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Intrinsic Bayes Factors for Exponential Model Comparison with Censored Data

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ABSTRACT

This paper addresses the Bayesian hypotheses testing for the comparison of exponential population under type II censoring. In Bayesian testing problem, conventional Bayes factors can not typically accommodate the use of noninformative priors which are improper and are defined only up to arbitrary constants. To overcome such problem, we use the recently proposed hypotheses testing criterion called the intrinsic Bayes factor. We derive the arithmetic, expected and median intrinsic Bayes factors for our problem. The Monte Carlo simulation is used for calculating intrinsic Bayes factors which are compared with P-values of the classical test.

Key Words : Exponential Distribution; Type II Censoring; Noninformative Priors; Intrinsic Bayes Factors; P-Values.

1. Introduction

In lifetime studies, the exponential distribution is one of the most frequently used distributions. There are a huge body of literatures concerned with exponential models in the lifetime and reliability analysis. Also many works have been done under the Bayesian approach since the middle of 1970.

The comparison of two lifetime distributions is often important in statistical analyses of lifetime data. When the distributions are one-parameter exponential distributions, this amounts to a comparison of their failure rates or means. Classical approach to this hypothesis testing problem for type II censored data has been well-summarized in the literatures (see for example Lawless (1982)). Recently Colosimo and Cordeiro (1998) have considered the Bartlett correction for

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the likelihood ratio (LR) tests for the equality of $k(\geq 2)$ exponential distributions based on type II censored samples.

The primary objective of this paper is to provide a Bayesian alternative to the classical test for the comparison of two exponential populations under type II censoring using noninformative priors. Although we are concerned exclusively with type II censoring, the general methods are applicable to other censoring schemes as well.

In Bayesian testing problem, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constrains often the use of noninformative priors. Since noninformative priors such as Jeffrey's (1961) priors or reference priors (Berger and Bernardo (1989, 1992)) are typically improper so that such priors are only up to arbitrary constants which affects the values of Bayes factors. Many people have made efforts to compensate for that arbitrariness. See Geisser and Eddy (1979), Spiegalhalter and Smith (1982), San Martini and Spezzaferri (1984) and O'Hagan (1995) for related works.

Berger and Pericchi (1996b) introduced a new model selection and hypotheses testing criterion, called the Intrinsic Bayes Factor (IBF) using a data-splitting idea, which would eliminate the arbitrariness of improper priors. This approach has shown to be quite useful in several statistical areas. (cf. Berger and Pericchi (1996a), Varshavsky (1996) and Lingham and Sivaganesan (1997)).

The outline of the remaining sections is as follows. In Section 2, we review the concept of the IBF methodology. In Section 3, we specifically derive expressions for IBFs to solve our problem. Finally, we give some numerical examples to illustrate our results and contrast it with classical method.

2. Preliminaries

Let $\mathbf{Y} = (Y_1, \cdots, Y_n)$ be an observation with density $f(\mathbf{y}|\theta)$, where $\theta \in \Theta$ is a vector of unknown parameters. Suppose that we wish to test the hypotheses $H_i : \theta \in \Theta_i$, where $\Theta_i \subseteq \Theta, i = 1, \cdots, q$. Let $\pi_i(\theta)$ be the prior distribution for θ under H_i, and let p_i be the prior probability of H_i, $i = 1, \cdots, q$. Then the posterior probability that H_i is true is

$$P(H_i|\mathbf{y}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} B_{ji}\right)^{-1} \tag{2.1}$$

where B_{ji}, the Bayes factor of H_j to H_i, is defined by

$$B_{ji} = \frac{m_j(\mathbf{y})}{m_i(\mathbf{y})} = \frac{\int_{\Theta_j} f(\mathbf{y}|\theta)\pi_j(\theta)d\theta}{\int_{\Theta_i} f(\mathbf{y}|\theta)\pi_i(\theta)d\theta}, \tag{2.2}$$

$m_i(\mathbf{y})$ being the marginal or predictive density of $\mathbf{Y}$ under H_i. The posterior probabilities in (2.1) are used to select the most plausible hypothesis.

If we use some noninformative priors $\pi_i^N(\theta)$, (2.2) becomes

$$B_{ji}^N = \frac{m_j^N(\mathbf{y})}{m_i^N(\mathbf{y})} = \frac{\int_{\Theta_j} f(\mathbf{y}|\theta)\pi_j^N(\theta)d\theta}{\int_{\Theta_i} f(\mathbf{y}|\theta)\pi_i^N(\theta)d\theta}. \tag{2.3}$$

A noninformative prior $\pi_i^N(\theta)$ is often improper, and is defined only up to arbitrary constants.

Hence, the corresponding Bayes factor, B_{ji}^N, is indeterminate. One solution to this indeterminancy problem, due to Berger and Pericchi (1996b), begins with the assumption that we can split the data vector $\mathbf{y}$ into $\mathbf{y}(l)$, the so-called *training sample*, and the remainder of the data $\mathbf{y}(-l)$, such that

$$0 < m_i^N(\mathbf{y}(l)) < \infty, ^\forall i = 1, \cdots, q. \tag{2.4}$$

In view of (2.4), the posteriors $\pi_i^N(\theta|\mathbf{y}(l))$ are well defined. Now, consider the Bayes factor, $B_{ji}(l)$, for the rest of the data $\mathbf{y}(-l)$, using $\pi_i^N(\theta|\mathbf{y}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int_{\Theta_j} f(\mathbf{y}(-l)|\theta, \mathbf{y}(l))\pi_j^N(\theta|\mathbf{y}(l))d\theta}{\int_{\Theta_i} f(\mathbf{y}(-l)|\theta, \mathbf{y}(l))\pi_i^N(\theta|\mathbf{y}(l))d\theta} = B_{ji}^N \times B_{ij}^N(\mathbf{y}(l)) \tag{2.5}$$

where B_{ji}^N is given by (2.3) and

$$B_{ij}^N(\mathbf{y}(l)) = \frac{m_i^N(\mathbf{y}(l))}{m_j^N(\mathbf{y}(l))}. \tag{2.6}$$

In (2.5), any arbitrary ratio, c_j/c_i say, that multiples B_{ji}^N would be cancelled by the ratio c_i/c_j forming the multiplicand in $B_{ij}^N(\mathbf{y}(l))$. Also, while the expression (2.6) renders $B_{ji}(l)$ in terms of the simpler marginal densties of $\mathbf{y}(l)$.

As training samples play a fundamental role in our testing $H_i, i = 1, \cdots, q$, we will need the following definition.

Definition 2.1 A training sample $\mathbf{y}(l)$, will called *proper* if (2.4) holds and *minimal* if it is proper and none of its subsets is proper.

Beger and Pericchi (1996b) advocated various summaries based on $B_{ji}(l)$'s in (2.5) from many training samples to test $H_i, i = 1, \cdots, q$. Generically termed the Intrinsic Bayes Factor (IBF) is given by the following definition.

Definition 2.2 The Arithmetic Intrinsic Bayes factor of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^{N} \cdot \frac{1}{L} \sum_{l=1}^{L} B_{ji}^{N}(\mathbf{y}(l)) \tag{2.7}$$

where L is the number of all possible minimal training samples.

When the sample size is small, the training sample average in (2.7) can has large variance (as statistics in frequentist sense), which indicates an instability of IBF's. Also, computation can be lengthy if L is large. As a way of overcoming these problems, Berger and Pericchi (1996b) recommands replacing the average in (2.7) by their expectation, evaluated at the MLE.

Definition 2.3 The Expected Arithmatic Intrinsic Bayes factor of H_j to H_i is

$$B_{ji}^{EAI} = B_{ji}^{N} \cdot \frac{1}{L} \sum_{l=1}^{L} E_{\hat{\theta}}^{H_j}[B_{ij}(\mathbf{y}(l))] \tag{2.8}$$

where $\hat{\theta}$ is the MLE of θ under H_j

Next we use the another intrinsic Bayes factor, is called median intrinsic Bayes factor. By Berger and Pericchi (1998), the median intrinsic Bayes factor seems to be a simple and very generally applicable intrinsic Bayes factor, which works well for nested or non-nested models, and even for small or moderate sample sizes.

Definition 2.4 The Median Intrinsic Bayes factor of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^{N} \cdot MED_l[B_{ji}^{N}(\mathbf{y}(l))] \tag{2.9}$$

where MED_l indicates the median taken for all $1 \le l \le L$.

One can calculate the posterior probability of H_i using (2.1), where B_{ji} is replaced by B_{ji}^{AI}, B_{ji}^{EAI} and B_{ji}^{MI} from (2.7), (2.8) and (2.9), respectively.

3. Main Results

The exponential model with parameter θ for an homogeneous population is given by

$$f(y|\theta) = \theta \exp(-\theta y), \tag{3.1}$$

where $y \geq 0$ and $\theta > 0$. A type II censored sample is one for which the r smallest observations in a random sample of n items are observed. We are interested in comparing two exponential lifetime data under type II censoring. We consider two samples of sizes n_1, n_2 from exponential populations with parameters θ_1, θ_2, respectively. Under type II censoring the observed data consist of the ordered failure times $y_{i1} \leq y_{i2} \leq \cdots \leq y_{ir_i}$ and $(n_i - r_i)$ survivors, where $i = 1, 2$. We want to test the hypotheses of $H_1 : \theta_1 = \theta_2$ vs. $H_2 : \theta_1 \neq \theta_2$. Epstein and Tsao (1953) considered the LR test for H_1 vs. H_2 which is based on $F(2r_1, 2r_2)$ distribution. Our interest is to develop a Bayesian test for H_1 vs. H_2 which is an alternative to the classical LR test.

In subsection 3.1, we determine the minimal training sample (MTS), given the data $\mathbf{y_1} = (y_{11}, \cdots, y_{1r_1})$ and $\mathbf{y_2} = (y_{21}, \cdots, y_{2r_2})$. In subsection 3.2, we derive expressions for the IBFs given by (2.7), (2.8) and (2.9) for MTS.

3.1 Minimal Training Sample

The goal here is to determine the set of all possible MTS's for the data $\mathbf{y_1}$ and $\mathbf{y_2}$. To this end, we use Definition 2.1 and the Jeffrey's priors $\pi_i^N(\theta), i = 1, 2$, say, corresponding respectively to $H_1 : \theta_1 = \theta_2 (= \theta)$, $H_2 : \theta_1 \neq \theta_2$. The Jeffrey's priors for $H_i, i = 1, 2$ are respectively given by

$$\pi_1^N(\theta) = \frac{1}{\theta} \tag{3.2}$$

and

$$\pi_2^N(\theta_1, \theta_2) = \frac{1}{\theta_1 \theta_2}. \tag{3.3}$$

We now derive the marginals with respect to the Jeffrey's priors given by (3.2) to (3.3). For this end, we first observe that the joint pdf of $(Y_1, \cdots, Y_r)$ is given by

$$f(y, \cdots, y_r|\theta) = \frac{n!}{(n-r)!}\theta^r \exp[-\theta(\sum_{i=1}^{r} y_i + (n-r)y_r)], 0 < y_1 < \cdots < y_r. \tag{3.4}$$

Moreover, the marginal pdf of $Y_k, 1 \le k \le r$, is given by

$$f(y_k|\theta) = \frac{n!}{(k-1)!(n-k)!}\theta[\exp(-\theta y_k)]^{n-k+1}[1-\exp(-\theta y_k)]^{k-1}. \tag{3.5}$$

Now, we introduce some notation for the marginals that we will use. For $i = 1, 2$, let $m_i(y_{1k}, y_{2k'})$ and $m_i(\mathbf{y_1}, \mathbf{y_2})$ be the marginal densities of $(Y_{1k}, Y_{2k'})$ and $(\mathbf{Y_1}, \mathbf{Y_2})$ under the hypothesis H_i, respectively. In the following lemma, we give the marginal densities for any one observation in each sample.

Lemma 3.1 We have the marginal density $m_1(y_{1k}, y_{2k'})$ under H_1 and the marginal density $m_2(y_{1k}, y_{2k'})$ under H_2 as follows.

$$\begin{aligned} &m_1(y_{1k}, y_{2k'}) \\ &= \frac{n_1!}{(k-1)!(n_1-k)!}\frac{n_2!}{(k'-1)!(n_2-k')!}\sum_{i=0}^{k-1}\sum_{j=0}^{k'-1}(-1)^{i+j} \\ &\cdot\binom{k-1}{i}\binom{k'-1}{j}\frac{1}{[(n_1-k+i+1)y_{1k}+(n_2-k'+j+1)y_{2k'}]^2} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &m_2(y_{1k}, y_{2k'}) \\ &= \frac{n_1!}{(k-1)!(n_1-k)!}\frac{n_2!}{(k'-1)!(n_2-k')!}\sum_{i=0}^{k-1}\sum_{j=0}^{k'-1}(-1)^{i+j} \\ &\cdot\binom{k-1}{i}\binom{k'-1}{j}\frac{1}{(n_1-k+i+1)y_{1k}}\frac{1}{(n_2-k'+j+1)y_{2k'}}, \end{aligned} \tag{3.7}$$

where $1 \le k \le r_1$ and $1 \le k' \le r_2$.

The proof involves the Binomial Theorem, and details are deferred to the Appendix. It is clear from Lemma 3.1 that the marginal density $(Y_{1k}, Y_{2k'})$ is finite for all $1 \le k \le r_1$ and $1 \le k' \le r_2$ under each hypothesis, and hence we conclude that any training sample of size one is an MTS.

3.2 Intrinsic Bayes Factors

The marginal densities corresponding to the whole data $(\mathbf{Y_1}, \mathbf{Y_2})$ can also be expressed in closed forms. We give these in the following lemma.

Lemma 3.2 For the whole data, we have the marginal density $m_1(\mathbf{y_1}, \mathbf{y_2})$

under H_1, and the marginal density $m_2(\mathbf{y_1}, \mathbf{y_2})$ under H_2 as follows.

$$m_1(\mathbf{y_1}, \mathbf{y_2}) = \frac{n_1!}{(n_1 - r_1)!} \frac{n_2!}{(n_2 - r_2)!} \cdot \frac{\Gamma(r_1 + r_2)}{(\sum_{i=1}^{r_1} y_{1i} + (n_1 - r_1) y_{1r_1} + \sum_{j=1}^{r_2} y_{2j} + (n_2 - r_2) y_{2r_2})^{r_1 + r_2}}, \tag{3.8}$$

and

$$m_2(\mathbf{y_1}, \mathbf{y_2}) = \frac{n_1!}{(n_1 - r_1)!} \frac{n_2!}{(n_2 - r_2)!} \cdot \frac{\Gamma(r_1)\Gamma(r_2)}{(\sum_{i=1}^{r_1} y_{1i} + (n_1 - r_1) y_{1r_1})^{r_1} (\sum_{j=1}^{r_2} y_{2j} + (n_2 - r_2) y_{2r_2})^{r_2}}. \tag{3.9}$$

The proof is routine and is omitted. Now, we give the expressions for the Bayes factors. In lines with the notation in Section 2, we let $B_{12}^N(\mathbf{y}_{k,k'}(l))$ and B_{21}^N represent the Bayes factors computed using the MTS, $\mathbf{y}_{k,k'}(l) = (y_{1k}, y_{2k'})$ and the whole data, respectively. Thus we get the following theorem from Lemmas 3.1 and 3.2.

Theorem 3.1 (i) The Bayes factor computed using the whole data is given by

$$B_{21}^N = \frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1 + r_2)} \frac{(\sum_{i=1}^{r_1} y_{1i} + (n_1 - r_1) y_{1r_1})^{r_1} (\sum_{j=1}^{r_2} y_{2j} + (n_2 - r_2) y_{2r_2})^{r_2}}{(\sum_{i=1}^{r_1} y_{1i} + (n_1 - r_1) y_{1r_1} + \sum_{j=1}^{r_2} y_{2j} + (n_2 - r_2) y_{2r_2})^{r_1 + r_2}}.$$

(ii) The Bayes factor computed using the $\mathbf{y}_{k,k'}(l) = (y_{1k}, y_{2k'})$ is givn by

$$B_{12}^N(\mathbf{y}_{k,k'}(l)) = \frac{\sum_{i=0}^{k-1} \sum_{j=0}^{k'-1} (-1)^{i+j} \binom{k-1}{i} \binom{k'-1}{j} \frac{1}{[(n_1-k+i+1)y_{1k} + (n_2-k'+j+1)y_{2k'}]^2}}{\sum_{i=0}^{k-1} \sum_{j=0}^{k'-1} (-1)^{i+j} \binom{k-1}{i} \binom{k'-1}{j} \frac{1}{(n_1-k+i+1)(n_2-k'+j+1) y_{1k} y_{2k'}}}.$$

From the Theorem 3.1 the arithematic intrinsic Bayes factor B_{ji}^{AI} is given by

$$B_{21}^{AI} = B_{21}^N \cdot \frac{1}{r_1 r_2} \sum_{k=1}^{r_1} \sum_{k'=1}^{r_2} B_{12}^N(\mathbf{y}_{k,k'}(l)). \tag{3.10}$$

Next for computing the expected arithmatic intrinsic Bayes factor, we need to the following lemma.

Lemma 3.3 The expectation of $B_{12}^N(\mathbf{y}_{k,k'}(l))$ is given by

$$
\begin{aligned}
& E_\theta^{H_2}[B_{12}^N(\mathbf{y}_{k,k'}(l))] \\
= & \frac{\theta_1\theta_2}{\sum_{i=0}^{k-1}\sum_{j=0}^{k'-1}(-1)^{i+j}\binom{k-1}{i}\binom{k'-1}{j}/(n_1-k+i+1)(n_2-k'+j+1)} \\
& \cdot\sum_{i=0}^{k-1}\sum_{j=0}^{k'-1}(-1)^{i+j}\binom{k-1}{i}\binom{k'-1}{j}\frac{n_1!}{(k-1)!(n_1-k)!}\frac{n_2!}{(k'-1)!(n_2-k')!} \\
& \cdot\sum_{l=0}^{k-1}\sum_{m=0}^{k'-1}(-1)^{l+m}\binom{k-1}{l}\binom{k'-1}{m}\frac{1}{(\theta_1 b_j c_l-\theta_2 a_i d_m)^3} \\
& \cdot[(\theta_1 b_j c_l+\theta_2 a_i d_m)\log(\theta_1 b_j c_l/\theta_2 a_i d_m)-2(\theta_1 b_j c_l-\theta_2 a_i d_m)], \qquad (3.11)
\end{aligned}
$$

where $\theta=(\theta_1,\theta_2)$, $a_i=(n_1-k+i+1)$, $b_j=(n_2-k'+j+1)$, $c_l=(n_1-k+l+1)$ and $d_m=(n_2-k'+m+1)$.

The proof involves the Binomial Theorem and the tranformation technique and is omitted. Thus we get the following theorem from Definition 3.3 and Lemma 3.3.

Theorem 3.2 The expected arithmatic intrinsic Bayes factor is given by

$$B_{21}^{EAI}=B_{21}^N\cdot\frac{1}{r_1r_2}\sum_{k=1}^{r_1}\sum_{k'=1}^{r_2}E_{\hat\theta}^{H_2}B_{12}^N(\mathbf{y}_{k,k'}(l)) \qquad (3.12)$$

where $\hat\theta_1=r_1/W_1, W_1=\sum_{i=1}^{r_1}y_i+(n_1-r_1)y_{r_1}$ and $\hat\theta_2=r_2/W_2, W_2=\sum_{j=1}^{r_2}y_j+(n_2-r_2)y_{r_2}$.

From the Definition 3.4 and Theorem 3.1, we dervie the median Bayes factors as follow:

$$B_{21}^{MI}=B_{21}^N\cdot MED_l[B_{12}^N(\mathbf{y}_{k,k'}(l))]. \qquad (3.13)$$

4. Numerical Examples

Example 1 : For the hypotheses $H_1:\theta_1=\theta_2$ vs. $H_2:\theta_1\neq\theta_2$, we want to compare the classical LR test with Bayesian test using noninformative priors based on P-values and posterior probabilities of H_1. To illustrate the difference between the frequentist method and the Bayesian test procedure, we examine the cases when $(\theta_1, \theta_2) = (1,1), (1,2), (1,3)$ and $(n_1, n_2) = (10,10), (10,20), (20,20)$ with 20% and 10% censoring. The P-values are computed based on LR test statistic using an F distribution with $2r_1$ and $2r_2$ degrees of freedom. The Bayes factors and the posterior probabilities of H_1 being true are computed assuming equal prior probabilities. The numerical values of P-value and Bayes factors for testing $H_1 : \theta_1 = \theta_2$ vs. $H_2 : \theta_1 \neq \theta_2$ are given in Table 1.

Table 1: P-values and Bayes factors for testing $H_1 : \theta_1 = \theta_2$ vs. $H_2 : \theta_1 \neq \theta_2$

		$(\theta_1, \theta_2) = (1,1)$			
(n_1, n_2)	(r_1, r_2)	P-value	B_{21}^{AI}	B_{21}^{ME}	B_{21}^{EAI}
(10,10)	(8,8)	0.6436	0.5733	0.5902	0.4902
	(9,9)	0.6407	0.5751	0.5920	0.4821
(10,20)	(8,16)	0.8358	0.5324	0.5483	0.4509
	(9,18)	0.7663	0.5429	0.5875	0.4496
(20,20)	(16,16)	0.9340	0.5237	0.5425	0.4429
	(18,18)	0.8425	0.5288	0.5418	0.4289
		$(\theta_1, \theta_2) = (1,2)$			
(n_1, n_2)	(r_1, r_2)	P-value	B_{21}^{AI}	B_{21}^{ME}	B_{21}^{EAI}
(10,10)	(8,8)	0.0723	1.3484	1.3154	1.5956
	(9,9)	0.0593	1.5128	1.5326	1.7985
(10,20)	(8,16)	0.0659	1.2586	1.1133	1.6155
	(9,18)	0.0413	1.7166	1.5307	2.1726
(20,20)	(16,16)	0.0448	1.6502	1.5203	2.2331
	(18,18)	0.0253	2.4455	2.2307	3.1861
		$(\theta_1, \theta_2) = (1,3)$			
(n_1, n_2)	(r_1, r_2)	P-value	B_{21}^{AI}	B_{21}^{ME}	B_{21}^{EAI}
(10,10)	(8,8)	0.0111	3.5943	2.7984	5.6998
	(9,9)	0.0075	4.6380	3.6923	7.5585
(10,20)	(8,16)	0.0048	5.8585	3.7360	10.1238
	(9,18)	0.0021	10.9359	6.8600	18.9017
(20,20)	(16,16)	0.0020	12.6557	9.0996	21.6959
	(18,18)	0.0007	28.0022	17.6593	48.3485

From Table 1, when $(\theta_1, \theta_2) = (1,2)$, the Bayes factors select H_2 properly, but the P-value does not select H_2 for some cases. Actually for this case, as the sample sizes become larger, the P-values will select H_2. The both P-values and Bayes factors support H_1 for the case of $(\theta_1, \theta_2) = (1, 1)$. Also they support H_2 for the case of $(\theta_1, \theta_2) = (1, 3)$. Thus the Bayes test procedure gives fairly reasonable answers.

Example 2 : The following data, given by Proschan (1963), are time intervals of successive failures of the air conditioning system in Boeing 720 jet airplanes. We assume that the time between successive failures for each plane is independent and exponentially distributed.

Three samples of sizes 24,16 and 15 are taken from Proschan (1963). The ordered observations in each case are given below.

Plane 1	3,5,5,13,14,15,22,22,23,30,36,39,44,46,50,72,79,88,97, 102,139,188,197,210
Plane 2	14,14,27,32,34,54,57,59,61,66,67,102,134,152,209,230
Plane 3	12,21,26,27,29,29,48,57,59,70,74,153,326,386,502

In Tables 2 and 3, we provide the P-value, Bayes factors and posterior probabilities for $H_1 : \theta_1 = \theta_2$ vs. $H_2 : \theta_1 \neq \theta_2$ for the first data set (the plane 1 and the plane 2), and also for the second data set (the plane 1 and the pane 3). P-values are computed based on $F(\frac{r_2/W_2}{r_1/W_1}; 2r_1, 2r_2)$, where $W_1 = \sum_{i=1}^{r_1} y_i + (n_1 - r_1)y_{r_1}, W_2 = \sum_{j=1}^{r_2} y_j + (n_2 - r_2)y_{r_2}$ (cf. Lawless(1982)).

For the first data set, there is no strong evidence for H_2 in terms of both the P-value and the posterior probability. But for the second data set, there is a disagreement between the P-value and Bayes factors. When we just look at the statistics r_i/W_i of each set of data, it seems that there is a strong evidence for supporting H_2. However, we can see that the particular observation 326 in plane 3 lessen the statistic r_2/W_2, which makes the P-value large. Meanwhile, Bayes factors give fairly reasonable answers.

It has been noticed that, more often than not, a P-value often does not agree with the posterior probability that the null hypothesis is correct. Delampady and Beger (1990) have showed that the lower bounds of Bayes factors and poterior probabilities in favor of null hypotheses are much larger than the corresponding P-values of the chi-squred goodness of fit test. Furthermore, P-values are computed based on sufficent statistics, which might be misleading for some cases. The arithmetic, expected and median intrinsic Bayes factors are computed based on entire observations so that they give accurate interpretations and fairly steady answers.

Table 2: P-values, Bayes factors and $P(H_1|\mathbf{y})$ for testing $H_1 : \theta_1 = \theta_2$ vs. $H_2 : \theta_1 \neq \theta_2$ based on the data for the planes 1 and 2; $(n_1, n_2, r_1, r_2, r_1/W_1, r_2/W_2)$=(24, 16, 19, 13, 0.01599, 0.01158)

P-value	B_{21}^{AI}	B_{21}^{EAI}	B_{21}^{M}	$P^{AI}(H_1\|\mathbf{y})$	$P^{EAI}(H_1\|\mathbf{y})$	$P^{M}(H_1\|\mathbf{y})$
0.3572	0.5339	0.6637	0.5339	0.6519	0.6010	0.6519

Table 3: P-values, Bayes factors and $P(H_1|\mathbf{y})$ for testing $H_1 : \theta_1 = \theta_2$ vs. $H_2 : \theta_1 \neq \theta_2$ based on the data for the planes 1 and 3; $(n_1, n_2, r_1, r_2, r_1/W_1, r_2/W_2)$=(24, 15, 21, 13, 0.01543, 0.008212)

P-value	B_{21}^{AI}	B_{21}^{EAI}	B_{21}^{M}	$P^{AI}(H_1\|\mathbf{y})$	$P^{EAI}(H_1\|\mathbf{y})$	$P^{M}(H_1\|\mathbf{y})$
0.0670	1.8115	1.5136	1.8115	0.3557	0.3978	0.3557

5. Appendix

Proof of Lemma 3.1. First we derive the marginal density $m_1(y_{1k}, y_{2k'})$ under H_1 using the Binomial Theorem.

$$m_1(y_{1k}, y_{2k'}) = \int_0^\infty f(y_{1k}|\theta) f(y_{2k'}|\theta)\pi_1^N(\theta)d\theta$$

$$= \int_0^\infty \frac{n_1!}{(k_1-1)!(n_1-k_1)!}\frac{n_2!}{(k_2-1)!(n_2-k_2)!}\theta[\exp(-\theta y_{1k})]^{n_1-k_1+1}$$

$$\times \quad [1-\exp(-\theta y_{1k})]^{k_1-1}\theta[\exp(-\theta y_{2k'})]^{n_2-k_2+1}[1-\exp(-\theta y_{2k'})]^{k_2-1}\frac{1}{\theta}d\theta$$

$$= \frac{n_1!}{(k_1-1)!(n_1-k_1)!}\frac{n_2!}{(k_2-1)!(n_2-k_2)!}\sum_{i=0}^{k_1-1}\sum_{j=0}^{k_2-1}(-1)^{i+j}\binom{k_1-1}{i}\binom{k_2-1}{j}$$

$$\times \quad \int_0^\infty \theta\exp[-\theta(iy_{1k} + jy_{2k'} + (n_1-k_1+1)y_{1k} + (n_2-k_2+1)y_{2k'})]d\theta$$

$$= \frac{n_1!}{(k_1-1)!(n_1-k_1)!}\frac{n_2!}{(k_2-1)!(n_2-k_2)!}\sum_{i=0}^{k_1-1}\sum_{j=0}^{k_2-1}(-1)^{i+j}$$

$$\times \quad \binom{k_1-1}{i}\binom{k_2-1}{j}[(n_1-k_1+i+1)y_{1k} + (n_2-k_2+j+1)y_{2k'}]^{-2}.$$

Similarly we derive the margimal density $m_2(y_k, y_{k'})$ under H_2 as follows.

$$m_2(y_{1k}, y_{2k'}) = \int_0^\infty \int_0^\infty f(y_{1k}|\theta_1) f(y_{2k'}|\theta_2) \pi_2^N(\theta_1, \theta_2) d\theta_1 d\theta_2$$

$$= \int_0^\infty \int_0^\infty \frac{n_1!}{(k_1-1)!(n_1-k_1)!} \frac{n_2!}{(k_2-1)!(n_2-k_2)!}$$

$$\times \quad \theta_1[\exp(-\theta_1 y_{1k})]^{n_1-k_1+1}[1-\exp(-\theta_1 y_{1k})]^{k_1-1}$$

$$\times \quad \theta_2[\exp(-\theta_2 y_{2k'})]^{n_2-k_2+1}[1-\exp(-\theta_2 y_{2k'})]^{k_2-1} \frac{1}{\theta_1\theta_2} d\theta_1 d\theta_2$$

$$= \frac{n_1!}{(k_1-1)!(n_1-k_1)!} \frac{n_2!}{(k_2-1)!(n_2-k_2)!}$$

$$\times \quad \sum_{i=0}^{k_1-1} (-1)^i \binom{k_1-1}{i} \int_0^\infty \exp[-\theta_1((n_1-k_1+i+1)y_{1k})] d\theta_1$$

$$\times \quad \sum_{j=0}^{k_2-1} (-1)^j \binom{k_2-1}{j} \int_0^\infty \exp[-\theta_2((n_2-k_2+j+1)y_{2k'})] d\theta_2$$

$$= \frac{n_1!}{(k_1-1)!(n_1-k_1)!} \frac{n_2!}{(k_2-1)!(n_2-k_2)!} \sum_{i=0}^{k_1-1} \sum_{j=0}^{k_2-1} (-1)^{i+j}$$

$$\times \quad \binom{k_1-1}{i} \binom{k_2-1}{j} [(n_1-k_1+i+1)y_{1k}]^{-1} [(n_2-k_2+j+1)y_{2k'}]^{-1}.$$

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