{10}	0.234D+07	0.234D+07	0.168D+07	0.193D+07	0.111
	(s.d. of B_{10})	(0.234D+08)	(0.234D+08)	(0.168D+08)	(0.193D+08)	(0.194)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.398	0.426	0.419	0.368	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.279)	(0.288)	(0.284)	(0.267)	
	power by B_{10}	0.600	0.570	0.570	0.630	
0.7	Mean of B_{10}	0.619D+01	0.560D+01	0.515D+01	0.699D+01	0.276
	(s.d. of B_{10})	(0.313D+02)	(0.284D+02)	(0.254D+02)	(0.345D+02)	(0.279)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.631	0.655	0.654	0.592	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.246)	(0.246)	(0.245)	(0.241)	
	power by B_{10}	0.230	0.210	0.210	0.270	
1.0	Mean of B_{10}	0.576D+00	0.516D+00	0.496D+00	0.708D+00	0.505
	(s.d. of B_{10})	(0.212D+01)	(0.189D+01)	(0.181D+01)	(0.248D+01)	(0.292)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.787	0.802	0.806	0.755	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.146)	(0.144)	(0.142)	(0.149)	
	power by B_{10}	0.050	0.050	0.040	0.060	
1.5	Mean of B_{10}	0.296D+02	0.273D+02	0.216D+02	0.315D+02	0.200
	(s.d. of B_{10})	(0.138D+03)	(0.140D+03)	(0.109D+03)	(0.157D+03)	(0.276)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.518	0.519	0.542	0.472	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.298)	(0.298)	(0.300)	(0.287)	
	power by B_{10}	0.420	0.420	0.410	0.470	
2.0	Mean of B_{10}	0.101D+06	0.750D+05	0.733D+05	0.111D+06	0.023
	(s.d. of B_{10})	(0.854D+06)	(0.588D+06)	(0.620D+06)	(0.941D+06)	(0.070)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.167	0.165	0.183	0.142	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.222)	(0.221)	(0.235)	(0.197)	
	power by B_{10}	0.900	0.900	0.860	0.920	
3.0	Mean of B_{10}	0.935D+09	0.787D+09	0.561D+09	0.887D+09	0.000
	(s.d. of B_{10})	(0.499D+10)	(0.431D+10)	(0.293D+10)	(0.474D+10)	(0.000)
	Mean of $P(H_0 \mid \boldsymbol{x})$	0.001	0.001	0.001	0.001	
	(s.d. of $P(H_0 \mid \boldsymbol{x})$)	(0.006)	(0.006)	(0.007)	(0.005)	
	power by B_{10}	1.000	1.000	1.000	1.000	

Table 5.4 : Testing results of $H_0: \mu_1 = \mu_2 = \mu$ versus $H_1: \mu_1 \neq \mu_2$ for two independent Poisson data of size $n_1 = n_2 = 30$ simulated from Poisson(5.0) and Poisson(μ) distribution.

μ		AIBF	MIBF	GIBF	FBF	p-value
2.0	Mean of B_{10}	0.186D+14	0.126D+14	0.963D+14	0.153D+14	0.000
	(s.d. of B_{10})	(0.113D+15)	(0.753D+15)	(0.588D+15)	0.924D+15	(0.000)
	Mean of $P(H_0 \| \boldsymbol{x})$	0.000	0.000	0.000	0.000	
	(s.d. of $P(H_0 \| \boldsymbol{x})$)	(0.001)	(0.001)	(0.002)	0.001	
	power by B_{10}	1.000	1.000	1.000	1.000	
3.0	Mean of B_{10}	0.147D+10	0.118D+10	0.762D+09	0.131D+10	0.007
	(s.d. of B_{10})	(0.147D+11)	(0.118D+11)	(0.762D+10)	(0.131D+11)	(0.029)
	Mean of $P(H_0 \| \boldsymbol{x})$	0.073	0.068	0.083	0.061	
	(s.d. of $P(H_0 \| \boldsymbol{x})$)	(0.154)	(0.147)	(0.168)	(0.134)	
	power by B_{10}	0.960	0.970	0.960	0.980	
4.0	Mean of B_{10}	0.318D+02	0.357D+02	0.225D+02	0.391D+02	0.162
	(s.d. of B_{10})	(0.198D+03)	(0.221D+03)	(0.138D+03)	(0.243D+03)	(0.214)
	Mean of $P(H_0 \| \boldsymbol{x})$	0.536	0.515	0.569	0.478	
	(s.d. of $P(H_0 \| \boldsymbol{x})$)	(0.274)	(0.273)	(0.272)	(0.266)	
	power by B_{10}	0.400	0.440	0.370	0.500	
5.0	Mean of B_{10}	0.288D+00	0.322D+00	0.241D+00	0.402D+00	0.505
	(s.d. of B_{10})	(0.327D+00)	(0.366D+00)	(0.262D+00)	(0.455D+00)	(0.300)
	Mean of $P(H_0 \| \boldsymbol{x})$	0.807	0.790	0.829	0.754	
	(s.d. of $P(H_0 \| \boldsymbol{x})$)	(0.123)	(0.128)	(0.114)	(0.134)	
	power by B_{10}	0.050	0.050	0.040	0.070	
6.0	Mean of B_{10}	0.106D+02	0.119D+02	0.805D+01	0.136D+02	0.212
	(s.d. of B_{10})	(0.503D+02)	(0.555D+02)	(0.366D+02)	(0.653D+02)	(0.267)
	Mean of $P(H_0 \| \boldsymbol{x})$	0.553	0.534	0.582	0.500	
	(s.d. of $P(H_0 \| \boldsymbol{x})$)	(0.294)	(0.292)	(0.293)	(0.285)	
	power by B_{10}	0.380	0.410	0.360	0.440	
7.0	Mean of B_{10}	0.487D+05	0.456D+05	0.300D+05	0.561D+05	0.024
	(s.d. of B_{10})	(0.368D+06)	(0.338D+06)	(0.231D+06)	(0.423D+06)	(0.071)
	Mean of $P(H_0 \| \boldsymbol{x})$	0.174	0.163	0.193	0.147	
	(s.d. of $P(H_0 \| \boldsymbol{x})$)	(0.229)	(0.220)	(0.244)	(0.203)	
	power by B_{10}	0.890	0.910	0.870	0.920	
8.0	Mean of B_{10}	0.155D+08	0.149D+08	0.888D+07	0.151D+08	0.001
	(s.d. of B_{10})	(0.846D+08)	(0.810D+08)	(0.489D+08)	(0.819D+08)	(0.007)
	Mean of $P(H_0 \| \boldsymbol{x})$	0.021	0.019	0.024	0.017	
	(s.d. of $P(H_0 \| \boldsymbol{x})$)	(0.072)	(0.067)	(0.078)	(0.062)	
	power by B_{10}	0.990	0.990	0.990	0.990	
9.0	Mean of B_{10}	0.246D+12	0.194D+12	0.121D+12	0.204D+12	0.000
	(s.d. of B_{10})	(0.201D+13)	(0.156D+13)	(0.977D+12)	(0.164D+13)	(0.000)
	Mean of $P(H_0 \| \boldsymbol{x})$	0.000	0.000	0.000	0.000	
	(s.d. of $P(H_0 \| \boldsymbol{x})$)	(0.000)	(0.000)	(0.000)	(0.000)	
	power by B_{10}	1.000	1.000	1.000	1.000	