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Design of On-line Process Control with Variable Measurement Interval †

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Abstract

A mixed model with a white noise process and an IMA(0,1,1) process is considered as a process model. It is assumed that the process is a white noise in the absence of a special cause and the process changes to an IMA(0,1,1) due to a special cause.

One useful scheme in measuring the process level is to use the variable measurement interval (VMI) between measurement times according to the value of the previous chart statistic. The advantage of the VMI scheme is to measure the process level infrequently when in control to save the measurement cost and to measure frequently when out of control to save the off-target cost.

This paper considers the VMI scheme in order to detect changes in the process model from a white noise to an IMA(0,1,1). The VMI scheme is shown to be effective compared to the standard fixed measurement interval (FMI) scheme in both statistical and economic contexts.

Key Words : statistical process control; special cause; white noise; IMA(0,1,1); expectecd cost per unit time; adjustment

1. Introduction

Statistical process control (SPC) is used to monitor the occurrence of special causes which make the state of the process change from an in-control state to an out-of-control state. The most common process model used in SPC is the Shewhart model defined as, at time t,

$$Z_t = \mu_t + a_t \tag{1.1}$$

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where μ_t is an unknown process mean and $a_t^{\prime}$s are iid $N(0,\sigma^2)$. The mean μ_t is assumed to be μ_0 when the process is in control and $\mu_1(\neq \mu_0)$ when out of control.

Suppose that a process follows a white noise model when in control, while the process follows an IMA(0,1,1) model when out of control. This model then is basically different from the Shewhart model since the level of the out-of-control process is not constant. Here we assume that a special cause makes the process level change from a constant mean process to a nonstationary process. An IMA(0,1,1) process, due to its inherent disturbances, takes on different values in different local segments of time. Box and Kramer (1992), and Montgomery (1999) have shown that an IMA(0,1,1) model can describe the wandering behavior of the process. Vander Wiel (1996) considered an IMA(0,1,1) process as a process model, where a step shift in the level is monitored.

An IMA(0,1,1) model is defined as

$$Z_t = Z_{t-1} + a_t - \theta a_{t-1} \tag{1.2}$$

where θ is a smoothing constant. Let A_0 denote the interval length from the start to the measurement time immediately before the occurrence of a special cause. Then the mixed model with a white noise and an IMA(0,1,1) can be expressed as follows.

$$N_t = \begin{cases} a_t, \text{ if } t \leq A_0 \\ a_t + \lambda \sum_{i=A_0}^{t-1} a_i, \text{ if } t \geq A_0 + 1 \end{cases} \tag{1.3}$$

where $\lambda = 1 - \theta$. The model N_t for $t \geq A_0 + 1$ is the same as the model Z_t in equation (1.2). We consider in this model that the process is iid in the absence of a special cause, but a special cause disturbs the process and makes the process nonstationary. Then the purpose of the control chart is to measure the process level and detect changes in the distribution as soon as possible.

In order to detect the occurrence of a special cause the process level is measured according to a measurement scheme. The standard measurement scheme which observes the process level at every fixed number of intervals is referred to the fixed measurement interval (FMI) scheme. In recent years it has been found that the performance of a control chart scheme can be improved by varying the sampling interval as a function of the control chart statistic. A typical way of varying the length between two consecutive monitorings is to use a short interval when there is an indication of a possible problem and to use a long interval when there is no indication of a problem. This monitoring scheme is referred to the variable sampling interval (VSI) scheme. A similar approach to VSI is to use the variable sample size (VSS), and the combination of VSI and VSS is referred to the variable sampling rates (VSR). Examples of VSI ,VSS, and VSR are Reynolds et al (1988), Reynolds (1989), Reynolds, Amin and Arnold (1990), Prabhu, Montgomery and Runger (1994), and Park and Reynolds (1994,1999).

Basically the same scheme as VSI is used here and is referred to the variable measurement interval (VMI) scheme. In this paper, properties of the two measurement schemes, FMI and VMI, are evaluated in statistical and economic contexts. Also the performances of two measurement schemes are compared to each other. We use the term 'measurement' instead of 'sampling' here, since we just measure the process level rather than taking a number of observations randomly from a larger number of possible outcomes.

2. Measurement Schemes : FMI and VMI

The FMI scheme is to measure the process level at every fixed number of intervals and to give an out-of-control signal if $|N_t| \geq c_f$ for a given control limit c_f. The interval length used in the FMI scheme is denoted as d_f.

In the VMI scheme, we first select values of measurement intervals to use. For administrative convenience, we consider only two values of measurement intervals. It has been shown by Reynolds(1988) and Park and Reynolds(1994,1999) that allowing only two values of sampling intervals in the VSI control procedures gives good performances in statistical and economic contexts.

It is also expected here that allowing only two values of measurement intervals would give good performances. The two interval lengths are denoted as d_1 and d_2 ($d_1 > d_2$). At the start of the process and after every false alarm, the shorter interval d_2 will be used in order to avoid waiting too long in case of possible unstable process conditions at the start-up or after each false alarm.

The procedure of the VMI scheme is to measure the process level after d_1 intervals if $|N_t| < c_1$ or to measure the process after d_2 intervals if $c_1 \leq |N_t| < c_v$ or to give an out-of-control signal if $|N_t| \geq c_v$, for a given threshold limit c_1 of two measurement intervals and a control limit c_v. Let R_t denote the interval length between t-th and (t+1)-st measurements on the condition that there is no signal, then the measurement interval is explicitly defined as

$$R_t = \begin{cases} d_1, \text{ if } |N_t| < c_1 \\ d_2, \text{ if } c_1 \leq |N_t| < c_v. \end{cases} \tag{2.1}$$

3. Statistical properties of the control scheme

Let M_0, S_0, and F_0 be the number of measurements, the sum of squared deviations from the target, and the number of false alarms, respectively, made during the in-control period. Also let A_1, M_1, and S_1 be the interval length, the number of measurements, and the sum of squared deviations from the target, respectively, made during the out-of-control period. We assume that the in-control period follows a geometric distribution with parameter p_s. The in-control period is usually assumed to follow an exponential distribution in the literature. A geometric distribution is a discrete analogue of an exponential distribution. A special cause occurring between the two consecutive measurement times under the exponential distribution is regarded as occurring on the second measurement time under the geometric distribution. We also assume that the in-control process changes to the out-of-control process immediately at the time that a special cause occurs.Thus the probability function of A_0 is

$$P(A_0 = x) = (1 - p_s)^x p_s \tag{3.1}$$

for $x = 0, 1, \cdots$.

Let the superscripts f and v denote application of FMI and VMI schemes, respectively. Since the interval length and the sum of squared deviations do not depend on the measurement scheme when the process is in control, we have the following expressions for the in-control average interval length (AIL) and expected sum of squared deviations (ESS).

$$E(A_0^f) = E(A_0^v) = \frac{1 - p_s}{p_s}. \tag{3.2}$$

$$E(S_0^f) = E(S_0^v) = \frac{(1 - p_s)\sigma^2}{p_s}. \tag{3.3}$$

The number of measurements depends on the measurement scheme and the in-control distribution. The in-control average numbers of measurements (ANM) are obtained as follows(see Appendix A for derivations).

$$E(M_0^f) = \frac{(1-p_s)^{d_f}}{1-(1-p_s)^{d_f}}. \tag{3.4}$$

$$E(M_0^v) = \frac{(1-p_s)^{d_2}}{1-p_1(1-p_s)^{d_1}-(1-p_1)(1-p_s)^{d_2}} \tag{3.5}$$

where $p_1 = 2\Phi(c_1) - 1$ for the standard normal distribution function Φ.

Since the conditional distribution of the number of false alarms given the number of measurements follows a binomial distribution in which the probability of success is equal to the type I error probability, the marginal average numbers of false alarms (ANF) are obtained as follows.

$$E(F_0^f) = E(M_0^f)\alpha^f \tag{3.6}$$

where $\alpha^f = 2\{1-\Phi(c_f)\}$.

$$E(F_0^v) = E(M_0^v)\alpha^v \tag{3.7}$$

where $\alpha^v = 2\{1-\Phi(c_v)\}$.

Let τ be the number of intervals from the M_0-th measurement to the interval immediately before the occurrence of a special cause, and R_{M_0} be the interval length from the M_0-th measurement to the next. In the VMI scheme, then, the

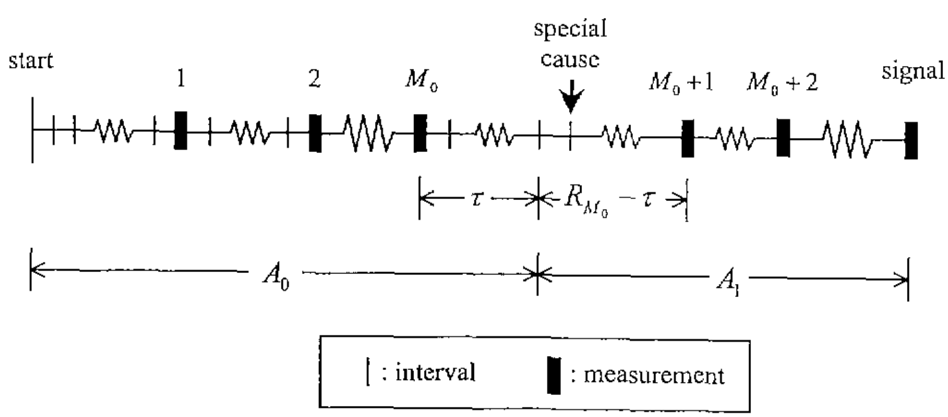

Figure 1 : Diagram of a cycle with intervals and measurements over time.

interval length from the change of model to the first measurement in the out-of-control state will be $R_{M_0} - \tau$, on which the out-of-control properties depend. The joint probability function of R_{M_0} and τ for the VMI scheme is obtained in Appendix B. We define a cycle of the process as the time from the start of the process to the true out-of-control signal. Then the cycle length is equal to $A_0 + A_1$. A cycle of the process is illustrated in Figure 1, where intervals and measurements as well as τ, R_{M_0}, A_0 and A_1 are expressed.

Derivations of expectations of A_1, M_1, and S_1 are not known theoretically. Instead, they are derived by simulation methods here. The conditional expectations of A_1, M_1, and S_1 given $R_{M_0} - \tau$ are obtained using simulations, and then the marginal expectations are evaluated according to the joint distribution of R_{M_0} and τ.

4. Economic properties of the control scheme

The expected cost per unit time is widely used in evaluating the properties of a control sheme economically. The economic design of a control chart was first developed by Duncan(1956) and a unified cost model is proposed by Lorenzen and Vance(1986). In order to improve statistical properties the economic design with statistical constraints was developed by Saniga(1989) and referred to the economic statistical design.

In calculating the expected cost per unit time, we need to define cost parameters associated with the cycle. We consider the following cost parameters.

C_M : cost per measurement

C_T : off-target cost in units of σ^2

C_F : cost per false alarm

Although there are some other time and cost parameters involved in a cycle of the process for more general economic models, we restrict our study to a model with the parameters defined here. In most cases, these parameters have important meanings and others are negligible.

The expected cost per cycle is expressed as

$$E(C) = C_M E(M_0 + M_1) + C_T E(S_0 + S_1) + C_F E(F_0). \qquad (4.1)$$

Then the expected cost per unit time is defined as the ratio of the expected cost per cycle to the expected cycle length, that is

$$E(L) = \frac{C_M E(M_0 + M_1) + C_T E(S_0 + S_1) + C_F E(F_0)}{E(A_0 + A_1)}. \qquad (4.2)$$

5. Comparison of FMI and VMI

The two measurement schemes are compared in statistical and economic contexts. Since applying the VMI scheme renders more complexity in administration, it would be worth considering only if there are substantial improvements in effectiveness compared to its counterpart, FMI.

In comparing the properties statistically, we usually set the in-control statistical properties of the two schemes the same and then compare the out-of-control statistical properties. In accomplishing the in-control properties the same, we need to set the followings.

$$\begin{aligned} E(M_0^f) &= E(M_0^v), \\ E(F_0^f) &= E(F_0^v). \end{aligned}$$

We also need to make the type I error probabilities of the two schemes the same. Because the type I error probability depends only on the control limit we need to set the two control limits the same, that is $c_f = c_v$. In order to make the in-control ANMs of the two schemes the same, we have, from equations (3.4) and (3.5),

$$c_1 = \Phi^{-1}\left(\frac{(1-p_s)^{d_2} - (1-p_s)^{d_f}}{2(1-p_s)^{d_f}\{(1-p_s)^{d_2} - (1-p_s)^{d_1}\}} + 0.5\right) \qquad (5.1)$$

for $d_2 < d_f < d_1$.

In Figures 2-4, the out-of-control properties of the FMI and VMI schemes are plotted for some choices of λ and p_s when $c_f = 3$, $d_f = 5$. The in-control properties are all set equal for the two schemes. The out-of-control properties are obtained by simulations in which the conditional out-of-control process given R_{M_0} and τ is repeated 20000 times. It was shown that the case for $d_2 = 4$ is optimal among cases for $d_2 = 1, 2, 3, 4$. Thus the case for $d_2 = 4$ is used in plotting the properties of VMI. The out-of-control properties of FMI are almost the same for different values of p_s, and thus only λ is used in representing the FMI scheme. The entries of parentheses in the legend of figures are (p_s, λ) for VMI and (λ) for FMI.

From all the three figures we can choose values for d_1 so that the VMI values are less than the FMI values. Thus we see that the performance of the FMI scheme can be improved substantially by using the VMI scheme. The values for d_1, however, which minimize the three averages are quite different and this makes it difficult to choose one value for d_1. Because of the different behavior of the three averages with respect to d_1, a certain criterion for determining the interval length is needed for the optimal performance of VMI. One such criterion for determining optimal chart parameters is to use an economic model.

We obtained expected costs per unit time in order to compare economic performances of the two schemes. The optimal chart parameters are obtained by solving nonlinear minimization problems where the expected cost per unit time is the object function of chart parameters. Since the out-of-control properties can not be expressed as explicit functions of chart parameters, we use finite difference approximation to partial derivatives by using simulated values of the properties.

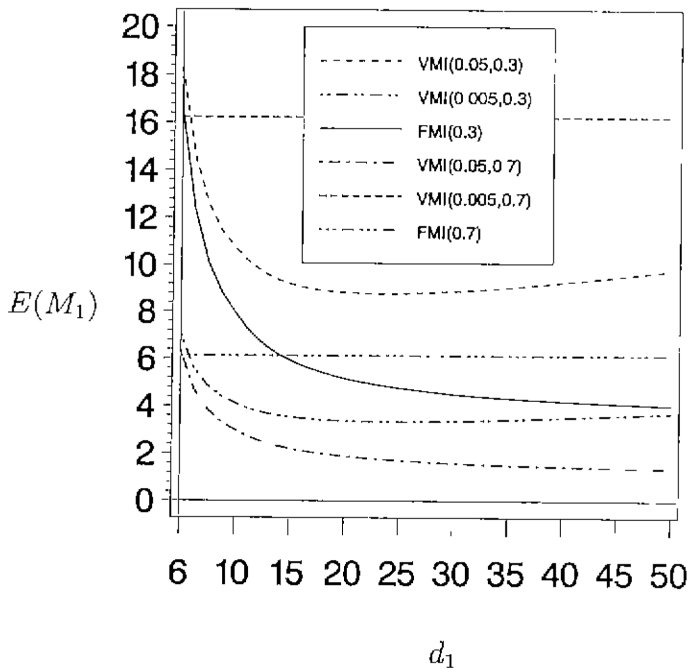

Figure 2 : Out-of-control ANM for $d_f = 5$, $c_f = 3.0$; VMI(p_s, λ), FMI(λ).

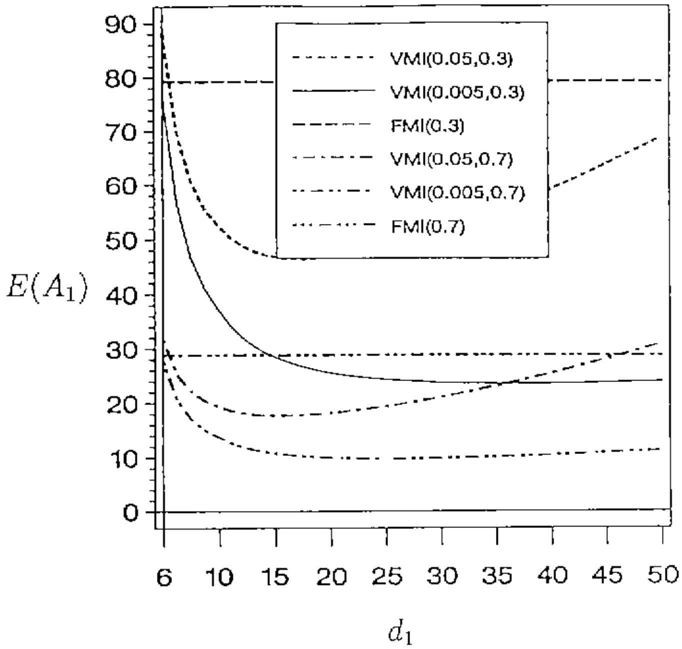

Figure 3 : Out-of-control AIL for $d_f = 5$, $c_f = 3.0$; VMI(p_s, λ), FMI(λ).

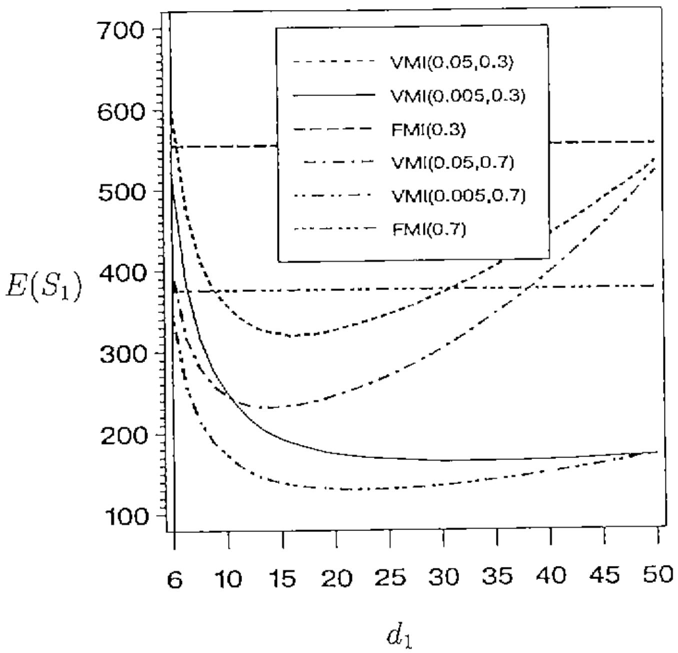

Figure 4 : Out-of-control ESS for $d_f = 5$, $c_f = 3.0$; VMI(p_s, λ), FMI(λ).

In Tables 1 and 2, the optimal chart parameters of the two schemes such as d_f, d_1, d_2 and c_f, c_1, c_v are selected to give the minimum expected cost per unit time. Also the expected cost per unit time as well as the out-of-control properties are obtained. It is seen that the cost of VMI is always substantially less than that of FMI. As the average in-control period increases (that is p_s decreases), the optimal values of c_f, c_1, c_v as well as d_1 increase. In each case the optimal value of d_2 is equal to 1 which is the smallest possible interval length. As the false alarm cost increases, the control limit also increases in order to reduce the false alarm rate. We see that the control limit of VMI, c_v, is larger than the control limit of FMI, c_f, and this makes the false alarm rate of VMI smaller than FMI.

Table 1. Optimal chart parameters and expected cost per unit time for $\lambda = 0.3$, $\sigma^2 = 1$, $C_T = 1$

p_s	C_M	C_F	d_f d_1, d_2	c_f c_1, c_v	EL^f EL^v	EM_1^f EM_1^v	EA_1^f EA_1^v	ES_1^f ES_1^v
0.005	1	50	12 19,1	1.640 1.552,3.390	1.890 1.120	3.583 1.143	37.550 10.945	146.819 22.178
		100	8 18,1	2.103 1.572,3.635	2.149 1.123	6.194 1.153	46.075 10.447	210.369 21.698
	5	50	30 30,1	1.000 2.132,3.251	2.067 1.306	1.790 1.139	39.573 17.290	156.695 40.449
		100	22 31,1	1.517 1.860,3.407	2.403 1.309	2.690 1.143	48.891 17.846	229.438 41.870
0.001	1	50	13 33,1	2.087 1.976,4.388	1.448 1.054	4.908 1.057	57.818 17.459	309.493 38.519
		100	8 31,1	2.600 2.146,4.847	1.582 1.054	9.137 1.062	69.604 16.468	441.896 36.338
	5	50	30 50,1	1.590 2.602,3.738	1.618 1.149	2.571 1.050	62.712 26.333	358.073 70.338
		100	30 50,1	1.835 2.600,4.641	1.751 1.151	2.949 1.054	74.059 26.388	483.197 72.458

Table 2. Optimal chart parameters and expected cost per unit time for $\lambda = 0.7$, $\sigma^2 = 1$, $C_T = 1$.

p_s	C_M	C_F	d_f / d_1, d_2	c_f / c_1, c_v	EL^f / EL^v	EM^f / EM^v	EA^f / EA^v	ES^f / ES^v
0.005	1	50	5	2.193	2.055	3.896	17.489	146.679
			10,1	2.128,4.016	1.180	1.155	5.893	20.627
		100	6	2.393	2.285	3.981	21.398	213.170
			11,1	2.021,3.905	1.181	1.145	6.400	23.029
	5	50	12	1.646	2.518	2.104	19.808	180.298
			18,1	2.546,3.804	1.487	1.120	10.183	48.192
		100	9	2.130	2.806	2.910	22.219	227.018
			17,1	2.406,4.216	1.492	1.126	9.650	44.757
0.001	1	50	7	2.600	1.475	4.105	25.739	300.137
			18,1	2.367,5.487	1.091	1.074	9.694	43.285
		100	8	2.603	1.534	3.835	27.186	331.779
			18,1	2.800,6.000	1.091	1.076	9.712	44.111
	5	50	19	1.954	1.759	2.096	30.850	409.296
			31,1	3.344,4.607	1.256	1.051	16.346	106.871
		100	18	2.233	1.861	2.358	33.977	495.507
			31,1	2.950,5.276	1.256	1.053	16.348	107.132

6. An example of a mixed model with a white noise and an IMA(0,1,1)

When a process follows an IMA(0,1,1) model due to its wandering behavior, it will be manipulated by some compensating variables to adjust the process level close to target. The adjustment is employed by feedback and/or feedforward control. This approach to process control is referred to engineering process control (EPC) or automatic process control (APC). Examples of recent works on EPC are Box and Kramer (1992), Box and Luceno (1994,1997). Recently integration of SPC and EPC has a great deal of interest since the two procedures are not divided clearly in modern manufacturing industries. Examples on this topic include Vander Wiel et al (1992), Montgomery et al (1994), Janakiram and Keats (1998), and Nembhard and Mastrangelo (1998).

Suppose that an IMA(0,1,1) process is adjusted by an automatic controller continuously at every interval. That is, an adjustment is made at every interval automatically. The adjustment made at time t for an IMA(0,1,1) process is the forecast $\widehat{Z}_t$ which is the exponentially weighted moving average (EWMA) of past data. Then the adjusted process will be a sequence of white noise when the controller works properly. If the controller breaks down (or deteriorates) at some time during the process, the adjusted process will no longer be a sequence of white noise. The breakdown of the controller is treated as a special cause, and states before and after the breakdown are regarded as in-control and out-of-control states, respectively. When the breakdown of the controller is detected, it will be replaced by a new one (or fixed) and the adjusted process will go back to a white noise process.

We assume, in this example, that the information about the process level used by the controller is not available for use in monitoring of the process. One possible example of this case is that the adjustment is made automatically as soon as the controller measures the process level, but there is no such device to transfer the observed deviation from the target to the operator. Thus we need to measure the process level according to a measurement rule with an additional cost.

Suppose that the process model without adjustment follows an IMA(0,1,1) model with a smoothing constant θ_0. The minimum mean squared error (MMSE) forecast of the process level at time t is the EWMA defined as

$$\widehat{Z}_t = \lambda_0 Z_{t-1} + \theta_0\{\lambda_0(Z_{t-2} + \theta_0 Z_{t-3} + \cdots)\} \tag{6.1}$$

where $\lambda_0 = 1 - \theta_0$.

If we assume that the process starts on the target, then we have, from equations (1.2) and (6.1),

$$Z_t = a_t + \lambda_0 \sum_{i=1}^{t-1} a_i \tag{6.2}$$

and

$$\widehat{Z}_t = \lambda_0 \sum_{i=1}^{t-1} a_i. \tag{6.3}$$

When the process is in control, that is, when the controller works properly, the compensation will be the same as $\widehat{Z}_t$ in (6.3). On the other hand, when the controller breaks down, the process can not be adjusted properly and thus the adjustment will not be the same as (6.3). Here we assume that the controller in the presence of a special cause *consistently underadjust the process. Thus it is* assumed that the adjustment in the presence of a special cause is as the following.

$$\widehat{Z}_t = \lambda_1 \sum_{i=A_0}^{t-1} a_i + \lambda_0 \sum_{i=1}^{A_0-1} a_i \tag{6.4}$$

for $t \geq A_0 + 1$ *and* $0 \leq \lambda_1 < \lambda_0$. Then the adjusted process regulated by its compensating variable will be $N_t = Z_t - \widehat{Z}_t$ and can be expressed the same as (1.3) for $\lambda = \lambda_0 - \lambda_1$. The case for $\lambda_1 = 0$ means that the controller stops working completely. When this happens the control action is constant and the problems with the controller can be easily detected by looking at the control action. Then there is no need to look at the process output and we exclude this case from our study.

It is seen that the adjusted IMA(0,1,1) process is a mixed model with a white noise and an IMA(0,1,1) processes. Thus the control procedure reduces to detecting changes of process model from a white noise to an IMA(0,1,1). In most process control procedures where adjustments are made by an automatic controller, the adjustment cost is negligible. Thus we set $C_A = 0$.

7. Conclusion and Remarks

In this paper, a mixed model with a white noise and an IMA(0,1,1) is considered as the process model. The purpose of the control procedure is to detect changes in the model as soon as possible.

In detecting the occurrence of a special cause, the two measurement schemes (FMI and VMI) are applied and their properties are obtained for comparison. It was shown that the performance of the VMI scheme is substantially better than that of the FMI scheme both in statistical and economic contexts.

The VMI scheme is designed to monitor infrequently when the process is in control and to monitor more frequently when out of control. The VMI scheme is especially useful when the use of long intervals does not produce much increase in the off-target cost while the process is in control, compared to the use of short intervals. Also VMI is useful when the gain in the off-target cost due to frequent measurements will be large enough to compensate for the increase in the measurement cost when the process is out of control.

The idea of VMI can be applied to more general problems of detecting changes in the process model from in-control to out-of-control. In these problems the VMI scheme will also be effective if deviations from the target when in control is stochastically smaller than those when out of control, so that long measurement intervals may be used to save the measurement cost when in control and short intervals may be used to save the off-target cost when out of control. The study of general cases is left for further research.

APPENDIX

Appendix A: Derivations of $E(M_0^f)$ and $E(M_0^v)$.

The in-control average number of measurements can be derived by using a Markov chain approach. The state of a Markov chain is defined at each measurement time during an in-control state according to the next interval length and the control state of the next measurement time. Let V_i denote the state of a Markov chain at i-th measurement time as follows.

$V_i = 1$ if $R_i = d_1$ and the process is in control at the $(i+1)$-st measurement time.

$V_i = 2$ if $R_i = d_2$ and the process is in control at the $(i+1)$-st measurement time.

$V_i = 3$ if $R_i = d_1$ and the process is out of control at the $(i+1)$-st measurement time.

$V_i = 4$ if $R_i = d_2$ and the process is out of control at the $(i+1)$-st measurement time.

The conditional probability of $V_i = 1$ given $V_{i-1} = 1$ is obtained as $\Pr(V_i = 1|V_{i-1} = 1) = p_1(1-p_s)^{d_1}$ and the other conditional probabilities are obtained similarly. Note that V_i is independent of V_{i-1}.Thus the transient state transition matrix is obtained as

$$\mathbf{Q} = \begin{bmatrix} p_1(1-p_s)^{d_1} & p_2(1-p_s)^{d_2} & p_1\{1-(1-p_s)^{d_1}\} & p_2\{1-(1-p_s)^{d_2}\} \\ p_1(1-p_s)^{d_1} & p_2(1-p_s)^{d_2} & p_1\{1-(1-p_s)^{d_1}\} & p_2\{1-(1-p_s)^{d_2}\} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $p_2 = 1 - p_1$. Note that we use the short interval d_2 after each false alarm.

Since the short interval d_2 is used at the start, we have the starting state probability vector,

$$\mathbf{s}' = (0,\ (1-p_s)^{d_2},\ 0,\ 1-(1-p_s)^{d_2}).$$

Using the transition matrix and starting state vector we have the probability function of M_0^v, for $x \geq 1$,

$$\begin{aligned}\Pr(M_0^v = x) &= \mathbf{s}'\mathbf{Q}^x(\mathbf{I}-\mathbf{Q})\mathbf{1} \\ &= (1-p_s)^{d_2}(1-e)^{x-1}e\end{aligned}$$

for a unit vector $\mathbf{1}$ and $e = 1 - p_1(1-p_s)^{d_1} - p_2(1-p_s)^{d_2}$. Note that e is the probability of going to states $V_i = 3$ or 4 from states $V_i = 1$ or 2. Thus the in-control ANM is obtained as

$$\begin{aligned}E(M_0^v) &= \mathbf{s}'\mathbf{Q}(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{1} \\ &= \frac{(1-p_s)^{d_2}}{e}.\end{aligned}$$

If we put $d_1 = d_2 = d_f$ in (3.5) we have the equation (3.4).

Appendix B: Derivation of the joint probability of R_{M_0} and τ.

The conditional probability of τ given R_{M_0} and M_0 is obtained as follows. For $y = d_1, d_2,\ x = 0, 1, \cdots, y-1$, and $k \geq 1$,

$$\Pr(\tau = x | R_{M_0} = y, M_0 = k) = \frac{(1-p_s)^x p_s}{1-(1-p_s)^y}.$$

Also, the conditional probability of R_{M_0} given $M_0 \geq 1$ is obtained as follows. For $y = d_1, d_2$, and $k \geq 1$,

$$\Pr(R_{M_0} = y, M_0 = k) = (1-p_s)^{d_2}(1-e)^{k-1}\{1-(1-p_s)^y\}\{p_1 I_{\{d_1\}}(y) + p_2 I_{\{d_2\}}(y)\}$$

where $I_{\{x\}}(y) = 1$ if $y = x$, and $= 0$ if $y \neq x$. Thus,

$$\begin{aligned}\Pr(R_{M_0} = y | M_0 = k) &= \Pr(R_{M_0} = y, M_0 = k) / \Pr(M_0 = k) \\ &= \frac{\{p_1 I_{\{d_1\}}(y) + p_2 I_{\{d_2\}}(y)\}\{1 - (1 - p_s)^y\}}{e}.\end{aligned}$$

Hence the joint probability function of R_{M_0} and τ given $M_0 \geq 1$ is, for $x = 0, 1, \cdots, y-1$,and $k \geq 1$,

$$\begin{aligned}\Pr(\tau = x, R_{M_0} = y | M_0 = k) &= \Pr(\tau = x | R_{M_0} = y, M_0 = k) \cdot \Pr(R_{M_0} = y | M_0 = k) \\ &= \frac{\{p_1 I_{\{d_1\}}(y) + p_2 I_{\{d_2\}}(y)\}(1 - p_s)^x p_s}{e}.\end{aligned}$$

Since the process starts with the short interval d_2, we have the following conditional probability given $M_0 = 0$. For $x = 0, 1, \cdots, y-1$,

$$P(\tau = x, R_{M_0} = y | M_0 = 0) = \frac{(1 - p_s)^x p_s}{1 - (1 - p_s)^y} I_{\{d_2\}}(y).$$

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