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On the strong law of large numbers for[1] pairwise negative quadrant dependent random variables

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Abstract

Petrov(1996) examined the connection between general moment conditions and the applicability of the strong law of large numbers to a sequence of pairwise independent and identically distributed random variables. In this note, we generalize Theorem 1 of Petrov(1996) and also show that this still holds under assumption of pairwise negative quadrant dependence(NQD).

1. Introduction

On the strong law of large numbers(SLLN) for a sequence of independent and identically distributed random variables there exist Kolmogorov's theorem and the Marcinkiewicz-Zygmund theorem(see e.g. Loéve, 1963 or Stout, 1974). In what follows, we put $S_n = \sum_{i=1}^{n} X_i$. According to Kolmogorov's theorem, there exists a constant b such that $S_n/n \to b$ almost surely if and only if $E|X_1| < \infty$. Here if $E|X_1| < \infty$, then $b = EX_1$. By the Marcinkiewicz-Zygmund's theorem for $0 < p < 2$ $(S_n - nb)/n^{1/p} \to 0$ almost surely if and only if $E|X_1|^p < \infty$ and $EX_1 = b$

Etemadi(1981) proved that Kolmogorov's theorem remains true if we replace the independence condition by the weaker condition of pairwise independence of random variables $X_1, X_2, \cdots$ and under the assumption that $\{X_n, n \geq 1\}$ is a sequence of pairwise independent and identically distributed random variables Martikainen(1995) showed that for $1 < r < 2$ the

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condition $E|X_1|^r \log^+|X_1|)^\tau < \infty$ (for some positive $\tau > 4r-6$) is sufficient for the SLLN $(S_n - ES_n)/n^{1/r} \to 0$ $a.s.$ However, it is not known that Marcinkiewicz-Zygmund's SLLN holds for pairwise independent and identically distributed random variables.

Two random variables X and Y are negative quadrant dependent(NQD) if, for all $x, y \in R$, $P[X>x, Y>y] \le P[X>x]P[Y>y]$ (or $P[X \le x, Y \le y] \le P[X \le x]P[Y \le y]$) and positive quadrant dependent(PQD) if $P[X>x, Y>y] \ge P[X>x]P[Y>y]$ (or $P[X \le x, Y \le y] \ge P[X \le x]P[Y \le y]$) for all $x, y \in R$. These concepts of quadrant dependence were introduced by Lehman(1966).

Matula(1992) proved that Kolmogorov's theorem still holds under the condition of pairwise negative quadrant dependence of random variables $X_1, X_2, \cdots$, that is, let $\{X_n, n \ge 1\}$ be a sequence of pairwise NQD random variables with the same distribution function $F(x)$ then $S_n/n \to a$ almost surely, for some constant $a \in \mathbb{R}$, if and only if $E|X_1| < \infty$. If $E|X_1| < \infty$, then $a = EX_1$.

Recently, Petrov(1996) examined the connection between general moment conditions and the applicability of the strong law of large numbers of identically distributed random variables. The following theorem is Theorem 1 of Petrov(1996).

Let $f(x)$ be an even continuous function that is positive and strictly increasing in the region $x > 0$ and satisfying the condition $f(x) \to \infty$ as $x \to \infty$. We put $a_n = f^{-1}(n)$ where f^{-1} is the inverse of f we have $a_n \uparrow \infty$.

Theorem 1. (Petrov, 1996) Let $\{X_n, n \ge 1\}$ be a sequence of pairwise independent identically distributed random variables. If $S_n / a_n \to 0$ $a.s.$ then $Ef(X_1) < \infty$.

In this note we generalize Theorem 1 and show that this remains true if we replace the pairwise independence condition by the condition of pairwise negative quadrant dependence.

2. Results

Lemma 2.1. Let X be a random variable and let $\{b_n, n \ge 1\}$ be a sequence of numbers satisfying $0 < b_n \uparrow \infty$. Let ϕ be any even function that is positive and strictly increasing in the region $x > 0$ and satisfying the condition that there are some constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \le \frac{\phi(b_n)}{n} \le c_2. \tag{2.1}$$

Then $E\phi(X) < \infty$ if and only if

$$\sum_{n=1}^{\infty} P(|X| > b_n) < \infty. \tag{2.2}$$

Proof. Let $b_0 = 0$. Assume $E\phi(X) < \infty$. From (2.1) and the assumption that $\phi(X)$ is increasing function we have

$$\begin{aligned} E\phi(X) &= \sum_{n=1}^{\infty} E[\phi(X) I(b_{n-1} < |X| \le b_n)] \\ &\ge \sum_{n=1}^{\infty} \phi(b_{n-1}) P(b_{n-1} < |X| \le b_n) \\ &\ge c_1 \sum_{n=1}^{\infty} (n-1) P(b_{n-1} < |X| \le b_n) \qquad (by\ (2.1)) \\ &= c_1 \sum_{n=1}^{\infty} n P(b_n < |X| \le b_{n+1}) = c_1 \sum_{n=1}^{\infty} P(|X| > b_n). \end{aligned}$$

Thus (2.2) holds. Assume that (2.2) holds. By the conditions as the above we have

$$\begin{aligned} E\phi(X) &\le \sum_{n=1}^{\infty} \phi(b_n) P(b_{n-1} < |X| \le b_n) \\ &\le c_2 \sum_{n=1}^{\infty} n P(b_{n-1} < |X| \le b_n) \qquad (by\ (2.1)) \\ &\le c_2 \{1 + \sum_{n=1}^{\infty} P(|X| > b_n)\} < \infty. \end{aligned}$$

Thus the proof is complete.

Remark. By putting $c_1 = c_2 = 1$, Lemma 2.1 becomes Lemma 3.3.2 in Stout (1974). In other words, Lemma 2.1 is a generalization of Lemma 3.3.2 of Stout (1974).

Theorem 2.2. Let $\{X_n, n \ge 1\}$ be a sequence of pairwise independent identically distributed random variables and let $\{b_n, n \ge 1\}$ be a sequence of numbers satisfying $0 < b_n \uparrow \infty$. Let ϕ be any even function that is positive and strictly increasing in the region $x > 0$ and satisfying (2.1). If

$$S_n / b_n \to 0 \ a.s. \tag{2.3}$$

then

$$E\phi(X_1) < \infty. \tag{2.4}$$

Proof. According to Lemma 1 in Petrov (1996) it follows from (2.3) that

$$\sum_{n=1}^{\infty} P(|X_1| \geq b_n) < \infty \tag{2.5}$$

since X_i's are identically distributed. Thus, from (2.5), (2.4) follows by Lemma 2.1.

Corollary 2.3. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise independent identically distributed random variables and $\{b_n, n \geq 1\}$ a sequence of positive numbers such that $0 < b_n \uparrow \infty$. Let ϕ be any even function that is positive and strictly increasing on $[0, \infty)$ and satisfying (2.1). If

$$\sum_{k=n}^{\infty} (1 / b_k) = O(n / b_n), \tag{2.6}$$

then (2.4) implies (2.3).

Proof. First note that by Lemma 2.1 it follows from (2.4) that $\sum_{n=1}^{\infty} P(|X_1| \geq b_n) < \infty$. Thus the desired result (2.3) follows by Lemma 2 in Petrov (1996).

Remarks. (1). It follows from Theorem 2.2 and Corollary 2.3 that if $\{X_n, n \geq 1\}$ is a sequence of pairwise independent identically distributed random variables and if (2.6) is satisfied then (2.3) and (2.4) are equivalent.

(2). By putting $c_1 = c_2 = 1$ in (2.1) $b_n = \phi^{-1}(n)$ and thus ϕ and b_n satisfy the conditions in Petrov(1996). Eventually, Theorem 2.2 is a genealization of Theorem 1 in Petrov(1996).

Lemma 2.4. (Matula, 1992) Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with the common distribution $F(x)$ and $\{b_n, n \geq 1\}$ a sequence of positive numbers such that $0 < b_n \uparrow \infty$. If $S_n / b_n \to 0$ *a. s.* then (2.5) holds.

Finally we show that Theorem 2.2 remains true if we replace independence condition by the weaker condition of negative quadrant dependence of random variables $X_1, X_2, \cdots$, that is, using Lemmas 2.1 and 2.4 we extend Theorem 2.2 to the pairwise NQD case;

Theorem 2.5. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with the same distribution $F(x)$ and $\{b_n, n \geq 1\}$ a sequence of positive numbers satisfying

$0 < b_n \uparrow \infty$. Let ϕ be any even strictly increasing function on $[0, \infty)$ satisfying (2.1). If

(2.3) is satisfied then (2.4) holds.

Proof. According to Lemma 2.4 (Lemma 3 in Matula (1992)) we have $\sum_{n=1}^{\infty} P[\,|X_1| \geq b_n] < \infty$. Thus the desired result follows by Lemma 2.1.

By using Lemma 2 of Petrov(1996) we also have the following result.

Corollary 2.6. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with the same distribution $F(x)$. Let $\{b_n, n \geq 1\}$ be a sequence of positive numbers such that $0 < b_n \uparrow \infty$. Let ϕ be any even function that is positive and strictly increasing on $[0, \infty)$ and satisfying (2.1). If (2.6) is satisfied then (2.3) and (2.4) are equivalent.

Corollary 2.7. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with the same distribution $F(x)$ and let $b_n = g(n)$ for all $n \geq 1$. Assume

(i) $x^2 / g(x) \uparrow \infty$ *as* $x \to \infty$,

(ii) $c_1 \leq b_n^2 / \{n\, g(b_n)\} \leq c_2$ for some positive constants c_1 and c_2,

(iii) $\sum_{k=n}^{\infty} (1/b_k) = O(n/b_n)$.

Then $E[X_1^2 / g(|X_1|)] < \infty$ if and only if $S_n / b_n \to 0$ *a.s.*

Proof Let $\phi(x) = x^2 / g(x)$. Then $x^2 / g(x)$ satisfies (2.1) and thus the desired result follows by Corollary 2.6.

We close this section by introducing an example of Corollary 2.6 :

Example 2.8. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with common distribution $F(x)$. Set $\phi(x) = |x|^p$ and $b_n = n^{\frac{1}{p}}$, $0 < p < 1$. Then condition (2.6) is satisfied. Hence, $E|X_1|^p < \infty$ if and only if $S_n / n^{1/p} \to 0$ *a. s.* Let $g(x)$ be an even function that is positive and strictly increasing on $[0, \infty)$ and satisfying $g(x) \uparrow \infty$ as $x \uparrow \infty$.

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