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A Note on a Family of Lattice Distributions

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ABSTRACT

In this note we use the Poisson Summation Formula to generalize a result of Harris and Park (1994) on lattice distributions induced by uniform $(0,1)$ random variables to those generated by random varibales with step functions as their probability density functions.

Keywords: Lattice Distributions, Random Variables, Poisson Summation Formula, Fourier Transform.

1. Introduction and Statements of the Main Theorems

In a recent paper by Harris and Park (1994), a family of lattice distributions induced by the probability density function of a sum of uniform $(0,1)$ random variables is examined. More specifically, let S_{n+1} denote the sum of $n+1$ independent $(0,1)$ random variables, then the probability density function can be written as (for example, see Feller (1971, p. 27):

$$f_{S_{n+1}} = \frac{1}{n!}\sum_{\nu=0}^{n+1}\binom{n+1}{\nu}(-1)^{\nu}(x-\nu)_+^n, \text{ for } 0 < x < n+1, \tag{1}$$

where

$$(x-\nu)_+ = \begin{cases} x-\nu & \text{if } x-\nu \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

A family of distributions f_{n+1}, induced by $f_{S_{n+1}}$, can be defined:

$$f_{n+1}(x) = \begin{cases} f_{S_{n+1}}(x) & \text{for } x = \delta, \delta+1, \ldots, \delta+n \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

It is shown in Harris and Park (1994) that the probability mass function f_{n+1} is a lattice distribution with carrier set $\{\delta, \delta+1, \ldots, \delta+n\}$.

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This paper generalizes the result of Harris and Park to a family of lattice distributions derived from a sum of independent random variables having a probability density function of the form

$$g(x) = \sum_{j=-\infty}^{\infty} a_j \chi_{I_j}(x), \tag{3}$$

where $\{I_j\}_{j=-\infty}^{\infty}$ is a uniform partition of $\mathbb{R}$ with $|I_j| = |I|$, $a_j \geq 0$ for each j, and $\sum_{j=-\infty}^{\infty} a_j = 1/|I|$.

Note that the uniform $(0,1)$ distribution and the typical probability histogram have forms given by (3). For $j = 0, \pm 1, \ldots$, let $I_j = [\alpha + j|I|, \alpha + (j+1)|I|]$. Let T_{n+1} be the sum of $n+1$ independent random variables each having probability density function of the form given by (3). Define a lattice distribution g_{n+1} derived from the probability density function $g_{T_{n+1}}$ by

$$g_{n+1}(x) = \begin{cases} g_{T_{n+1}}(x) & \text{for } x = \alpha + \delta + j|I|, j = 0, \pm 1, \ldots, 0 \leq \delta < |I| \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Theorem 1. *For $0 \leq \delta < |I|$, the lattice distribution as defined by (4) is a probability mass function with carrier set $\{\alpha + \delta + j|I| : j = 0, \pm 1, \ldots\}$, that is,*

$$\sum_{j=-\infty}^{\infty} g_{n+1}(\alpha + \delta + j|I|) = \frac{1}{|I|}.$$

Theorem 1 is a generalization of the result in Harris and Park (1994).

Theorem 2. *Let $V_{n+1,\delta}$ be a random variable having the lattice distribution g_{n+1}. The moments of order $0, \ldots, n$ of $V_{n+1,\delta}$ are the same as the corresponding moments of T_{n+1}, that is, for $k = 0, \ldots, n$ and $0 \leq \delta < |I|$,*

$$E(V_{n+1,\delta}^k) = E(T_{n+1}^k).$$

The proofs of the theorems will be given in the next section.

2. Proofs of Theorems

Let

$$\hat{g}_{T_{n+1}}(s) = \int_{-\infty}^{\infty} g_{T_{n+1}}(t) e^{-2\pi i s t} dt$$

be the Fourier transform of $g_{T_{n+1}}$.

Proof of Theorem 1: To prove Theorem 1, we apply the Poisson Summation Formula (see for example Dym and McKean (1972) or Feller (1971, p.629)) to $g_{T_{n+1}}$ to obtain

$$\sum_{j=-\infty}^{\infty} g_{n+1}(\delta + j|I|) = \frac{1}{|I|} \sum_{k=-\infty}^{\infty} \hat{g}_{T_{n+1}}(\frac{k}{|I|}) e^{\frac{2\pi i k\delta}{|I|}}. \tag{5}$$

Since $g_{T_{n+1}}$ is the $(n+1)$-fold convolution of g, we have $\hat{g}_{T_{n+1}} = \hat{g}^{n+1}$. The Fourier transform of g can be written as

$$\hat{g}(s) = \sum_{j=-\infty}^{\infty} a_j e^{-2\pi i s} [\frac{e^{-2\pi i j|I|s} - e^{-2\pi i(j+1)|I|s}}{2\pi i s}]$$

for $s \neq 0$ and $\hat{g}(0) = 1$. Since $e^{-2\pi i k} = 1$ for all integers k, it follows that $\hat{g}(k/|I|) = 0$ for $k \neq 0$. Therefore $\hat{g}_{T_{n+1}}(k/|I|) = \hat{g}^{n+1}(k/|I|) = 0$ for $k \neq 0$ and $\hat{g}_{T_{n+1}}(0) = 1$. The proof of Theorem 1 is complete.

Remark 1. Theorem 1 clearly holds for any absolutely continuous probability density function g such that its Fourier transform $\hat{g}$ satisfies $\hat{g}(k/|I|) = 0$ for $k \neq 0$.

Proof of Theorem 2: We apply the Poisson Summation Formula to $g_{T_{n+1}}(t)e^{-2\pi i st}$ to obtain

$$\sum_{j=-\infty}^{\infty} g_{n+1}(\delta + j|I|) e^{-2\pi i s(\delta + j|I|)} = \frac{1}{|I|} \sum_{k=-\infty}^{\infty} \hat{g}_{T_{n+1}}(s + \frac{k}{|I|}) e^{\frac{2\pi i k\delta}{|I|}}. \tag{6}$$

Let $\hat{g}^{(\ell)}_{T_{n+1}}$ be the ℓth derivative of $\hat{g}_{T_{n+1}}$. Since $\hat{g}_{T_{n+1}} = \hat{g}^{n+1}$ and $\hat{g}(k/|I|) = 0$ for $k \neq 0$, it follows that $\hat{g}^{(\ell)}_{T_{n+1}}(k/|I|) = 0$ for $k \neq 0$ and $\ell \leq n$. Now differentiate (6) with respect to s and evaluate at $s = 0$ to obtain

$$(-2\pi i)^\ell \sum_{j=-\infty}^{\infty} (\delta + j|I|)^\ell g_{n+1}(\delta + j|I|) = \frac{1}{|I|} \hat{g}^{(\ell)}_{T_{n+1}}(0),$$

which is a constant independent of δ. Furthermore, for $\ell = 0, \ldots, n$, we have

$$\begin{aligned} \frac{1}{(-2\pi i)^\ell} \hat{g}^{(\ell)}_{T_{n+1}}(0) &= E(T^\ell_{n+1}) \\ &= \sum_{j=-\infty}^{\infty} (\delta + j|I|)^\ell g_{n+1}(\delta + j|I|)|I| \\ &= E(V^\ell_{n+1,\delta}). \end{aligned}$$

The proof of Theorem 2 is complete.

References

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