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Edgeworth Expansion and Bootstrap Approximation for Survival Function Under Koziol-Green Model

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Abstract

Confidence intervals for survival function give useful information about the lifetime distribution. In this paper, we develop Edgeworth expansions as approximation to the true and bootstrap distributions of normalized nonparametric maximum likelihood estimator of survival function in the Koziol-Green model and then use these results to show that the bootstrap approximations have second order accuracy.

1. Introduction

Lifetime data with incomplete observations often arise in medical research and reliability analysis. In some clinical trials, each subject is observed from some entry time until a particular event happens. Often it is impossible to observe complete lifetime of the subject. With this type of right censored data, Inferences for survival function of the lifetime distribution is important. In particular, confidence intervals for survival function give an useful information about the lifetime distribution. Typically such intervals are obtained by using first order normal approximation of an estimator of survival function. But, In many instances, the applications in which these approximations are used have small sample sizes, hence the appropriateness of such intervals is suspected. A primary goal of this paper is derived better confidence limits for survival function than those constructed based on the normal approximations via Edgeworth expansion.

The bootstrap method is an alternative to normal approximation. The bootstrap procedure to the random censorship model was first discussed by Efron(1981). Subsequently, many researchers have been made a study of the bootstrap approximation. On the other hand, the higher order asymptotics have been devised to increase the accuracy of the approximation of the exact distribution of statistics. A well known method is to use the first few terms of an Edgeworth expansion. Recently, some important progresses have been accomplished in the way of producing accurate approximations to the distribution of censored data. Lai and Wang(1993) provided general Edgeworth expansions for the true and bootstrap distributions of

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an asymptotic U-statistic. Chen and Lo(1996) obtained one-term Edgeworth expansions for the distributions of the studentized Nelson-Aalen(NA) and Kaplan-Meier(KM) estimators and their bootstrap versions. Gross and Lai(1996), using the results of Lai and Wang(1993), derived the Edgeworth expansion of asymptotic U-statistic representation for the studentized KM estimator and its bootstrap distributions with left-truncated and right-censored data.

In this paper, we develop Edgeworth expansions as approximations to the true and bootstrap distributions of normalized nonparametric maximum likelihood estimator(ACL estimator) of survival function in the Koziol-Green model which is proposed by Abdushukurov(1984) and Cheng and Lin(1984, 1987) and then use these results to show that the bootstrap approximations have second order accuracy.

2. Preliminary and Assumptions

Let $T_1, T_2, \cdots, T_n$ be independent and identically distributed(*i.i.d.*) random variables with a continuous distribution function F. These are censored on the right by the *i.i.d.* random variables $C_1, C_2, \cdots, C_n$ with a continuous distribution function G, so that the observations available consist of the pairs $X_i = (Z_i, \delta_i)$ for $i = 1, \cdots, n$, where $Z_i = T_i \wedge C_i$ and $\delta_i = I_{\{T_i \le C_i\}}$. Here and in the sequel, $a \wedge b = \min(a, b)$ and $I_{\{A\}}$ denotes the indicator function of the event A.

In the usual random censorship model one assumes that the lifetimes and censoring sequences are independent. Thus the observed sequence $\{Z_i, i = 1, \cdots, n\}$ is *i.i.d.* random variable having distribution function $H(t) = 1 - S(t) = 1 - \overline{F}(t)\overline{G}(t)$, where $\overline{F}(t) = 1 - F(t)$ and $\overline{G}(t) = 1 - G(t)$.

Koziol and Green(1976) introduced the appealing and useful special model in which there exists a positive constant λ, the censoring parameter, such that

$$\overline{G}(t) = \overline{F}^{\lambda}(t), \qquad 0 < t < \infty. \tag{2.1}$$

It is often referred to as the Koziol-Green(KG) model of random censorship.

Under the this model, it is easy to show that if F is continuous, then

$$\begin{aligned} E(\delta_i) &= P\{T_i \le C_i\} \\ &= \int_0^{\infty} P\{C_i \ge t \mid T_i = t\} \, dF(t) \\ &= \int_0^{\infty} \overline{G}(t) \, dF(t) \\ &= \frac{1}{1+\lambda} \equiv \theta. \end{aligned}$$

and $\overline{F}(t) = [S(t)]^{\theta}$. Hence θ is the expected proportion of uncensored observation. The case $\lambda = 0$ $(\theta = 1)$ corresponds to no censoring and the expected number of the censored observations increases as λ increases.

In most studies of the KG model, the KM estimator of survival function plays the key role in estimation and hypothesis testing. An obvious question is whether the information through the censoring parameter λ is fully acknowledged by using KM estimator. Then Abdushukurov(1984) and Cheng and Lin(1984, 1987) independently proposed the ACL estimator of survival function as follows :

$$\overline{F}_n(t) = [S_n(t)]^{\theta_n}, \quad \text{for each } t,$$

where

$$S_n(t) = \frac{1}{n}\sum_{i=1}^{n} I_{\{Z_i > t\}} \quad \text{and} \quad \theta_n = \frac{1}{n}\sum_{i=1}^{n}\delta_i .$$

The ACL estimator is preferred to KM estimator of survival function whenever (2.1) holds, since the asymptotic variance of ACL estimator is strictly smaller than that of the corresponding functional based on the KM estimator. Moreover, they showed that the sequence of random functions $\{A_n(t) = \sqrt{n}(\overline{F}_n(t) - \overline{F}(t))\}$, for any bounded interval of the form $[0, T]$, converges weakly to the mean zero Gaussian process $W(t)$ with covariance function, for $0 \le s \le t \le T$,

$$\begin{aligned} Cov(W(s), W(t)) = \; & \theta^2[S(s)]^{\theta-1}[S(t)]^{\theta}[1 - S(s)] \\ & + \theta(1-\theta)[S(s)S(t)]^{\theta}[\ln S(s)][\ln S(t)] . \end{aligned} \tag{2.2}$$

On the bootstrap approximation in KG model, Dikta and Ghorai(1990) resampled the bootstrap samples, $X_i^* = (Z_i^*, \delta_i^*)$, $i = 1, \cdots, n$, using the independence of Z and δ. The bootstrap analog to $\overline{F}_n(t)$ is

$$\overline{F}_n^*(t) = [S_n^*(t)]^{\theta_n^*} ,$$

where

$$S_n^*(t) = \frac{1}{n}\sum_{i=1}^{n} I_{\{Z_i^* > t\}} \quad \text{and} \quad \theta_n^* = \frac{1}{n}\sum_{i=1}^{n}\delta_i^* .$$

Dikta and Ghorai(1990) considered the process

$$\{A_n^*(t) = \sqrt{n}(\overline{F}_n^*(t) - \overline{F}_n(t)), \; 0 < t < T\}$$

and showed that the process $A_n^*(t)$ converges weakly to the Gaussian process $W(t)$ with covariance given by (2.2).

3. Edgeworth Expansion and Bootstrap Approximation

Lai and Wang(1993) devised an Edgeworth expansion for normalized NA estimator after proving that the statistic has an asymptotic U-statistic representation. Using these results we, firstly, show that the normalized ACL estimator has an asymptotic U-statistic.

Theorem 3.1 Let $\overline{F}_n(t) = [\, S_n(t)\,]^{\theta_n}$ be an estimator of the survival function $\overline{F}(t)$ in Koziol-Green model. Then $U_n(t) = \sqrt{n}(\,\overline{F}_n(t) - \overline{F}(t))$ has an asymptotic U-statistic representation as follow :

$$\begin{aligned} U_n(t) = \sum_{i=1}^{n}\left\{\frac{\alpha(X_i)}{n^{1/2}} + \frac{\alpha'(X_i)}{n^{3/2}}\right\} &+ \sum_{1\le i<j\le n}\frac{\beta(X_i, X_j)}{n^{3/2}} \\ &+ \sum_{1\le i<j<k\le n}\frac{\gamma(X_i, X_j, X_k)}{n^{5/2}} + R_n, \end{aligned} \tag{3.1}$$

where $X_i = (Z_i, \delta_i)$ and $\alpha, \alpha', \beta, \gamma$ are nonrandom Borel function which are invariant under permutation of the arguments and which satisfy assumptions (A1)-(A4) of Lai and Wang(1993).

Proof. Taking a three-term Taylor expansion of $\overline{F}_n(t) = [\, S_n(t)\,]^{\theta_n}$ about $S_n(t) = S(t)$ and $\theta_n = \theta$, and let

$$\begin{aligned} p_1 &= \theta[S(t)]^{\theta-1}, \\ p_2 &= [S(t)]^{\theta}\ln S(t), \\ p_3 &= \theta(\theta-1)[S(t)]^{\theta-2}, \\ p_4 &= 2[S(t)]^{\theta-1}[1+\theta\ln S(t)], \\ p_5 &= [S(t)]^{\theta}[\ln S(t)]^2, \\ p_6 &= \theta(\theta-1)(\theta-2)[S(t)]^{\theta-3}, \\ p_7 &= 3[S(t)]^{\theta-2}\{2\theta+\theta^2\ln S(t)-\theta\ln S(t)-1\}, \\ p_8 &= 3[S(t)]^{\theta-1}\{2\ln S(t)+\theta[\ln S(t)]^2\}, \\ p_9 &= [S(t)]^{\theta}[\ln S(t)]^3. \end{aligned}$$

Then we may write

$$[S_n(t)]^{\theta_n} - [S(t)]^{\theta} = p_1[S_n(t)-S(t)] + p_2[\theta_n-\theta]$$
$$+\frac{1}{2!}\{p_3[S_n(t)-S(t)]^2 + p_4[S_n(t)-S(t)][\theta_n-\theta] + p_5[\theta_n-\theta]^2\}$$
$$+\frac{1}{3!}\{p_6[S_n(t)-S(t)]^3 + p_7[S_n(t)-S(t)]^2[\theta_n-\theta]$$
$$+p_8[S_n(t)-S(t)][\theta_n-\theta]^2 + p_9[\theta_n-\theta]^3\} + R_n.$$

Let $\omega_i(t) = I_{\{Z_i>t\}} - S(t)$ and $\nu_i = I_{\{\delta_i=1\}} - \theta$, then

$$U_n(t) = \sum_{i=1}^{n}\left\{\frac{\alpha(X_i)}{n^{1/2}} + \frac{\alpha'(X_i)}{n^{3/2}}\right\} + \sum_{1\le i<j\le n}\frac{\beta(X_i,X_j)}{n^{3/2}} + \sum_{1\le i<j<k\le n}\frac{\gamma(X_i,X_j,X_k)}{n^{5/2}} + R_n,$$

where

$$\alpha(X_i) = p_1\omega_i(t) + p_2\nu_i,$$
$$\alpha'(X_i) = \frac{1}{2}\{[p_3(1-2S(t)) + p_6S(t)(1-S(t)) + p_8\theta(1-\theta)]\omega_i(t)$$
$$+2p_4\omega_i(t)\nu_i + [p_5(1-2\theta) + p_7S(t)(1-S(t)) + p_9\theta(1-\theta)p_9]\nu_i(t)\},$$
$$\beta(X_i,X_j) = p_3\omega_i(t)\omega_j(t) + p_4[\omega_i(t)\nu_j + \omega_j(t)\nu_i] + p_5\nu_i\nu_j,$$

and

$$\gamma_1(X_i,X_j,X_k) = p_6\omega_i(t)\omega_j(t)\omega_k(t) + p_9\nu_i\nu_j\nu_k + p_7[\omega_i(t)\omega_j(t)\nu_k + \omega_i(t)\omega_k(t)\nu_j$$
$$+\omega_j(t)\omega_k(t)\nu_i] + p_8[\nu_i\nu_j\omega_k(t) + \nu_i\nu_k\omega_j(t) + \nu_j\nu_k\omega_i(t)].$$

An application of the exponential inequality and exponential bounds for empirical process can be used to show that $P\{|R_n| \ge n^{-1-\varepsilon}\} = o(n^{-1})$ for $0<\varepsilon<1/2$. □

We next derive an Edgeworth expansion for normalized ACL estimator. Since $U_n(t)$ has an asymptotic U-statistic, we show that $\alpha(X_i)$ satisfies Cramér's condition and second-degree U-statistic component satisfies Condition (C) in Lai and Wang(1993).

From the definition of $\alpha(X_i)$, it follows that Cramér's condition holds if F has non-vanishing absolutely continuous component with respect to Lebesque measure. To show that Condition (C) holds, let $f_k(X_j) = [S(t)]^k\{\omega_j(t) + \nu_j\}$ and $X_1 = (Z_1, \delta_1)$. Then

$$W_k = E\{\beta(X_1,X_2)f_k(X_2)|X_1\}$$
$$= \theta(\theta-1)[S(t)]^{k+\theta-1}[2-S(t)+\theta\ln S(t)]\omega_1(t) \tag{3.2}$$
$$+[S(t)]^{k+\theta}\{(1-S(t))(1+\theta\ln S(t)) + \theta(\theta-1)\ln S(t)\}\nu_1 .$$

Hence, for any $K\ge 1$ and constants $a_1,\cdots,a_K$, $\sum_{k=1}^{K} a_kW_k = 0$ $a.s.$ implies that

$$\sum_{k=1}^{K} a_k[S(t)]^k\{\theta(\theta-1)[S(t)]^{\theta-1}[2-S(t)+\theta\ln S(t)]\omega_1(t)\}$$
$$=-\sum_{k=1}^{K} a_k[S(t)]^k\{[S(t)]^{\theta}[(1-S(t))(1+\theta\ln S(t))+\theta(\theta-1)\ln S(t)]\nu_1\} \quad a.s.. \tag{3.3}$$

Taking variance on both sides of (3.3), then

$$\sum_{k=1}^{K}\{a_k[S(t)]^k\}^2\{\theta(\theta-1)[S(t)]^{\theta-1}[2-S(t)+\theta\ln S(t)]\}^2 S(t)(1-S(t))$$
$$=-\sum_{k=1}^{K}\{a_k[S(t)]^k\}^2\{[S(t)]^{\theta}[(1-S(t))(1+\theta\ln S(t))+\theta(\theta-1)\ln S(t)]\}^2\theta(1-\theta). \tag{3.4}$$

Since $S(t)$ is continuous, this implies that $a_1=\cdots=a_K=0$. Note that $\beta_1(X_1,X_2)$ is bounded random variable. Hence Condition (C) holds, and the linear operator L defined by $(Lf)(y)=E\{\beta(y,Y_2)f(Y_2)\}$ has infinitely many nonzero eigenvalues.

From Theorem 3.1 and above results, we develop the Edgeworth expansion of the normalized ACL estimator.

Theorem 3.2 Let the $U_n(t)$ be an asymptotic U-statistic defined by (3.1) and $\sigma(t)=[E\{\alpha^2(X)\}]^{1/2}$. Define $a_3=E\{\alpha^3(X)\}$, $a_4=E\{\alpha^4(X)\}$, $a'=E\{\alpha(X)\alpha'(X)\}$, $b=E\{\alpha(X_1)\alpha(X_2)\beta(X_1,X_2)\}$, $\kappa_3=a_3+3b$, $c=E\{\alpha(X_1)\alpha(X_2)\alpha(X_3)\gamma(X_1,X_2,X_3)\}$, and $\kappa_4=a_4-3\sigma^4+4c+12E\{\alpha^2(X_1)\alpha(X_2)\beta(X_1,X_2)+\alpha(X_1)\alpha(X_2)\beta(X_1,X_3)\beta(X_2,X_3)\}$. In addition, let

$$P_1(z)=\frac{\kappa_3}{6\sigma^3}(z^2-1),$$

and

$$P_2(z)=\left\{a'+\frac{E\beta^2(Y_1,Y_2)}{4}\right\}\frac{z}{\sigma^2}+\frac{\kappa_4}{24\sigma^4}(z^3-3z)+\frac{\kappa_3^{\,2}}{72\sigma^6}(z^5-10z^3+15z).$$

Then the expansion

$$P\left\{\frac{U_n(t)}{\sigma(t)}\le z\right\}=\Phi(z)-n^{-1/2}P_1(z)\phi(z)-n^{-1}P_2(z)\phi(z)+o(n^{-1}) \tag{3.5}$$

holds uniformly for $-\infty<z<\infty$.

Proof. From the result of Theorem 3.1 and (3.2)-(3.4), the asymptotic U-statistic $U_n(t)=\sqrt{n}(\overline{F}_n(t)-\overline{F}(t))$ satisfies the conditions of Theorem 1 of Lai and Wang(1993), and consequently $P\{U_n(t)/\sigma(t)\le z\}=\Phi(z)-n^{-1/2}P_1(z)\phi(z)-n^{-1}P_2(z)\phi(z)+o(n^{-1})$ uniformly for $z\in R$. □

-Let Q_n denote the empirical distribution that puts probability $1/n$ at each $X_i = (Z_i, \delta_i)$, $i=1,\cdots,n$. The bootstrap sample consists of *i.i.d.* random samples $X_1^*,\cdots, X_n^*$ with common distribution Q_n . The simple bootstrap method estimates the sampling distribution $P\{\sqrt{n}(\overline{F}_n(t)-\overline{F}(t))/\sigma(t)\le z\}$ by $P\{\sqrt{n}(\overline{F}_n^*(t)-\overline{F}_n(t))/\sigma_n(t)\le z|X_1,\cdots,X_n\}$. In most applications, $\sqrt{n}(\overline{F}_n(t)-\overline{F}(t))/\sigma(t)$ can be expressed as an asymptotic U-statistic which has a limiting standard normal distribution. The same argument can be used to represent bootstrap statistic as an asymptotic U-statistic. Thus, under the Cramér's condition and Condition (C) of Lai and Wang(1993), asymptotic U-statistics and their simple bootstrap versions have Edgeworth expansions whose difference is of the order $O(n^{-1})$, establishing the second order accuracy of the bootstrap approximation.

Theorem 3.3 Let $U_n^*(t) = \sqrt{n}(\overline{F}_n^*(t) - \overline{F}_n(t))$. Suppose that α satisfies Cramér's condition. Let Q_n put weight $1/n$ on each of the $X_i = (Z_i, \delta_i)$, $i=1,\cdots,n$, and let $X_1^*,\cdots, X_n^*$ be *i.i.d.* random variables with common distribution Q_n . Then

$$\sup |P\{U_n(t)/\sigma(t)\le z\} - P\{ U_n^*(t)/\sigma_n(t)\le z| Q_n\}| = O(n^{-1}) , \tag{3.6}$$

where $\sigma_n^2(t)$ is defined by

$$\sigma_n^2(t) = \theta_n^2[S_n(t)]^{2\theta_n-1}[1-S_n(t)]+\theta_n(1-\theta_n)[S_n(t)]^{2\theta_n}[\ln S_n(t)]^2 . \tag{3.7}$$

Proof. Let $\omega_i^*(t)=I_{\{Z_i^*>t\}}-S_n(t)$ and $\nu_i^* = I_{\{\delta_i^*=1\}} - \theta_n$, then the same argument as that Theorem 3.1 show that $U_n^*(t)$ has the asymptotic U-statistic representation

$$U_n^*(t) = \sum_{i=1}^n\left\{\frac{\widehat{\alpha}(X_i^*)}{n^{1/2}}+\frac{\widehat{A_n}(X_i^*)}{n^{3/2}}\right\} + \sum_{1\le i<j\le n}\frac{\widehat{\beta_n}(X_i^*,X_j^*)}{n^{3/2}} + \sum_{1\le i<j<k\le n}\frac{\widehat{\gamma_n}(X_i^*,X_j^*,X_k^*)}{n^{5/2}} + R_n^*$$

with

$$\widehat{\alpha}_n(X_i^*)=p_1^*\omega_i^*(t)+p_2^*\nu_i^*,$$
$$\widehat{\beta}_n(X_i^*,X_j^*)=p_3^*\omega_i^*(t)\omega_j^*(t)+p_4^*[\omega_i^*(t)\nu_j^*+\omega_j^*(t)\nu_i^*]+p_5^*\nu_i^*\nu_j^*,$$

where

$$p_1^* = \theta_n [S_n(t)]^{\theta_n - 1},$$

$$p_2^* = [S_n(t)]^{\theta_n} \ln S_n(t),$$

$$p_3^* = \theta_n(\theta_n - 1)[S_n(t)]^{\theta_n - 2},$$

$$p_4^* = 2[S_n(t)]^{\theta_n - 1}\{1 + \theta_n \ln S_n(t)\},$$

$$p_5^* = [S_n(t)]^{\theta_n}[\ln S_n(t)]^2.$$

Since $\widehat{A_n}$ and $\widehat{\gamma_n}$ be bounded in absolute values by some nonrandom constant C and $\sup|S_n(t) - S(t)| = O(n^{-1/2}) = \sup|\theta_n - \theta|$, the condition of Theorem 2 of Lai and Wang(1993) are satisfied. Therefore, applying Theorem 2 of Lai and Wang(1993), the proof is complete. □

4. Numerical Results

To investigate the accuracy of the normal, Edgeworth and bootstrap approximations to the exact point in a artificial data setting, we carry out the Monte Carlo simulation studies.

We assume that the distributions of lifetimes are exponential. Since the lifetimes are subject to be censored to the right and the survival function of censoring time is some power of the survival function of lifetime, we set that the censoring times are also distributed as an exponential distribution whose parameters are selected to make censoring rate to be 20, 40, 60 and 80%. Using the IMSL package, we generate n (20, 30 and 50) lifetimes and censoring times from distributions F and G, and calculate $\overline{F}_n(t) = [S_n(t)]^{\theta_n}$ and $\sigma_n(t)$ for fixed time point based on the simulated data set.

The exact value, $P(z)$, of $P\{U_n(t)/\sigma(t) \le z\}$ is computed by the Monte Carlo method using 100,000 simulations and the Edgeworth approximation, $PE(z)$, is one-term Edgeworth correction $\Phi(z) - n^{-1/2}P_1(z)\phi(z)$ to the normal approximation $\Phi(z)$, which is accurate to the order of $O(n^{-1})$ by Theorem 3.1. The bootstrap approximation, $PB(z)$, is based on a single random sample of n observations as described by Dikta and Ghorai(1990) and then 20,000 bootstrap samples for the evaluation of $P\{U_n^*(t)/\sigma_n(t) \le z\}$ by simulation.

The results of these simulations are given in Table 1 - 4. From the tables, we see that the accuracies of Edgeworth and bootstrap approximations even outperform the those of normal approximation. Also, the improvement is particularly apparent when there is non-negligible discrepancy between the exact value and the normal approximation, e.g., at $z = \pm 1.96$ or ± 1.64. and the probabilities of Edgeworth and bootstrap approximations still tend to close the exact value despite heavy censoring. The accuracies of Edgeworth expansion are similar to those of bootstrap approximations in most cases.

Table 1. Values of exact $P(z)$, normal $\Phi(z)$, Edgeworth $PE(z)$ and bootstrap $PB(z)$ approximations for exponential model ($F \sim$ Exp (0.8), t = 0.4, CR = 20%).

	n=20				n=30				n=50			
z	P(z)	Φ(z)	PE(z)	PB(z)	P(z)	Φ(z)	PE(z)	PB(z)	P(z)	Φ(z)	PE(z)	PB(z)
-1.96	0.035	0.025	0.034	0.044	0.035	0.025	0.033	0.036	0.031	0.025	0.031	0.032
-1.64	0.057	0.051	0.060	0.073	0.055	0.051	0.058	0.063	0.056	0.051	0.057	0.058
-1.45	0.084	0.074	0.082	0.101	0.082	0.074	0.081	0.084	0.078	0.074	0.079	0.077
-1.28	0.115	0.100	0.107	0.104	0.107	0.100	0.105	0.106	0.103	0.100	0.104	0.105
0.00	0.505	0.500	0.478	0.448	0.493	0.500	0.482	0.504	0.490	0.500	0.486	0.489
1.28	0.915	0.900	0.906	0.917	0.911	0.900	0.905	0.907	0.906	0.900	0.904	0.908
1.45	0.939	0.926	0.935	0.945	0.939	0.926	0.933	0.934	0.934	0.926	0.931	0.931
1.64	0.963	0.950	0.959	0.972	0.958	0.950	0.957	0.958	0.957	0.950	0.956	0.958
1.96	0.984	0.975	0.984	0.996	0.983	0.975	0.983	0.983	0.981	0.975	0.981	0.982

Table 2. Values of exact $P(z)$, normal $\Phi(z)$, Edgeworth $PE(z)$ and bootstrap $PB(z)$ approximations for exponential model ($F \sim$ Exp (0.6), t = 0.4, CR = 40%).

	n=20				n=30				n=50			
z	P(z)	Φ(z)	PE(z)	PB(z)	P(z)	Φ(z)	PE(z)	PB(z)	P(z)	Φ(z)	PE(z)	PB(z)
-1.96	0.041	0.025	0.040	0.045	0.034	0.025	0.037	0.043	0.035	0.025	0.034	0.035
-1.64	0.064	0.051	0.066	0.064	0.066	0.051	0.063	0.065	0.059	0.051	0.060	0.062
-1.45	0.091	0.074	0.087	0.104	0.087	0.074	0.085	0.090	0.079	0.074	0.082	0.084
-1.28	0.120	0.100	0.110	0.108	0.117	0.100	0.108	0.114	0.105	0.100	0.107	0.106
0.00	0.460	0.500	0.465	0.492	0.479	0.500	0.471	0.466	0.483	0.500	0.478	0.493
1.28	0.908	0.900	0.909	0.915	0.911	0.900	0.908	0.912	0.904	0.900	0.906	0.906
1.45	0.948	0.926	0.940	0.950	0.940	0.926	0.938	0.941	0.935	0.926	0.935	0.937
1.64	0.968	0.950	0.965	0.979	0.965	0.950	0.962	0.970	0.962	0.950	0.959	0.959
1.96	0.992	0.975	0.990	0.996	0.987	0.975	0.987	0.991	0.986	0.975	0.984	0.985

Table 3. Values of exact $P(z)$, normal $\Phi(z)$, Edgeworth $PE(z)$ and bootstrap $PB(z)$ approximations for exponential model ($F \sim$ Exp (0.4), t = 0.4, CR = 60%).

	n=20				n=30				n=50			
z	P(z)	Φ(z)	PE(z)	PB(z)	P(z)	Φ(z)	PE(z)	PB(z)	P(z)	Φ(z)	PE(z)	PB(z)
-1.96	0.045	0.025	0.046	0.049	0.042	0.025	0.042	0.040	0.038	0.025	0.038	0.035
-1.64	0.076	0.051	0.073	0.067	0.070	0.051	0.069	0.063	0.064	0.051	0.065	0.064
-1.45	0.093	0.074	0.093	0.101	0.091	0.074	0.090	0.093	0.083	0.074	0.086	0.083
-1.28	0.118	0.100	0.115	0.121	0.120	0.100	0.112	0.109	0.109	0.100	0.109	0.105
0.00	0.466	0.500	0.449	0.453	0.474	0.500	0.458	0.485	0.478	0.500	0.468	0.481
1.28	0.927	0.900	0.914	0.922	0.913	0.900	0.911	0.919	0.907	0.900	0.909	0.908
1.45	0.960	0.926	0.946	0.954	0.947	0.926	0.943	0.949	0.943	0.926	0.939	0.939
1.64	0.982	0.950	0.972	0.978	0.970	0.950	0.968	0.973	0.968	0.950	0.964	0.960
1.96	0.997	0.975	0.996	0.997	0.993	0.975	0.992	0.992	0.990	0.975	0.988	0.986

Table 4. Values of exact $P(z)$, normal $\Phi(z)$, Edgeworth $PE(z)$ and bootstrap $PB(z)$ approximations for exponential model ($F \sim \text{Exp}(0.2)$, $t = 0.4$, $CR = 80\%$).

	n=20				n=30				n=50			
z	P(z)	Φ(z)	PE(z)	PB(z)	P(z)	Φ(z)	PE(z)	PB(z)	P(z)	Φ(z)	RE(z)	PB(z)
-1.96	0.060	0.025	0.054	0.053	0.048	0.025	0.049	0.052	0.046	0.025	0.044	0.043
-1.64	0.073	0.051	0.081	0.063	0.080	0.051	0.076	0.074	0.076	0.051	0070	0.072
-1.45	0.096	0.074	0.101	0.104	0.086	0.074	0.096	0.095	0.089	0.074	0.091	0.087
-1.28	0.122	0.100	0.120	0.111	0.124	0.100	0.116	0.120	0.113	0.100	0.113	0.119
0.00	0.437	0.500	0.430	0.439	0.467	0.500	0.443	0.474	0.460	0.500	0.456	0.476
1.28	0.946	0.900	0.920	0.918	0.920	0.900	0.916	0.932	0.918	0.900	0.912	0.923
1.45	0.975	0.926	0.954	0.959	0.959	0.926	0.949	0.959	0.950	0.926	0.944	0.948
1.64	0.988	0.950	0.980	0.983	0.980	0.950	0.975	0.981	0.972	0.950	0.969	0.976
1.96	0.999	0.975	0.999	0.997	0.998	0.975	0.999	0.997	0.994	0.975	0.994	0.995

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