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Bayes Factors for Independence and Symmetry in Freund's Bivariate Exponential Model with Censored Data

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Abstract

In this paper, we consider the Bayesian hypotheses testing for independence and symmetry in Freund's bivariate exponential model with censored data. In Bayesian testing problem, we use the noninformative priors for parameters which are improper and are defined only up to arbitrary constants. And we use the recently proposed hypotheses testing criterion called the intrinsic Bayes factor. Also we derive the arithmetic and median intrinsic Bayes factors and use these results to analyze some data sets.

1. Introduction

Let's consider a life testing experiment in which multiple two-component shared parallel systems are put on test. In many cases of life testing and reliability analysis, two components are assumed to have independent life time distributions. However, in many life testing situations it is more realistic to assume some form of positive dependence among components. This positive dependece among component life time arises from common environmental stresses and shocks, from components depending on common sources of power, and so on. As an example, we consider the paired organs like kidneys, eyes, ears or any other paired organs in an individual as two component system. In these cases, each paired organ is correlated each other. Freund (1961) formulated a bivariate extension of the exponential model as a model for a system where the failure times of the two components may depend on each other.

For complete data set, Kunchur and Munoli (1994) obtained minimum variance unbiased estimator for the system reliability. Hanagal (1996) et.al. obtained estimator of system reliability from stress-strength relationship. Hanagal and Kale (1992) considered statistical hypothesis testing for independence and symmetry from a frequentist viewpoint.

In Bayesian testing problem, the Bayes factor depend rather strongly on the prior

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distributions, much more so than in, say, estimation. So, the Bayes factor under proper priors have been very successful. Frequently, however, elicitation of subjective prior distributions is impossible, because of time or cost limitations, or resistance or lack of training of clients. Also subjective elicitation can easily result in poor prior distribution and statistical analysis is often required to appear objective. So, the literature on noninformative priors has grown enormously over recent years. There have been several excellent books or review articles that have been concerned with discussing or comparing different approaches to developing noninformative priors(See Ghosh and Mukerjee, 1992).

However, noninformative priors such as Jeffrey's (1961) priors or reference priors (Berger and Bernardo (1989,1992)) are typically improper so that such priors are defined only up to arbitrary constants which affects the values of Bayes factors. So, Geisser and Eddy (1979), Spiegalhalter and Smith (1982), San Martini and Spezzaferri (1984) and O'Hagan (1995) have made efforts to compensate for that arbitrariness.

Berger and Pericchi (1996b) introduced a new model selection and hypotheses testing criterion, called the Intrinsic Bayes Factors (IBF's) using a data-splitting idea, which would eliminate the arbitrariness of improper priors. These can be constructed in very general situations-nested, nonnested, and even irregular problems-and they seem to correspond to actual Bayes factors, at least asymptotically. This approach has shown to be quite useful (Berger and Pericchi (1996a), Varshavsky (1996) and Lingham and Sivaganesan (1997)).

In this paper, we consider a Bayesian approach to test independence and symmetry in Freund's bivariate exponential model with bivariate censored data as extension of complete data. Here we use noninformative priors as improper priors. Also we derive intrinsic Bayes factors to solve our problem and give some numerical results to illustrate our results.

2. Preliminaries

The random variables (X, Y) are said to follow Freund's bivariate exponential model with parameters $\theta=(\alpha, \alpha', \beta, \beta')$ if the joint probability density function is given as

$$f(x, y : \theta) = \begin{cases} \alpha\beta' \exp[-\beta' y-(\alpha+\beta-\beta')x], & y>x>0, \\ \alpha'\beta \exp[-\alpha' x-(\alpha+\beta-\alpha')y], & x>y>0. \end{cases} \quad (2.1)$$

As a matter of convenience, we introduce the following notations.

t_{x_i}, $i=1,2,\cdots,n$: fixed censoring time of ith observation for the first component X.

t_{y_i}, $i=1,2,\cdots,n$: fixed censoring time of ith observation for the second component Y.

$G_{1i}=I(X_i>t_{x_i})$, $G^*_{1i}=1-G_{1i}$, $i=1,2,\cdots,n$.

$G_{2i}=I(Y_i>t_{y_i}),\ \ G_{2i}^*=1-G_{2i},\ \ i=1,2,\cdots,n.$

$R_i=I(X_i<Y_i),\ \ R_i^*=1-R_i,\ \ i=1,2,\cdots,n.$

In system testing, we might observe:

(1) failures of both components($G_{1i}^*G_{2i}^*=1$).

(2) failure of one component and censoring of other component($G_{1i}G_{2i}^*+G_{1i}^*G_{2i}=1$).

(3) censoring of both components($G_{1i}G_{2i}=1$).

Then lifetime (x_i,y_i) of ith item is observed as follows;

$$(x_i,y_i)=(\min(x_i,t_{x_i}),\ \ \min(y_i,t_{y_i})). \tag{2.2}$$

Hence, likelihood function is given as

$$\begin{aligned}f(\mathbf{x},\mathbf{y}\mid\theta)=&\prod_{i=1}^{n}\{[f(x_i,y_i)]^{G_{1i}^*G_{2i}^*}\cdot[\overline{F}(x_i,y_i)]^{G_{1i}G_{2i}}\cdot\\&[\overline{F}_{X\mid Y=y}(x_i)f_Y(y_i)]^{G_{1i}G_{2i}^*}\cdot[\overline{F}_{Y\mid X=x}(y_i)f_X(x_i)]^{G_{1i}^*G_{2i}}\}^{(R_i+R_i^*)}\\=&\alpha^{n_1+n_4}\cdot\beta^{n_2+n_5}\cdot\alpha'^{n_2}\cdot\beta'^{n_1}\cdot\exp[-\alpha(\sum_{i\in S_{14}}x_i+\sum_{i\in S_{25}}y_i+\sum_{i\in S_6}(x_i+y_i))]\\&\cdot\exp[-\beta(\sum_{i\in S_{14}}x_i+\sum_{i\in S_{25}}y_i+\sum_{i\in S_6}(x_i+y_i))]\cdot\exp\Big[-\alpha'(\sum_{i\in S_{25}}x_i-\sum_{i\in S_2}y_i)\Big]\\&\cdot\exp\Big[-\beta'(\sum_{i\in S_{14}}(y_i-x_i))\Big].\end{aligned}\tag{2.3}$$

Here, $n_1=\sum_{i=1}^{n}R_iG_{1i}^*G_{2i}^*,\ \ n_2=\sum_{i=1}^{n}R_i^*G_{1i}^*G_{2i}^*,\ \ n_4=\sum_{i=1}^{n}R_iG_{1i}^*G_{2i},\ \ n_5=\sum_{i=1}^{n}R_i^*G_{1i}G_{2i}^*,$

$n_6=\sum_{i=1}^{n}G_{1i}G_{2i},\ \ S_{14}=\{i\mid R_iG_{1i}^*G_{2i}^*=1\ \text{ or }\ R_iG_{1i}^*G_{2i}=1\},$

$S_{25}=\{i\mid R_i^*G_{1i}^*G_{2i}^*=1\ \text{ or }\ R_i^*G_{1i}G_{2i}^*=1\},\ \ S_2=\{i\mid R_i^*G_{1i}^*G_{2i}^*=1\},$
$S_6=\{i\mid G_{1i}G_{2i}=1\}.$

We can obtain the maximum likelihood estimators for the parameters as follows;

$$\hat{\alpha}=\frac{n_1+n_4}{\sum_{i\in S_{14}}x_i+\sum_{i\in S_{25}}y_i+\sum_{i\in S_6}(x_i+y_i)},\quad \hat{\beta}=\frac{n_2+n_5}{\sum_{i\in S_{14}}x_i+\sum_{i\in S_{25}}y_i+\sum_{i\in S_6}(x_i+y_i)},$$

$$\hat{\alpha}'=\frac{n_2}{\sum_{i\in S_{25}}x_i-\sum_{i\in S_2}y_i},\quad \hat{\beta}'=\frac{n_1}{\sum_{i\in S_{11}}(y_i-x_i)}.$$

$\sqrt{n}(\hat{\theta}-\theta)$ has the asymptotic multivariate normal distribution with mean vector zero and covariance matrix $\Sigma=((I^{ij}),\ i,j=1,2,3,4)$ where I^{ij} is the (i,j)th element of inverse matrix of Fisher information $(I=(I_{ij}),i,j=1,2,3,4)$ with

$$I_{11}=\frac{n_1+n_4}{\alpha^2},\quad I_{22}=\frac{n_2+n_5}{\beta^2},\quad I_{33}=\frac{n_2}{\alpha'^2},\quad I_{44}=\frac{n_1}{\beta'^2}\ \text{and}\ I_{ij}=0,\ \text{for}\ i\neq j=1,2,3,4.$$

Now, we introduce the intrinsic Bayes factor in the general hypotheses testing. As a matter of convenience, we introduce the following notations.

$\boldsymbol{X}=(X_1,\cdots,X_n)$: observation with density $f(\boldsymbol{x}\mid\theta)$, where $\theta\in\Theta$ is a finite dimensional parameter and Θ is parameter space.

Θ_i: parameter space under ith hypothesis H_i , $i=1,2,\cdots,q$.

$f(x\mid\theta_i)$: the density under H_i , $i=1,2,\cdots,q$.

$\pi_i(\theta_i)$: the prior distribution under H_i , $i=1,2,\cdots,q$.

$m_i(\boldsymbol{x})$: the marginal density of $\boldsymbol{X}$ under H_i when use $\pi_i(\theta_i)$, $i=1,2,\cdots,q$.

p_i : the prior probability of H_i being true, $i=1,2,\cdots,q$.

$\pi_i^N(\theta_i)$: the improper prior distribution under H_i , $i=1,2,\cdots,q$.

$m_i^N(\boldsymbol{x})$: the marginal density of $\boldsymbol{X}$ under H_i when use $\pi_i^N(\theta_i)$, $i=1,2,\cdots,q$.

Then $\pi_i^N(\theta_i)$ is usually written as $\pi_i^N(\theta_i)\ \propto h_i(\theta_i)$, where h_i is a function whose integral over the Θ_i-space diverges. Formally, we can write $\pi_i^N(\theta_i)=c_i\,h_i(\theta_i)$, although the normalizing constant c_i does not exist, but treating it as an unspecified constant.

The posterior probability that H_i is true is given as

$$P(H_i\mid\boldsymbol{x})=\Big(\sum_{j=1}^{q}\frac{p_j}{p_i}B_{ji}\Big)^{-1}, \tag{2.4}$$

where B_{ji}, the Bayes factor of H_j to H_i, is defined by

$$B_{ji}=\frac{m_j(\boldsymbol{x})}{m_i(\boldsymbol{x})}=\frac{\int_{\Theta_j}f(\boldsymbol{x}\mid\theta_j)\pi_j(\theta_j)d\theta_j}{\int_{\Theta_i}f(\boldsymbol{x}\mid\theta_i)\pi_i(\theta_i)d\theta_i}. \tag{2.5}$$

The posterior probabilities in (2.4) are then used to select the most plausible hypothesis.
If one were to use some noninformative priors, then (2.5) becomes

$$B_{ji}^N=\frac{m_j^N(\boldsymbol{x})}{m_i^N(\boldsymbol{x})}=\frac{\int_{\Theta_j}f(\boldsymbol{x}\mid\theta_j)\pi_j^N(\theta_j)d\theta_j}{\int_{\Theta_i}f(\boldsymbol{x}\mid\theta_i)\pi_i^N(\theta_i)d\theta_i}. \tag{2.6}$$

Hence, the corresponding Bayes factor, B_{ji}^N, is indeterminate. One solution to this indeterminancy problem is to use part of the data as a training sample. Let $\mathbf{x}(l)$ denote the part of the data to be so used and let $\mathbf{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\mathbf{x}(l)) < \infty, \quad i=1,\cdots,q. \tag{2.7}$$

In view (2.7), the posteriors $\pi_i^N(\theta_i \mid \mathbf{x}(l))$ are well defined. Now, consider the Bayes factor, $B_{ji}(l)$, for the rest of the data $\mathbf{x}(-l)$, using $\pi_i^N(\theta_i \mid \mathbf{x}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int_{\Theta_j} f(\mathbf{x}(-l) \mid \theta_j, \mathbf{x}(l)) \pi_j^N(\theta_j \mid \mathbf{x}(l)) d\theta_j}{\int_{\Theta_i} f(\mathbf{x}(-l) \mid \theta_i, \mathbf{x}(l)) \pi_i^N(\theta_i \mid \mathbf{x}(l)) d\theta_i} = B_{ji}^N \times B_{ij}^N(\mathbf{x}(l)), \tag{2.8}$$

where B_{ji}^N is given by (2.6) and

$$B_{ij}^N(\mathbf{x}(l)) = \frac{m_i^N(\mathbf{x}(l))}{m_j^N(\mathbf{x}(l))}. \tag{2.9}$$

In (2.8), any arbitrary ratio, c_j/c_i say, that multiples B_{ji}^N would be cancelled by the ratio c_i/c_j forming the multiplicand in $B_{ij}^N(\mathbf{x}(l))$. Also, while the expression (2.9) renders $B_{ji}(l)$ in terms of the simpler marginal densities of $\mathbf{x}(l)$.

As training samples, Arithmetic and Median Intrinsic Bayes Factor play a fundamental role in our testing $H_i, i=1,\cdots,q$, we introduce the following definitions.

Definition 1.(Berger and Pericchi(1996b)) A training sample $\mathbf{x}(l)$, will called *proper* if (2.7) holds and *minimal* if it is proper and none of its subsets is proper.

Definition 2.(Berger and Pericchi(1996b)) The Arithmetic Intrinsic Bayes factor of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^N \cdot \frac{1}{L} \sum_{l=1}^{L} B_{ij}^N(\mathbf{x}(l)), \tag{2.10}$$

where L is the number of all possible minimal training samples.

Definition 3.(Berger and Pericchi(1998)) The Median Intrinsic Bayes factor of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^N \cdot ME[B_{ij}^N(\mathrm{x}(l))], \tag{2.11}$$

where ME indicates the median, here to be taken over all the training sample Bayes factors.

We can also calculate the posterior probability of H_i using (2.4), where B_{ji} are replaced by B_{ji}^{AI} and B_{ji}^{MI} from (2.10) and (2.11).

3. Bayesian Hypothesis Test

In Freund's bivariate exponential model, we want to test the hypotheses of symmetry and independence test. That is, the hypotheses of symmetry is $H_1 : \alpha = \beta$, $\alpha' = \beta'$ v.s. H_2 : not H_1, and the hypotheses of independence is $H_3 : \alpha = \alpha'$, $\beta = \beta'$ v.s. H_4 : not H_3. Consider samples of sizes n from Freund's model with parameters $\underline{\theta} = (\alpha, \alpha', \beta, \beta')$.

To test the hypothesis of symmetry based on the M.L.E.'s, Hanagal and Kale(1992) obtained the test statistics as follows;

$$\text{if } \frac{n(\hat{\alpha}-\hat{\beta})^2}{(\hat{\alpha}+\hat{\beta})^2} + \frac{n(\hat{\alpha}'-\hat{\beta}')^2}{(\hat{\alpha}+\hat{\beta})\left[\frac{\hat{\alpha}'^2}{\hat{\beta}}+\frac{\hat{\beta}'^2}{\hat{\alpha}}\right]} \geq \chi^2_{(2,1-\gamma)},$$

then reject H_1 with significance level γ.

In similar method, to test the hypothesis of independence based on the M.L.E.'s, they obtained the test statistics as follows;

$$\text{if } \frac{n(\hat{\alpha}-\hat{\alpha}')^2\hat{\beta}}{(\hat{\alpha}+\hat{\beta})(\hat{\alpha}\hat{\beta}+\hat{\alpha}'^2)} + \frac{n(\hat{\beta}-\hat{\beta}')^2\hat{\alpha}}{(\hat{\alpha}+\hat{\beta})(\hat{\alpha}\hat{\beta}+\hat{\beta}'^2)} \geq \chi^2_{(2,1-\gamma)},$$

then reject H_3 with significance level γ.

3.1 Symmetry Test

Here, the goal is to determine the set of all possible minimal training sample(MTS) for the data $(\boldsymbol{x}, \boldsymbol{y})$ to test $H_1 : \alpha = \beta$, $\alpha' = \beta'$ v.s. H_2 : not H_1. Here, $\theta_1 = (\alpha, \alpha')$ and $\theta_2 = (\alpha, \alpha', \beta, \beta')$.

The noninformative priors for $H_1 : \alpha = \beta$, $\alpha' = \beta'$ v.s. H_2 : not H_1 are respectively given by

$$\pi_1^N(\theta_1) = \frac{1}{\alpha\alpha'} \tag{3.1}$$

and

$$\pi_2^N(\theta_2) = \frac{1}{\alpha\alpha'\beta\beta'}. \tag{3.2}$$

To derive the marginals with respect to the noninformative priors given by (3.1) and (3.2), we first observe that the joint pdf of $(\boldsymbol{x}, \boldsymbol{y})$ is given by

$$f(\boldsymbol{x}, \boldsymbol{y}) = \alpha^{n_1+n_4} \cdot \beta^{n_2+n_5} \cdot \alpha'^{n_2} \cdot \beta'^{n_1} \cdot \exp[-\alpha(\sum_{i\in S_{14}} x_i + \sum_{i\in S_{25}} y_i + \sum_{i\in S_6}(x_i+y_i))]$$

$$\cdot \exp[-\beta(\sum_{i\in S_{14}} x_i + \sum_{i\in S_{25}} y_i + \sum_{i\in S_6}(x_i+y_i))] \cdot \exp\Big[-\alpha'(\sum_{i\in S_{25}} x_i - \sum_{i\in S_{25}} y_i)\Big]$$
$$\cdot \exp\Big[-\beta'(\sum_{i\in S_{14}}(y_i - x_i))\Big]. \tag{3.3}$$

Moreover, the joint pdf of any four paired observations, $((x_j, y_j), (x_k, y_k), (x_l, y_l), (x_m, y_m))$,$1\le j< k< l< m\le n$, is given by

$$\prod_{i\in\{j,k,l,m\}} f(x_i, y_i) = \prod_{i\in\{j,k,l,m\}} \{[f(x_i, y_i)]^{G^*_{1i}G^*_{2i}} \cdot [\overline{F}(x_i, y_i)]^{G_{1i}G_{2i}}$$
$$\cdot [\overline{F}_{X\mid Y=y}(x_i) f_Y(y_i)]^{G_{1i}G^*_{2i}} \cdot [\overline{F}_{Y\mid X=x}(y_i) f_X(x_i)]^{G^*_{1i}G_{2i}}\}^{(R_i+R^*_i)}$$
$$= \alpha^{n'_1+n'_4} \cdot \beta^{n'_2+n'_5} \cdot \alpha'^{n'_2} \cdot \beta'^{n'_1} \cdot \exp[-\alpha(\sum_{i\in S'_{14}} x_i + \sum_{i\in S'_{25}} y_i + \sum_{i\in S'_6}(x_i+y_i))]$$
$$\cdot \exp[-\beta(\sum_{i\in S'_{14}} x_i + \sum_{i\in S'_{25}} y_i + \sum_{i\in S'_6}(x_i+y_i))] \cdot \exp\Big[-\alpha'(\sum_{i\in S'_{25}} x_i - \sum_{i\in S'_{25}} y_i)\Big]$$
$$\cdot \exp\Big[-\beta'(\sum_{i\in S'_{14}}(y_i - x_i))\Big]. \tag{3.4}$$

Here, $n'_1 = \sum_{i\in\{j,k,l,m\}} R_i G^*_{1i} G^*_{2i}$, $n'_2 = \sum_{i\in\{j,k,l,m\}} R^*_i G^*_{1i} G^*_{2i}$, $n'_4 = \sum_{i\in\{j,k,l,m\}} R_i G^*_{1i} G_{2i}$,

$n'_5 = \sum_{i\in\{j,k,l,m\}} R^*_i G_{1i} G^*_{2i}$, $n'_6 = \sum_{i\in\{j,k,l,m\}} G_{1i} G_{2i}$,

$S'_{14} = \{i \mid R_i G^*_{1i} G^*_{2i} = 1 \text{ or } R_i G^*_{1i} G_{2i} = 1, \ i\in\{j,k,l,m\}\}$,

$S'_{25} = \{i \mid R^*_i G^*_{1i} G^*_{2i} = 1 \text{ or } R^*_i G_{1i} G^*_{2i} = 1, \ i\in\{j,k,l,m\}\}$,

$S'_2 = \{i \mid R^*_i G^*_{1i} G^*_{2i} = 1, \ i\in\{j,k,l,m\}\}$, $S_6 = \{i \mid G_{1i} G_{2i} = 1\}$.

In the following lemma, we give the marginal densities for any four paired observations.

Lemma 1. For the minimal training sample case, we have the marginal density $m^N_i((x_j, y_j), (x_k, y_k), (x_l, y_l), (x_m, y_m))$, $i=1,2$ under $H_i, i=1,2$ as follows.

$$m^N_1((x_j, y_j), (x_k, y_k), (x_l, y_l), (x_m, y_m))$$
$$= \Gamma(n'_1+n'_2+n'_4+n'_5) \cdot \Gamma(n'_1+n'_2)$$
$$\cdot \left(\frac{1/2}{\sum_{i\in S_{14}} x_i + \sum_{i\in S_{25}} x_i + \sum_{i\in S_6}(x_i+y_i)}\right)^{n'_1+n'_2+n'_4+n'_5} \cdot \left(\frac{1}{\sum_{i\in S_{14}}(y_i-x_i) + \sum_{i\in S_{25}}(x_i-y_i)}\right)^{n'_1+n'_2} \tag{3.5}$$

and

$$m^N_2((x_j, y_j), (x_k, y_k), (x_l, y_l), (x_m, y_m))$$
$$= \Gamma(n'_1+n'_4) \cdot \Gamma(n'_2+n'_5) \cdot \Gamma(n'_2) \cdot \Gamma(n'_1)$$

$$\cdot\left(\frac{1}{\sum_{i\in S_{14}} x_i+\sum_{i\in S_{25}} y_i+\sum_{i\in S_6}(x_i+y_i)}\right)^{n_1'+n_4'}\cdot\left(\frac{1}{\sum_{i\in S_{14}} x_i+\sum_{i\in S_{25}} y_i+\sum_{i\in S_6}(x_i+y_i)}\right)^{n_2'+n_5'}$$

$$\cdot\left(\frac{1}{\sum_{i\in S_{25}}(x_i-y_i)}\right)^{n_2'}\cdot\left(\frac{1}{\sum_{i\in S_{14}}(y_i-x_i)}\right)^{n_1'}. \qquad (3.6)$$

Since the marginal densities $m_1^N((x_j,y_j),(x_k,y_k),(x_l,y_l),(x_m,y_m))$ and $m_2^N((x_j,y_j),(x_k,y_k),(x_l,y_l),(x_m,y_m))$ are finite for all $1\le j< k< l< m\le n$ under each hypothesis, we conclude that any training sample of size four is an MTS. That is, the marginal distribution is improper when the size of MTS is less 4. To prove this fact, we must prove that the marginal distribution is improper when the size of MTS is 3. If the size of MTS is equal to 3, that is, ($n_1'+n_2'+n_4'+n_5'+n_6'=3$), we can see that $n_1'+n_4'\le 1$ and $n_2'+n_5'=2$, or $n_1'+n_4'=2$ and $n_2'+n_5'\le 1$. In the case of $n_1'+n_4'\le 1$ and $n_2'+n_5'=2$, we can see that the marginal distribution of (3.4) with respect to the noninformative priors given by (3.1) and (3.2) is infinite. In the case of $n_1'+n_4'=2$ and $n_2'+n_5'\le 1$, we can check easily by similar method.

The marginal densities corresponding to the full data $(\boldsymbol{X},\boldsymbol{Y})$ for test $H_1:\alpha=\beta$, and $\alpha'=\beta'$ v.s. $H_2:$ not H_1 can also be expressed in the following lemma.

Lemma 2. For the full data, we have the marginal density $m_i^N(\boldsymbol{x},\boldsymbol{y})$ under $H_i, i=1,2$ as follows.

$$m_1^N(\boldsymbol{x},\boldsymbol{y})$$
$$=\Gamma(n_1+n_2+n_4+n_5)\cdot\Gamma(n_1+n_2)$$
$$\cdot\left(\frac{1/2}{\sum_{i\in S_{14}} x_i+\sum_{i\in S_{25}} x_i+\sum_{i\in S_6}(x_i+y_i)}\right)^{n_1+n_2+n_4+n_5}\cdot\left(\frac{1}{\sum_{i\in S_{14}}(y_i-x_i)+\sum_{i\in S_{25}}(x_i-y_i)}\right)^{n_1+n_2} \qquad (3.7)$$

and

$$m_2^N(\boldsymbol{x},\boldsymbol{y})$$
$$=\Gamma(n_1+n_4)\cdot\Gamma(n_2+n_5)\cdot\Gamma(n_2)\cdot\Gamma(n_1)$$

$$\cdot\left(\frac{1}{\sum_{i\in S_{14}} x_i+\sum_{i\in S_{25}} y_i+\sum_{i\in S_6}(x_+y_i)}\right)^{n_1+n_4}\cdot\left(\frac{1}{\sum_{i\in S_{14}} x_i+\sum_{i\in S_{25}} y_i+\sum_{i\in S_6}(x_+y_i)}\right)^{n_2+n_5}$$

$$\cdot\left(\frac{1}{\sum_{i\in S_{25}} x_i-\sum_{i\in S_{25}} y_i}\right)^{n_2}\cdot\left(\frac{1}{\sum_{i\in S_{14}}(y_i-x_i)}\right)^{n_1}. \qquad (3.8)$$

To test $H_1 : \alpha=\beta, \alpha^{'}=\beta^{'}$ v.s. H_2 : not H_1, we get the following theorem from Lemmas 1 and 2.

Theorem 1. (i) The Bayes factor using the full data is given by

$$B_{21}^N = \frac{m_2^N((\boldsymbol{x}, \boldsymbol{y}))}{m_1^N((\boldsymbol{x}, \boldsymbol{y}))}. \tag{3.9}$$

(ii) The Bayes factor using the minimal training sample $(\boldsymbol{x}, \boldsymbol{y})(l) = ((x_j, y_j), (x_k, y_k), (x_l, y_l), (x_m, y_m))$ is given by

$$B_{12}^N((\boldsymbol{x}, \boldsymbol{y})(l)) = \frac{m_1^N((\boldsymbol{x}, \boldsymbol{y})(l))}{m_2^N((\boldsymbol{x}, \boldsymbol{y})(l))}. \tag{3.10}$$

From the Theorem 1, the arithmatic intrinsic Bayes factor B_{21}^{AI} to test $H_1 : \alpha=\beta$, $\alpha^{'}=\beta^{'}$ v.s. H_2 : not H_1 is given by

$$B_{21}^{AI} = B_{21}^N \cdot \frac{1}{\binom{n}{4}} \sum_l B_{12}^N((\boldsymbol{x}, \boldsymbol{y})(l)). \tag{3.11}$$

Next we use the another intrinsic Bayes factor called median intrinsic Bayes factor(Berger and Pericchi(1998)). They showed that the median intrinsic Bayes factor seems to be a simple and very generally applicable intrinsic Bayes factor, which works well for nested or non-nested models, and even for small or moderate sample sizes.

From the Definition 3, Lemma 1, Lemma 2 and Theorem 1, we derive the median Bayes factors to test $H_1 : \alpha=\beta, \alpha^{'}=\beta^{'}$ v.s. H_2 : not H_1 as follow:

$$B_{21}^{MI} = B_{21}^N \cdot ME[B_{12}^N((\boldsymbol{x}, \boldsymbol{y})(l))]. \tag{3.12}$$

3.2 Independence Test

The goal here is to determine the set of all possible minimal training sample(MTS) for the data $(\boldsymbol{x}, \boldsymbol{y})$ to test $H_3 : \alpha=\alpha^{'}, \beta=\beta^{'}$ v.s. H_4 : not H_3.

To test $H_3 : \alpha=\alpha^{'}, \beta=\beta^{'}$ v.s. H_4 : not H_3, we must to determine the set of all possible MTS's for the data $(\boldsymbol{x}, \boldsymbol{y})$. Here, $\theta_3=(\alpha, \beta)$ and $\theta_4=(\alpha, \alpha^{'}, \beta, \beta^{'})$. The noninformative priors for $H_3 : \alpha=\alpha^{'}, \beta=\beta^{'}$ v.s. H_4 : not H_3 are respectively given by

$$\pi_3^N(\theta_3) = \frac{1}{\alpha\beta} \tag{3.13}$$

and

$$\pi_4^N(\theta_4) = \frac{1}{\alpha\alpha^{'}\beta\beta^{'}}. \tag{3.14}$$

In the following lemma, we now derive the marginals with respect to the noninformative priors given by (3.13) and (3.14).

Lemma 3. For the minimal training sample case, we have the marginal density $m_i^N((x_j, y_j),(x_k, y_k),(x_l, y_l), (x_m, y_m))$, $i=3,4$ under $H_i, i=3,4$ as follows.

$$\begin{aligned} & m_3^N((x_j, y_j),(x_k, y_k),(x_l, y_l),(x_m, y_m)) \\ & = \Gamma(n_1^{'}+n_2^{'}+n_4^{'})\cdot\Gamma(n_1^{'}+n_2^{'}+n_5^{'})\cdot\left(\frac{1}{\sum_{i\in S_{14}^{'}} x_i+\sum_{i\in S_{25}^{'}} x_i+\sum_{i\in S_6^{'}}(x_i+y_i)}\right)^{n_1^{'}+n_2^{'}+n_4^{'}} \\ & \cdot\left(\frac{1}{\sum_{i\in S_{14}^{'}} y_i+\sum_{i\in S_{25}^{'}} y_i+\sum_{i\in S_6^{'}}(x_i+y_i)}\right)^{n_1^{'}+n_2^{'}+n_5^{'}} \end{aligned} \tag{3.15}$$

and $m_4^N((x_j, y_j),(x_k, y_k),(x_l, y_l),(x_m, y_m))$ is the same as that of Lemma 1.

Also, we conclude that any training sample of size three is an MTS.

Nextly the marginal densities corresponding to the full data (X, Y) for test $H_3 : \alpha=\alpha^{'}$, $\beta=\beta^{'}$ v.s. H_4 : not H_3 can also be expressed in the following lemma.

Lemma 4. For the full data, we have the marginal density $m_i^N(x, y)$, $i=3,4$ under $H_i, i=3,4$ as follow.

$$\begin{aligned} m_3^N((x, y)) &= \Gamma(n_1+n_2+n_4)\cdot\Gamma(n_1+n_2+n_5)\cdot\left(\frac{1}{\sum_{i\in S_{14}} x_i+\sum_{i\in S_{25}} x_i+\sum_{i\in S_6}(x_i+y_i)}\right)^{n_1+n_2+n_4} \\ & \cdot\left(\frac{1}{\sum_{i\in S_{11}} y_i+\sum_{i\in S_{25}} y_i+\sum_{i\in S_6}(x_i+y_i)}\right)^{n_1+n_2+n_5} \end{aligned} \tag{3.16}$$

and $m_4^N((x, y))$ is the same as that of Lemma 2.

Nextly we get the following theorem from Lemmas 3 and 4.

Theorem 2. (i) The Bayes factor using the full data is given by

$$B_{43}^N = \frac{m_4^N((x, y))}{m_3^N((x, y))}. \tag{3.17}$$

(ii) The Bayes factor using the $(\mathbf{x}, \mathbf{y})(l) = ((x_j, y_j), (x_k, y_k), (x_l, y_l), (x_m, y_m))$ is given by

$$B_{34}^{N}((\mathbf{x}, \mathbf{y})(l)) = \frac{m_3^N((\mathbf{x}, \mathbf{y})(l))}{m_4^N((\mathbf{x}, \mathbf{y})(l))}. \tag{3.18}$$

From the Theorem 2, the arithematic and median intrinsic Bayes factor B_{43}^{AI} to test $H_3: \alpha = \alpha', \ \beta = \beta'$ v.s. H_4 : not H_3 is given by

$$B_{43}^{AI} = B_{43}^{N} \cdot \frac{1}{\binom{n}{4}} \sum_{l} B_{34}^{N}((\mathbf{x}, \mathbf{y})(l)). \tag{3.19}$$

and

$$B_{43}^{MI} = B_{43}^{N} \cdot ME[B_{34}^{N}((\mathbf{x}, \mathbf{y})(l))]. \tag{3.20}$$

4. Simulation Study

In this section, we present some examples to illustrate for our findings regarding the test (i) $H_1: \alpha = \beta \ \alpha' = \beta'$ v.s. H_2 : not H_1 and (ii) $H_3: \alpha = \alpha' \ \beta = \beta'$ v.s. H_4 : not H_3. We take the prior probability of H_i being true, $p_i = 0.5, \ i = 1,2,3,4$.

The data are simulated from Freund's bivariate exponential model of size 10 for the parameters $\underline{\theta} = (\alpha, \alpha', \beta, \beta') = (0.06, 0.10, 0.11, 0.15)$ and the censoring times $t_{x_i} = t_{y_i}, \ i = 1, 2, \cdots, n$ are taken previous fixed values for convenience' sake.

Table 1 indicates the generated Freund's bivariate exponential data. Here * denotes the censored data.

Table 2 indicates Bayes factors, the posterior probabilities $P(H_2 \mid \mathbf{x}, \mathbf{y})$ and p-value based on M.L.E.'s for testing $H_1: \alpha = \beta \ \alpha' = \beta'$ v.s. H_2 : not H_1. Also Table 3 indicates Bayes factors, the posterior probabilities $P(H_4 \mid \mathbf{x}, \mathbf{y})$ and p-value based on M.L.E.'s for testing $H_3: \alpha = \alpha' \ \beta = \beta'$ v.s. H_4 : not H_3.

<Table 1> Freund's Bivariate Exponential Data

i	1	2	3	4	5	6	7	8	9	10
x_i	18.147	0.201	7.984	5.173	43.900	7.694	36.995*	6.203	1.290	27.024
y_i	13.281	4.359	8.595	4.533	39.833	24.481*	7.966*	19.958	21.904	39.748

<Table 2> Bayes Factors, the Posterior Probabilities of for testing and *p-values* for H_1 : $\alpha=\beta\ \alpha^{'}=\beta^{'}$ v.s. H_2 : not H_1

Tests	B_{21}^{AI}	B_{21}^{MI}	$P^{AI}(H_2 \mid \boldsymbol{x}, \boldsymbol{y})$	$P^{MI}(H_2 \mid \boldsymbol{x}, \boldsymbol{y})$	*p-value*
H_1 *v.s.* H_2	4.4101	4.0664	0.8151	0.8026	0.2426

<Table 3> Bayes Factors, the Posterior Probabilities of for testing and *p-values* for H_3 : $\alpha=\alpha^{'}\ \beta=\beta^{'}$ v.s. H_4 : not H_3

Tests	B_{43}^{AI}	B_{43}^{MI}	$P^{AI}(H_4 \mid \boldsymbol{x}, \boldsymbol{y})$	$P^{MI}(H_4 \mid \boldsymbol{x}, \boldsymbol{y})$	*p-value*
H_3 *v.s.* H_4	2.9140	2.5165	0.7445	0.7156	0.0915

First, from the table 2, the Bayes factors B_{21}^{AI} = 4.4101 and B_{21}^{MI} = 4.0664 and the posterior probability $P^{AI}(H_2 \mid \boldsymbol{x},\boldsymbol{y})$ = 0.8151 and $P^{MI}(H_2 \mid \boldsymbol{x},\boldsymbol{y})$=0.8026. Also the p-value based on M.L.E.'s is 0.2426. That is, there is evidence for H_2 in terms of the posterior probabilities based on Bayes factors. But there is no evidence for H_2 in terms of the p-value based on M.L.E.'s under significance level 0.05 and 0.01.

Second, from the table 3, the Bayes factors B_{43}^{AI} = 2.9140 and B_{43}^{MI} = 2.5165 and the posterior probability $P^{AI}(H_4 \mid \boldsymbol{x},\boldsymbol{y})$ = 0.7445 and $P^{MI}(H_4 \mid \boldsymbol{x},\boldsymbol{y})$=0.7156. Also the p-value based on M.L.E.'s is 0.0915. That is, there is evidence for H_4 in terms of the posterior probabilities based on Bayes factors. But there is no evidence for H_2 in terms of the p-value based on M.L.E.'s under significance level 0.05. However, there is evidence for H_2 in terms of the p-value based on M.L.E.'s under significance level 0.10.

5. Concluding Remarks

As we see from the numerical results, the arithmetic and median intrinsic Bayes factors are computed based on entire observations so that they give accurate interpretations and fairly steady answers.

In general, there has been a considerable amount of literature on the controversy between a p-value and a Bayes factor. From the numerical results, it has been noticed that a p-value does not agree with the posterior probability that H_2 and H_4 are correct. From the given parameters $\underline{\theta}=(\alpha,\alpha',\beta,\beta')=(0.06,0.10,0.11,0.15)$, we can see that the model deviates from independence and symmetry.

In conclusion, IBF's are completely automatic Bayes factors, in that they are based only on the data and noninformative priors. IBF methodology can be easily applied to nonnested as well as to irregular problems. They can also be applied in general when the samples come from any distribution.

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