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Generalized One-Level Rotation Designs with Finite Rotation Groups Part II : Variance Formulas of Estimators †

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ABSTRACT

Rotation design is a sampling technique to reduce response burden and to estimate the population characteristics varying in time. Park and Kim(1999) discussed a generation of one-level rotation design which is called as $r_1^m - r_2^{m-1}$ design has more applicable form than existing before. In the structure of $r_1^m - r_2^{m-1}$ design, we derive the exact variances of generalized composite estimators for level, change and aggregate level characteristics of interest, and optimal coefficients minimizing their variances. Finally numerical examples are shown by the efficiency of alternative designs relative to widely used $4-8-4$ rotation design. This is continuous work of Part I studied by Park and Kim(1999).

Key Words: Rotation sampling design; One-level rotation design; Generalized composite estimator; Variance; Optimal weights; Efficiency.

1. Introduction

In a rotation sampling design, sample units have a restricted rotation group life ; as they leave the rotation group, new units are added. Such rotation design has been used in sample survey since the 1950's(Hansen, 1955; Woodruff, 1963; Rao and Graham, 1964; Cochran,1977; Wolter, 1979).

In one-level rotation design, some sample units in a rotation group drop out, are replaced by other units in the same rotation group, and return again to that sample later time(Hansen, 1955; Rao and Graham, 1964; Cantwell, 1990). This type of design is used for the Canadian Labor Force Survey(LFS) conducted by Statistics Canada, the Current Population Survey(CPS) at the U.S. Bureau

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of Census and the Labor Force Survey in Japan. The CPS is a representative survey using one-level rotation design. Park and Kim(1999) proposed a $r_1^m - r_2^{m-1}$ design in the sense that once a final sampling unit is selected, it is interviewed for consecutive r_1 survey periods, drop out for the next r_2 succeeding months, and returns to the sample for another r_1 months; this process is repeated in m times before the unit is out from sample completely. This design has more applicable form than existing before.

(i) mr_1 rotation groups are required and there is a positive integer satisfying $r_2 = lr_1$,

(ii) the same number of final sampling units from each of mr_1 rotation groups are selected so that mr_1 rotation groups are included in sample for each survey month.

(iii) For any given survey month, mr_1 sets of final sampling units from mr_1 rotation groups can be identified by their appearance time to sample from the first time to the mr_1th time and hence mr_1 sets of final sampling units consist of monthly sample.

It is ensured by (iii) that the overlapping percentage between any two survey periods in any number of $r_1^m - r_2^{m-1}$ designs depend on only time lag. They also provided an algorithm which automatically satisfies the above conditions.
The generalized composite estimator(GCE)(Breau and Ernst, 1983) is the most efficient one among several composite estimators developed for rotation sampling. Cantwell(1990) derived variances of GCE's for some characteristics of interest in his balanced rotation design. However variance formulas of Cantwell have a weak point which is crucial since the variance formulas contain unknown coefficients. Our findings are presented in the following four sections. Overlapping formula reflecting from arrangement of rotation group is introduced in Section 2. This formula plays an importance role in deriving the variance formulas of GCE's. In Section 3, the variance formulas of four type GCE's are derived as explicit forms in $r_1^m - r_2^{m-1}$ designs and optimal coefficients minimizing the variances of GCE's are derived. Finally, in Section 4 we investigate the efficiency of alternative designs by comparing them to the usual 4-8-4 design which is being used in CPS.

2. Overlapping Reflecting from Allocation

Following the definition of sample designation by Part I of this paper(Park nd Kim's, 1999), we derive another overlapping formula to be caused by arrangement of rotation group. Although this formula gives the same result which was shown in Part I of this paper, gives a clue to derive the variance of GCE. As defined in Part I of this paper, Let P_α be the αth panel consisting of mr_1 rotation groups numbered by $1, 2, \cdots, mr_1$. We assume that these P_α are well arranged by the algorithm given in Part I of this paper. Figure 2.1 is one example constructed by the algorithm.

In Figure 2.1, to determine the sample for the month $t=$ Jun.Year1 $= 6$, the necessary rotation groups are (6 7 8 3 4 5 6 7 8 1 2 1 2 3) where the first 3 rotation groups (6 7 8) come from P_1, the next 8 rotation groups, (3 4 5 6 7 8 1 2) from P_2 and the last 3 rotation groups (1 2 3) from P_3. Denote these 14 groups by a position vector $\mathcal{G}_B^t = (6\,7\,0\,0\,4\,5\,0\,0\,8\,1\,0\,0\,2\,3)'$ for $t = 6$, and its elements are indexed by $r = 1, \ldots, 14$. The rth element in $\mathcal{G}_B^t$ is 0 if the rotation group in the rth position is not in sample at t. Similarly, the position vector for the month $t + t^*$=Oct.Year1 $= 10$ is $\mathcal{G}_B^{t+t^*} = (4\,5\,0\,0\,8\,1\,0\,0\,2\,3\,0\,0\,6\,7)'$ in which (4 5 8 1) come from P_2 and (2 3 6 7) from P_3. Thus the 6 rotation groups (4 5 8 1) of P_2 and (2 3) of P_3 appear both in $\mathcal{G}_B^t$ and $\mathcal{G}_B^{t+t^*}$. Hence we have an overlapping between months $t = 6$ and $t = 10$. Based on the position vector $\mathcal{G}_B^t$, define this overlapping between months t and $t + t^*$ by a vector

$$F_{t,t+t^*} = (0\,0\,0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1\,1)'$$

in which the rth element is 1 if the same group in the same panel occupies the rth position in both position vectors and 0 otherwise. Similarly, one can see $F_{t,t} = (1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1\,1)'$, t=Jun.Year1=6. Since the 4 rotation groups (4 5 8 1) from P_2 and two (2 3) from P_3 are in both the month t=Jun.Year1=6 and month $t + t^* =$ Oct.Year1 =10, the overlapping percentage between these month is $(6/8) \times 100$. Note that 6 is the number of positions with 1 in both $F_{t,t}$ and $F_{t,t+t^*}$. Therefore this 6 can be also obtained by, $6 = F_{t,t}'F_{t,t+t^*} = (1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1\,1)'(0\,0\,0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1\,1)$. It is easy to check that this observation always holds for any $t^* \geq 0$.

Figure 2.1: $2^4 - 2^3$ system.

		Panels and rotation groups																															
		P_1								P_2								P_3								P_4							
YEAR	MON	1	2	3	4	5	6	7	8	3	4	5	6	7	8	1	2	1	2	3	4	5	6	7	8	3	4	5	6	7	8	1	2
Y1	JAN	o	o			o	o			o	o			o	o																		
	FEB		o	o			o	o			o	o			o	o																	
	MAR			o	o			o	o			o	o			o	o																
	APR				o	o			o	o			o	o			o	o															
	MAY					o	o			o	o			o	o			o	o														
	JUN						o	o			o	o			o	o			o	o													
	JUL							o	o			o	o			o	o			o	o												
	AUG								o	o			o	o			o	o			o	o											
	SEP									o	o			o	o			o	o			o	o										
	OCT										o	o			o	o			o	o			o	o									
	NOV											o	o			o	o			o	o			o	o								
	DEC												o	o			o	o			o	o			o	o							
Y2	JAN													o	o			o	o			o	o			o	o						
	FEB														o	o			o	o			o	o			o	o					

With an appropriate rearrangement of rotation groups in P_α as given in the algoritm provided in Part I of this paper, any $r_1^m - r_2^{m-1}$ design satisfying (i), (ii) and (iii) has the same rotation pattern as $2^4 - 2^3$ design in Figure 2.1(for details, see Park and Kim, 1999). For any $r_1^m - r_2^{m-1}$ design, we introduce the backward shift matrix $\mathbf{L}^{t^*}$ of dimension $(mr_1 + (m-1)r_2) \times (mr_1 + (m-1)r_2)$ with the (r, s)th element $l_{rs} = 1$ if $r - s = t^*$ and $l_{rs} = 0$ if $r - s \neq t^*$ (Horn and Johnson, 1985). Then it is easy to see that $F_{t,t+t^*} = \mathbf{L}^{t^*} F_{t,t}$. For example, the overlapping percentage between month t and $t + t^*$, $t^* \geq 0$ is simply

$$\frac{F'_{t,t}\mathbf{L}^{t^*}F_{t,t}}{8} \times 100$$

where $8 = mr_1$ is the number of rotation groups in $2^4 - 2^3$ design.

The general statement of the overlapping between months t and $t+t^*$, $t^* \geq 0$ for the $r_1^m - r_2^{m-1}$ design is presented as following: The rth element of $F_{t,t}$ is 1 if it is within the ranges from $r = [(i-1)(r_1+r_2)+1]$ to $r = [(i-1)(r_1+r_2)+r_1]$ if $1 \leq i \leq m$ and 0 if it is within the ranges from $r = [ir_1 + (i-1)r_2 + 1]$ to $r = [ir_1 + (i-1)r_2 + r_1]$ if $1 \leq i \leq m-1$. Using the backward shift matrix $\mathbf{L}^{t^*}$ of dimension $(mr_1 + (m-1)r_2) \times (mr_1 + (m-1)r_2)$, the general overlapping rate between the two samples at time t and $t + t^*$ for $t^* \geq 0$ is expressed by

$$O(t, t^*) = \frac{F'_{t,t}F_{t,t+t^*}}{mr_1} = \frac{F'_{t,t}\mathbf{L}^{t^*}F_{t,t}}{mr_1} \tag{2.1}$$

where $F_{t,t+t^*} = \mathbf{L}^{t^*}F_{t,t}$. This relation $F_{t,t+t^*} = \mathbf{L}^{t^*}F_{t,t}$ is valid only when mr_1 rotation groups at any survey month are well ordered by their appearance time to the sample for the 1st time to the mr_1th time from the right to the left.

3. Generalized Composite Estimator(GCE)

The estimators from rotation sampling have been developed during the past several decades with the fixed weights (Rao and Graham, 1965; Hansen, 1978; Huang and Ernst, 1981; Kumar and Lee, 1983). The rotation groups are assumed to be independent, and sampling units in a rotation group are also assumed to be independent. Therefore, measurements from sampling units in different rotation groups or panels are independent. But the measurements from the same sampling units in any two different survey months are related with a correlation structure. As we used the models to obtain parameters, we do not specify the sampling design except that it is a probability measure on the set of all possible samples.

Let $x_{t,(\alpha(i),\gamma(i))}$ be an estimator of a characteristic of interest in which the estimator is measured on the sampling units which appears to the sample for the ith time at month t where $\gamma(i)$ denotes the rotation group providing the sampling units and $\alpha(i)$ is the panel $P_{\alpha(i)}$ containing $\gamma(i)$. Hence $x_{t,(\alpha(i),\gamma(i))}$, $i = 1, 2, \cdots, mr_1$ are independent since all rotation groups at a fixed time t are different, and $x_{t,(\alpha(i),\gamma(i))}$ and $x_{t^*,(\alpha(i'),\gamma(i'))}$, $t \neq t^*$ are correlated only if $\alpha(i) = \alpha(i')$ and $\gamma(i) = \gamma(i')$ since that two measurements, $x_{t,(\alpha(i),\gamma(i))}$ and $x_{t^*,(\alpha(i'),\gamma(i'))}$, are measured from the same rotation group and panel implies that they are measured from the same sampling units; otherwise independent. Since once a sampling unit appearing in the sample for the ith time at time t is determined, $(\alpha(i), \gamma(i))$ is uniquely determined by allocation algorithm given Part I of this paper, we use $x_{t,i}$ instead of $x_{t,(\alpha(i),\gamma(i))}$ for notational simplicity. Since we have mr_1 sets of sampling units in the $r_1^m - r_2^{m-1}$ design for any survey month as discussed by Park and Kim(1999), we can define the GCE y_t for a characteristic X of interest at month t as

$$y_t = \sum_{i=1}^{mr_1} a_i x_{t,i} - \omega \sum_{i=1}^{mr_1} b_i x_{t-1,i} + \omega y_{t-1} \tag{3.1}$$

where the weight ω is bounded as $0 \leq \omega < 1$. The other weights a_i's and b_i's may take any value subject to $\sum_{i=1}^{mr_1} a_i = \sum_{i=1}^{mr_1} b_i = 1$. As we expect, the GCE becomes to be simple estimator if $\omega = 0$, where the overlapping and previous measurement at month $t-1$ are ignored.

3.1. Variance

We assume there exist at least first two moments so that the expectation $E(x_{t,i}) = \mu_t$ for all i, and a common variance $Var(x_{t,i}) = \sigma^2$ for all t and i. The estimator y_t is an unbiased estimator of μ_t since y_t converges in mean square. However y_t may not be unbiased when a rotation group bias is presented(Bailar, 1975 ;Kumar and Lee ,1983; Breau and Ernst, 1983). Thus we assume in this paper that such a bias is not exposed. Further, from the above discussion we assume the following covariance between $x_{t,i}$ and $x_{t',i'}$ for $t, t' = 1, 2, \ldots,$ and $i, i' = 1, \ldots, mr_1$ as

$$Cov(x_{t,i},\ x_{t',i'}) = \begin{cases} \sigma^2 & \text{if } t = t',\ \alpha(i) = \alpha(i') \text{ and } \gamma(i) = \gamma(i') \\ \rho_{tt'}\sigma^2 & \text{if } t \neq t',\ \alpha(i) = \alpha(i') \text{ and } \gamma(i) = \gamma(i') \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

where the subscript in $\rho_{tt'}$ is $tt' = |t - t'|$ so that the covariance is stationary.

Consider $Cov(\sum_{i=1}^{mr_1} a_i x_{t,i}, \sum_{i'=1}^{mr_1} b_{i'} x_{t-t^*,i'})$, for $t^* \geq 1$ in $r_1^m - r_2^{m-1}$ design. Let A_{t,t^*} be the set of ($x_{t,i}$ and $x_{t-t^*,i'}$)'s for $i, i' = 1, 2, \ldots, mr_1$ with $\alpha(i) = \alpha(i')$ and $\gamma(i) = \gamma(i')$. Then by (3.2)

$$Cov\left(\sum_{i=1}^{mr_1} a_i x_{t,i}, \sum_{i'=1}^{mr_1} b_{i'} x_{t-t^*,i'}\right) = \rho_{t^*}\sigma^2 \sum_{i=1}^{mr_1}\sum_{i'=1}^{mr_1} a_i b_{i'} I_{i,i'}(A_{t,t^*})$$

where $I_{i,i'}(A_{t,t^*}) = 1$ if $(x_{t,i}, x_{t-t^*,i'}) \in A_{t,t^*}$ and 0 otherwise, and $\rho_0 \equiv 1$. This $\sum_{i=1}^{mr_1}\sum_{i'=1}^{mr_1} a_i b_{i'} I_{i,i'}(A_{t,t^*})$ can be found similarly with a modified $F_{t,t}$ as discussed in Section 2. Denote the vector $\mathbf{a}_i = (a_{ir_1}, a_{ir_1-1}, \ldots, a_{(i-1)r_1+1})'$, $\mathbf{b}_{i'} = (b_{i'r_1}, b_{i'r_1-1}, \ldots, b_{(i'-1)r_1+1})'$ and $\mathbf{0}_{r_2}$ be $r_2 \times 1$ null vector. Let

$$\begin{aligned} \mathbf{a^o} &\equiv F_{a,t,t} = (\mathbf{a}_1', \mathbf{0}_{r_2}', \mathbf{a}_2', \mathbf{0}_{r_2}', \cdots, \mathbf{a}_m')' \text{ and} \\ \mathbf{b^o} &\equiv F_{b,t-t^*,t-t^*} = (\mathbf{b}_1', \mathbf{0}_{r_2}', \mathbf{b}_2', \mathbf{0}_{r_2}', \cdots, \mathbf{b}_m')' \end{aligned}$$

where $F_{a,t,t}$ and $F_{b,t-t^*,t-t^*}$ has the same interpretation as in $F_{t,t}$ except that the element with 1's in $F_{t,t}$ are replaced by a_i's for $F_{a,t,t}$ and by $b_{i'}$'s for $F_{b,t-t^*,t-t^*}$.

Since $F_{b,t-t^*,t-t^*}$ will be $F_{b,t-t^*,t} = \mathbf{L}^{t^*} F_{b,t-t^*,t-t^*}$ after t^* month later, and A_{t,t^*} is defined by the set of the positions with nonzero in both $F_{a,t,t}$ and $F_{b,t-t^*,t}$, the term $\sum_{i=1}^{mr_1}\sum_{i'=1}^{mr_1} a_i b_{i'} I_{i,i'}(A_{t,t^*})$ is simply $F_{a,t,t}' \mathbf{L}^{t^*} F_{b,t-t^*,t-t^*}$. Therefore,

$$Cov\left(\sum_{i=1}^{mr_1} a_i x_{t,i}, \sum_{i'=1}^{mr_1} b_{i'} x_{t-t^*,i'}\right) = \rho_{t^*}\sigma^2 F_{a,t,t}' \mathbf{L}^{t^*} F_{b,t-t^*,t-t^*} = \rho_{t^*}\sigma^2 \mathbf{a^o}' \mathbf{L}^{t^*} \mathbf{b^o} \tag{3.3}$$

The $\rho_{t^*}\sigma^2\mathbf{L}^{t^*}$ in (3.3) contains also the covariances not occurred between time t and $t-t^*$. These meaningless terms are purged out by $\mathbf{0}_{r_2}$'s in $\mathbf{a^o}, \mathbf{b^o}$. We reform (3.3) without meaningless terms as follows

$$\rho_{t^*}\sigma^2\mathbf{a^o}'\mathbf{L}^{t^*}\mathbf{b^o} \equiv \rho_{t^*}\sigma^2\mathbf{a}'\mathbf{R}\mathbf{L}^{t^*}\mathbf{R}'\mathbf{b} \qquad (3.4)$$

where $\mathbf{a} = (\mathbf{a}_1', \mathbf{a}_2', \cdots, \mathbf{a}_m')'$ and $\mathbf{b} = (\mathbf{b}_1', \mathbf{b}_2', \cdots, \mathbf{b}_m')'$ have the size $mr_1 \times 1$ and $\mathbf{R}$ is a $mr_1 \times (mr_1 + (m-1)r_2)$ matrix with the (i,j)th element, $(\mathbf{R})_{ij} = 1$ if $j - i = (k-1)r_2$ where $(k-1)r_1 + 1 \le i \le kr_1$ for $k = 1, 2, \ldots, m$; and $(\mathbf{R})_{ij} = 0$, otherwise. This $\mathbf{R}$ matrix removes irrelevant rows of $\mathbf{L}^{t^*}$ corresponding $\mathbf{0}_{r_2}$'s in $\mathbf{a^o}$ by premultiplying $\mathbf{R}\mathbf{L}^{t^*}$. Similarly, $\mathbf{L}^{t^*}\mathbf{R}'$ eliminates irrelevant columns of $\mathbf{L}^{t^*}$ corresponding $\mathbf{0}_{r_2}$'s in $\mathbf{b^o}$.

Using the above equation, right side of (3.4), we obtain the variances of y_t and $y_t - y_{t-t^*}$ as follows:

Theorem 3.1. *Suppose that a $r_1^m - r_2^{m-1}$ design satisfies the conditions in Section 1 and is constructed by the allocation algorithm given in Part I. Then using the covariance structure (3.2),*

$$Var(y_t) = \frac{\sigma^2}{1-\omega^2}\Big[\mathbf{a}'\Big(\mathbf{I} + 2\mathbf{Q}_1\Big)\mathbf{a} + \omega^2\mathbf{b}'\Big(\mathbf{I} + 2\mathbf{Q}_1\Big)\mathbf{b} - 2\mathbf{a}'\Big(\omega^2\mathbf{I} + \omega^2\mathbf{Q}' + \mathbf{Q}_1\Big)\mathbf{b}\Big]$$

$$Var(y_t - y_{t-t^*}) = 2(1-\omega^{t^*})Var(y_t)$$

$$- 2\sigma^2\left(\frac{1-\omega^{t^*}}{1-\omega^2}\right)\Big[\mathbf{a}'\mathbf{Q}^*_{t^*,0}\mathbf{a} - \mathbf{a}\Big(\mathbf{Q}^*_{t^*,1} + \omega^2\mathbf{Q}^{*\prime}_{t^*,-1}\Big)\mathbf{b} + \omega^2\mathbf{b}'\mathbf{Q}^*_{t^*,0}\mathbf{b}\Big]$$

where the $\mathbf{I}_{mr_1 \times mr_1}$ is an identity matrix, $T = mr_1 + (m-1)r_2$, $1 \le t^* \le T-1$, and $\mathbf{Q}_i = \sum_{j=i}^{T-1}\omega^j\rho_j\mathbf{L}^j$, $i = 0, 1, 2, \ldots, T-1$ with $\mathbf{L}^0 \equiv \mathbf{I}$ and

$$\mathbf{Q^*}_{k,q} = \left[\frac{\omega^k}{1-\omega^k}\sum_{i=1}^{k}\left(\frac{1}{\omega^{2i}} - 1\right)\omega^{i+q}\rho_{1,i+q}\mathbf{L}^{i+q} + \left(\frac{1}{\omega^k} + 1\right)\sum_{i=k+1+q}^{T-1}\omega^i\rho_{1,i}\mathbf{L}^i\right] \qquad (3.5)$$

Often of primary importance are the aggregate level and level change over a certain length of time(Cantwell,1990 ; Fuller, 1990). Denote by S_t^α the sum of the GCE's and $S_t^\alpha - S_{t-t^*}^\alpha$, $\alpha \ge t^*$ the sum change of the GCE's as

$$\begin{aligned} S_t^\alpha &= y_t + y_{t-1} + \cdots + y_{t-\alpha+1} \\ S_t^\alpha - S_{t-t^*}^\alpha &= y_t + y_{t-1} + \cdots + y_{t-\alpha+1} - (y_{t-t^*} + y_{t-t^*-1} + \cdots + y_{t-t^*-\alpha+1}). \end{aligned}$$

Theorem 3.2. *Under the same assumption given in Theorem 3.1,*

$$\begin{aligned} Var(S_t^\alpha) &= \alpha^2 Var(y_t) - \sum_{j=1}^{\alpha-1}(\alpha - j)Var(y_t - y_{t-j}) \\ Var(S_t^\alpha - S_{t-t^*}^\alpha) &= \sum_{j=-\alpha+1}^{\alpha-1}(\alpha - |j|)Var(y_t - y_{t-t^*+j}) - 2\sum_{j=1}^{\alpha-1} Var(y_t - y_{t-j}) \end{aligned}$$

The proofs of Theorem 3.1 and 3.2 are given in the Appendix.

3.2. Optimal coefficients a and b

The $Var(y_t)$ and $Var(y_t - y_{t-t^*})$ depend on coefficients or weights $a_1, a_2 \cdots, a_{mr_1}$, $b_1, b_2, \cdots, b_{mr_1}$ as well as ω and $\rho_{tt'}$ where the indices in a_i and b_i indicate the corresponding visiting time to sample at a survey month t. Previously the composite estimators are shown with fixed weights (Cantwell, 1990; Breau and Ernst, 1983). Since $Var(y_t)$ is a special case of $Var(S_t^\alpha)$ when we put $\alpha = 1$, and $Var(y_t - y_{t-t^*})$ is also obtained from $Var(S_t^\alpha - S_{t-t^*}^\alpha)$ with $\alpha = 1$, we only show the optimum weights of a_i's and b_i's, in the estimator by minimizing respective variances of $Var(S_t^\alpha)$ and $Var(S_t^\alpha - S_{t-t^*}^\alpha)$. Let λ_1 and λ_2 be Lagrange multipliers. In the terms of variances and Lagrange multipliers, we set the object function O_1 for the variance of S_t^α to be minimized, and O_2 for the variance of $S_t^\alpha - S_{t-t^*}^\alpha$ to be minimized

$$\begin{aligned} O_1 &= Var(S_t^\alpha) - 2\lambda_1(\mathbf{1}'\mathbf{a} - \mathbf{1}) - 2\lambda_2(\mathbf{1}'\mathbf{b} - 1), \\ O_2 &= Var(S_t^\alpha - S_{t-t^*}^\alpha) - 2\lambda_1(\mathbf{1}'\mathbf{a}^* - 1) - 2\lambda_2(\mathbf{1}'\mathbf{b}^* - 1). \end{aligned}$$

$\mathbf{1}$ of the size $(mr_1 \times 1)$ is unit vector. The scalar $(\mathbf{1}'\mathbf{a} - 1)$ and $(\mathbf{1}'\mathbf{b} - 1)$ from the constraints of $\sum_{i=1}^{mr_1} a_i = 1$ and $\sum_{i=1}^{mr_1} b_i = 1$, and matrices $(\mathbf{1}'\mathbf{a}^* - 1)$ and $(\mathbf{1}'\mathbf{b}^* - 1)$ from the constraints of $\sum_{i=1}^{mr_1} a_i^* = 1$ and $\sum_{i=1}^{mr_1} b_i^* = 1$.
The two variances, $Var(S_t^\alpha)$ and $Var(S_t^\alpha - S_{t-t^*}^\alpha)$, in Theorem 3.2 can be compressed as

$$Var(S_t^\alpha) = \frac{\sigma^2}{1-\omega^2}\left[\mathbf{a}'\mathbf{Q_{m,1}}\mathbf{a} + \mathbf{a}'\mathbf{Q_{m,2}}\mathbf{b} + \mathbf{b}'\mathbf{Q_{m,3}}\mathbf{b}\right] \text{ and}$$

$$Var(S_t^\alpha - S_{t-t^*}^\alpha) = \frac{2\sigma^2}{1-\omega^2}\left[\mathbf{a}^{*'}\mathbf{Q_{c,1}}\mathbf{a}^* + \mathbf{a}^{*'}\mathbf{Q_{c,2}}\mathbf{b}^* + \mathbf{b}^{*'}\mathbf{Q_{c,3}}\mathbf{b}^*\right], \text{ respectively,}$$

where $\mathbf{Q}_{m,1} = \Big(\omega_c(\mathbf{I}+2\mathbf{Q}_1) + 2\sum_{j=1}^{\alpha-1}(\alpha-j)(1-\omega^j)\mathbf{Q}^*_{j,0}\Big)$,

$$\mathbf{Q}_{m,2} = -2\Big(\omega_c(\omega^2\mathbf{I}+\mathbf{Q}_1+\omega^2\mathbf{Q}'_1) + \sum_{j=1}^{\alpha-1}(\alpha-j)(1-\omega^j)\Big(\mathbf{Q}^*_{j,1}+\omega^2\mathbf{Q}^{*'}_{j,-1}\Big)\Big),$$

$\mathbf{Q}_{m,3} = \omega^2\mathbf{Q}_{m,1}$, where $\omega_c = \dfrac{\alpha(1-\omega^2)-2\omega(1-\omega^\alpha)}{(1-\omega)^2}$ and

$$\mathbf{Q}_{c,1} = \Big(\sum_{j=-(\alpha-1)}^{\alpha-1}(\alpha-|j|)(1-\omega^{t^*-j})\Big(\mathbf{I}+2\mathbf{Q}_1-\mathbf{Q}^*_{t^*-j,0}\Big) - 2\sum_{j=1}^{\alpha-1}(\alpha-j)(1-\omega^j)\Big(\mathbf{I}+2\mathbf{Q}_1-\mathbf{Q}^*_{j,0}\Big)\Big)$$

$$\mathbf{Q}_{c,2} = \Big(-\sum_{j=-(\alpha-1)}^{\alpha-1}(\alpha-|j|)(1-\omega^{t^*-j})\Big(2(\omega^2\mathbf{I}+\mathbf{Q}_1+\omega^2\mathbf{Q}'_1)-\mathbf{Q}^*_{t^*-j,1}-\omega^2\mathbf{Q}^{*'}_{t^*-j,-1}\Big)$$

$$+2\sum_{j=1}^{\alpha-1}(\alpha-j)(1-\omega^j)\Big(2(\omega^2\mathbf{I}+\mathbf{Q}_1+\omega^2\mathbf{Q}'_1)-\mathbf{Q}^*_{j,1}-\omega^2\mathbf{Q}^{*'}_{j,-1}\Big)\Big),$$

$\mathbf{Q}_{c,3} = \omega^2\mathbf{Q}_{c,1}$

Define $\mathbf{A} = \mathbf{G}\big(\mathbf{J}\mathbf{G}-\mathbf{I}\big)$. For O_1, where $\mathbf{G} = \big(\mathbf{Q}_{m,1}+\mathbf{Q}'_{m,1}\big)^{-1}$ and $\mathbf{J} = \mathbf{1}\mathbf{1}'$. Using above expressions, it can be shown that the optimal coefficients of $\mathbf{a}$ and $\mathbf{b}$ are given by

$$\widehat{\mathbf{a}} = [\mathbf{I} - \frac{1}{\omega^2}\mathbf{A}\mathbf{Q}_{m,2}\mathbf{A}\mathbf{Q}'_{m,2}]^{-1}[\mathbf{I}+\mathbf{A}\mathbf{Q}_{m,2}]\mathbf{G}\mathbf{1} \qquad (3.6)$$

$$\widehat{\mathbf{b}} = \frac{1}{\omega^2}\mathbf{A}\mathbf{Q}'_{m,2}\widehat{\mathbf{a}} + \mathbf{G}\mathbf{1}. \qquad (3.7)$$

Similarly, from O_2, the optimal coefficients of $\mathbf{a}^*$ and $\mathbf{b}^*$ for $Var(S_t^\alpha - S_{t-t^*}^\alpha)$ can be obtained from $\widehat{\mathbf{a}}$ and $\widehat{\mathbf{b}}$ by replacing $\mathbf{Q}_{m,1}$, $\mathbf{Q}_{m,2}$, and $\mathbf{Q}_{m,3}$ with $\mathbf{Q}_{c,1}$, $\mathbf{Q}_{c,2}$, and $\mathbf{Q}_{c,3}$, respectively. The process is rather too complex to digest, and the example below may help to see the logics of the results.

Example 3.1. Table 3.1 shows the optimal coefficients for the 3^2-9^1 design. The optimal coefficients $\widehat{\mathbf{a}} = (\widehat{a}_1,\widehat{a}_2,\ldots,\widehat{a}_6)$ and $\widehat{\mathbf{b}} = (\widehat{b}_1,\widehat{b}_2,\ldots,\widehat{b}_6)$ for the variance $Var(y_t)$, and $\widehat{\mathbf{a}}^* = (\widehat{a}^*_1,\widehat{a}^*_2,\ldots,\widehat{a}^*_6)$ and $\widehat{\mathbf{b}}^* = (\widehat{b}^*_1,\widehat{b}^*_2,\ldots,\widehat{b}^*_6)$ for the variance $Var(y_t-y_{t-1})$ are obtained by the formulas (3.6) and (3.7). These coefficients are calculated under two correlation structures along with a fixed $\omega=0.5$ and $\rho=0.5, 0.7, 0.9$: the first is the exponential correlation $\rho_{tt'} = \rho^{|t-t'|}$ and the second is the arithmetic correlation $\rho_{tt'} = \rho+(1-|t-t'|)\times 0.01$. These correlations are applied to (3.2) to obtain the coefficients or weights in the Table 3.1. The weights $\widehat{a}_i$ and $\widehat{a}^*_i$ are the coefficients of $x_{t,i}$ and $\widehat{b}_i$ and $\widehat{b}^*_i$ are the coefficients of $x_{t-1,i}$. The coefficients of $\widehat{\mathbf{a}}$, $\widehat{\mathbf{a}}^*$, $\widehat{\mathbf{b}}$, $\widehat{\mathbf{b}}^*$ depend on the overlapping

Table 3.1: Weights for $3^2 - 9^1$ design under two correlation patterns with fixed $\omega = 0.5$

GCE	weights	exponential correlation			arithmetic correlation		
		0.5	0.7	0.9	0.5	0.7	0.9
	a_1	0.15625	0.14115	0.11044	0.15225	0.13490	0.10573
	a_2	0.17188	0.17943	0.19480	0.17642	0.18771	0.20589
	a_3	0.17188	0.17943	0.19480	0.17137	0.17747	0.18852
	a_4	0.15625	0.14114	0.11037	0.15221	0.13482	0.10556
	a_5	0.17188	0.17943	0.19480	0.17640	0.18768	0.20584
y_t	a_6	0.17188	0.17943	0.19480	0.17134	0.17742	0.18846
	b_1	0.21875	0.25052	0.29790	0.24893	0.28579	0.32831
	b_2	0.21875	0.25052	0.29790	0.20554	0.23496	0.27982
	b_3	0.06250	-0.00103	-0.09570	0.04576	-0.02044	-0.10779
	b_4	0.21875	0.25052	0.29790	0.24889	0.28572	0.32822
	b_5	0.21875	0.25052	0.29790	0.20543	0.23483	0.27969
	b_6	0.06250	-0.00105	-0.09589	0.04546	-0.02085	-0.10825
	a_1^*	0.11229	0.08924	0.06758	0.11996	0.09695	0.07118
	a_2^*	0.19276	0.20580	0.21929	0.18751	0.19865	0.21185
	a_3^*	0.19495	0.20495	0.21313	0.19255	0.20441	0.21696
	a_4^*	0.11229	0.08925	0.06762	0.11997	0.09698	0.07124
	a_5^*	0.19276	0.20580	0.21928	0.18750	0.19863	0.21185
$y_t - y_{t-1}$	a_6^*	0.19495	0.20495	0.21310	0.19251	0.20437	0.21692
	b_1^*	0.25101	0.28085	0.30644	0.23015	0.25762	0.28799
	b_2^*	0.22811	0.25610	0.28433	0.23514	0.26503	0.29597
	b_3^*	0.02088	-0.03694	-0.09076	0.03460	-0.02277	-0.08404
	b_4^*	0.25101	0.28084	0.30639	0.23016	0.25762	0.28795
	b_5^*	0.22811	0.25609	0.28430	0.23519	0.26508	0.29598
	b_6^*	0.02088	-0.03694	-0.09070	0.03476	-0.02258	-0.08385

between months t and $t-1$. The coefficients $\widehat{b}_3$, $\widehat{b}_6$, $\widehat{b}_3^*$ and $\widehat{b}_6^*$ are small or even negative, and the coefficients $\widehat{b}_i$, $\widehat{b}_i^*$, $i \neq 3$ or 6 are relatively large for fixed ρ. The reason is that the measurements $x_{t-1,3}$ and $x_{t-1,6}$, corresponding to the coefficients $\widehat{b}_3$, $\widehat{b}_6$, $\widehat{b}_3^*$ and $\widehat{b}_6^*$, will be rotated out at month t; hence, they are not in the sample at month t. But the measurement $x_{t-1,i}$ for $i \neq 3$ or 6, corresponding to the coefficients $\widehat{b}_i$, $\widehat{b}_i^*$, are in the sample at month t. Similarly, corresponding to $\widehat{\mathbf{a}}$ and $\widehat{\mathbf{a}}^*$, the measurements $x_{t,1}$ and $x_{t,4}$, are in the sample at month t, but not in the sample at month $t-1$. Therefore, the coefficients $\widehat{a}_1, \widehat{a}_4$, $\widehat{a}_1^*, \widehat{a}_4^*$ are relatively smaller than other coefficients $\widehat{a}_i$ and $\widehat{a}_i^*$, $i \neq 1$ or 4. Since $y_t - y_{t-1}$ depends on the overlapping between months t and $t-1$ more than y_t does for each fixed ρ, the optimal coefficients $\widehat{b}_i^* < \widehat{b}_i$ for $i = 3, 6$, while $\widehat{b}_i^* > \widehat{b}_i$ for $i = 1, 2, 4, 5$, and $\widehat{a}_i^* < \widehat{a}_i$ for $i = 1, 4$ and $\widehat{a}_i^* > \widehat{a}_i$ for $i = 2, 3, 5, 6$. It appears that the coefficients increase with the overlapping and increasing correlation ρ, while they decrease with no overlapping and increasing ρ. The coefficients with overlapping under the arithmetic correlation are larger than those coefficients under the exponential correlation; on the other hand, the coefficients without overlapping under the arithmetic correlation are smaller than those under the exponential correlation. Generally, it is true for most of the $r_1^m - r_2^{m-1}$ designs.

4. Design Efficiency

The efficiency of an alternative design is defined by the ratio of the two variances, one for an estimate from the alternative design and the other from the standard $4^2 - 8^1$ design, and it measures the comparative efficiency of the alternative designs. Tables 3, 4, 5, and 6 show the design efficiency of 14 alternative designs for the estimate y_t of the month t, the difference $y_t - y_{t-1}$ for the monthly change, S_t^3 of quarter sum, and $S_t^3 - S_{t-12}^3$ of yearly change for quarter sum, respectively. Computing the efficiency of an alternative design only requires the variance formula of GCE, we assume several simplifying conditions for this example. $x_{t,i} = \sum_{s=1}^{n_c} x_{t,i,s}/n_c$ where the measurement $x_{t,i,s}$ is of the sth unit in $x_{t,i}$ of size $n_{t,i}$, and $n_{t,i} = n_c$ for simplicity of calculation. We also assume the covariance below between $x_{t,i,s}$ and $x_{t',i',s'}$ equals to $\sigma^2\rho^{|t-t'|}$ when $s = s'$ and these two x's are from the same rotation group and panel. And hence

$$Cov(x_{t,i}, x_{t',i'}) = \begin{cases} \frac{\sigma^2}{n_c}\rho^{|t-t'|} & \text{if } \alpha(i) = \alpha(i') \text{ and } \gamma(i) = \gamma(i') \\ 0 & \text{otherwise} \end{cases}$$

Table 4.1: Efficiency of alternative design compared to $4^2 - 8^1$ design for y_t with equal sample size under exponential correlation(optimum value of ω)

Designs	ρ				
	0.5	0.6	0.7	0.8	0.9
$2^2 - 10^1$	0.9830(0.3)	0.9744(0.3)	0.9624(0.4)	0.9463(0.5)	0.9157(0.6)
$2^3 - 4^2$	0.9831(0.3)	0.9745(0.3)	0.9636(0.4)	0.9527(0.5)	0.9449(0.6)
$2^4 - 2^3$	0.9843(0.2)	0.9788(0.2)	0.9746(0.3)	0.9802(0.3)	1.0176(0.5)
$2^4 - 4^3$	0.9831(0.3)	0.9745(0.3)	0.9637(0.4)	0.9535(0.5)	0.9489(0.6)
$2^5 - 2^4$	0.9844(0.2)	0.9789(0.2)	0.9752(0.3)	0.9814(0.3)	1.0230(0.4)
$2^6 - 2^5$	0.9845(0.2)	0.9790(0.2)	0.9757(0.3)	0.9822(0.3)	1.0259(0.4)
$3^2 - 9^1$	0.9905(0.3)	0.9853(0.4)	0.9782(0.5)	0.9684(0.6)	0.9488(0.7)
$3^3 - 3^2$	0.9908(0.3)	0.9870(0.4)	0.9849(0.4)	0.9893(0.5)	1.0211(0.6)
$3^4 - 3^3$	0.9909(0.3)	0.9872(0.4)	0.9854(0.4)	0.9914(0.5)	1.0292(0.6)
$5^2 - 5^1$	1.0079(0.4)	1.0121(0.5)	1.0189(0.6)	1.0322(0.6)	1.0653(0.7)
$5^2 - 10^1$	1.0078(0.4)	1.0119(0.5)	1.0175(0.6)	1.0261(0.7)	1.0378(0.8)
$6^2 - 6^1$	1.0141(0.4)	1.0216(0.5)	1.0325(0.6)	1.0518(0.7)	1.0936(0.8)
$7^2 - 7^1$	1.0190(0.4)	1.0294(0.5)	1.0442(0.6)	1.0682(0.7)	1.1150(0.8)
$8^2 - 8^1$	1.0230(0.4)	1.0358(0.5)	1.0542(0.6)	1.0829(0.7)	1.1355(0.8)

The variances in Theorem 3.1 and 3.2 with the optimum coefficients of (3.6) and (3.7) are now considered as a function of only ω and ρ. The ω takes a value which minimizes the variances for each design as shown in the parenthesis in the tables below, and ranges from 0.1 to 0.9 with the increment of 0.1. The correlation ρ

takes the values $\rho = 0.5, 0.6, 0.7, 0.8$, or 0.9. The values in each Table provide the efficiency of alternative designs with the same sample size for both alternative and $4^2 - 8^1$ designs.

For example, 24 units are the size of the sample for both of $3^2 - 9^1$ and $4^2 - 8^1$ designs. That is, $n_c = 4$ for $3^2 - 9^1$ design and $n_c = 3$ for $4^2 - 8^1$ design. Smaller value than 1 implies that the alternative design is more efficient than $4^2 - 8^1$ design for the given optimal ω and correlation.

Table 4.2: Efficiency of alternative design compared to $4^2 - 8^1$ design for $y_t - y_{t-1}$ with equal sample size under exponential correlation(optimum value of ω)

Designs	ρ				
	0.5	0.6	0.7	0.8	0.9
$2^2 - 10^1$	1.1586(0.6)	1.2036(0.6)	1.2556(0.7)	1.3163(0.8)	1.3955(0.9)
$2^3 - 4^2$	1.1584(0.6)	1.2030(0.6)	1.2542(0.7)	1.3144(0.8)	1.3946(0.9)
$2^4 - 2^3$	1.1559(0.5)	1.1983(0.6)	1.2476(0.6)	1.3048(0.7)	1.3848(0.8)
$2^4 - 4^3$	1.1584(0.6)	1.2029(0.6)	1.2540(0.7)	1.3142(0.8)	1.3943(0.9)
$2^5 - 2^4$	1.1557(0.5)	1.1979(0.6)	1.2467(0.6)	1.3036(0.7)	1.3826(0.8)
$2^6 - 2^5$	1.1556(0.5)	1.1977(0.6)	1.2462(0.6)	1.3028(0.7)	1.3811(0.8)
$3^2 - 9^1$	1.0469(0.6)	1.0586(0.7)	1.0721(0.8)	1.0858(0.8)	1.1031(0.9)
$3^3 - 3^2$	1.0464(0.6)	1.0577(0.7)	1.0706(0.7)	1.0830(0.8)	1.1018(0.9)
$3^4 - 3^3$	1.0463(0.6)	1.0576(0.7)	1.0703(0.7)	1.0827(0.8)	1.1015(0.9)
$5^2 - 5^1$	0.9742(0.6)	0.9683(0.7)	0.9618(0.8)	0.9544(0.9)	0.9471(0.9)
$5^2 - 10^1$	0.9742(0.6)	0.9684(0.7)	0.9619(0.8)	0.9542(0.9)	0.9475(0.9)
$6^2 - 6^1$	0.9578(0.6)	0.9486(0.7)	0.9384(0.8)	0.9264(0.9)	0.9153(0.9)
$7^2 - 7^1$	0.9465(0.7)	0.9350(0.7)	0.9224(0.8)	0.9076(0.9)	0.8940(0.9)
$8^2 - 8^1$	0.9382(0.7)	0.9251(0.7)	0.9109(0.8)	0.8941(0.9)	0.8787(0.9)

Table 4.1 shows the relative efficiency of the 14 alternative designs at time t for the 5 correlation coefficients and optimal ω's in the parentheses. The 5 correlations between $x_{t,i}$ and $x_{t-1,i'}$ from same sampling units range from 0.5 to 0.9 as shown in the first row. For the correlation from 0.5 to 0.8, the first 9 alternative designs are more efficient than the $4^2 - 8^1$ design, while the last 5 designs are less efficient. For the correlation 0.9, the 4 alternative designs are more efficient, but the remaining 10 are less efficient than the standard $4^2 - 8^1$ design. Table 4.2 shows the efficiency of the 14 alternative designs for the monthly changes. The last 5 alternative designs from $5^2 - 5^1$ to $8^2 - 8^1$ are consistently better than standard design for all ranges of correlations, while the upper 9 designs are worse. For the monthly change with fixed ρ, the designs with a smaller overlapping from one month to the next have larger variance, while those with a larger overlapping have smaller variance. Table 4.3 shows the efficiency of the alternative designs for the quarter changes. The designs, $2^4 - 2^3$, $2^5 - 2^4$, $2^6 - 2^5$ and those from $5^2 - 5^1$ to $8^2 - 8^1$ are more efficient than the $4^2 - 8^1$ design. Again, this happens because the variance of quarter change becomes smaller when the

Table 4.3: Efficiency of alternative design compared to $4^2 - 8^1$ design for $y_t - y_{t-3}$ with equal sample size under exponential correlation(optimum value of ω)

Designs	ρ				
	0.5	0.6	0.7	0.8	0.9
$2^2 - 10^1$	1.0150(0.6)	1.0368(0.7)	1.0731(0.7)	1.1366(0.8)	1.2548(0.9)
$2^3 - 4^2$	1.0139(0.6)	1.0342(0.7)	1.0659(0.7)	1.1226(0.7)	1.2261(0.8)
$2^4 - 2^3$	0.9754(0.5)	0.9709(0.6)	0.9728(0.6)	0.9961(0.6)	1.0849(0.7)
$2^4 - 4^3$	1.0138(0.6)	1.0339(0.6)	1.0649(0.7)	1.1199(0.7)	1.2205(0.8)
$2^5 - 2^4$	0.9727(0.5)	0.9666(0.6)	0.9660(0.6)	0.9848(0.6)	1.0657(0.7)
$2^6 - 2^5$	0.9709(0.5)	0.9636(0.5)	0.9616(0.6)	0.9774(0.6)	1.0521(0.6)
$3^2 - 9^1$	1.0058(0.6)	1.0133(0.6)	1.0245(0.7)	1.0443(0.8)	1.0773(0.9)
$3^3 - 3^2$	1.0012(0.5)	1.0030(0.5)	1.0032(0.6)	1.0069(0.6)	1.0289(0.7)
$3^4 - 3^3$	1.0006(0.5)	1.0013(0.5)	1.0001(0.6)	0.9987(0.6)	1.0096(0.6)
$5^2 - 5^1$	0.9975(0.6)	0.9931(0.7)	0.9858(0.8)	0.9758(0.8)	0.9617(0.9)
$5^2 - 10^1$	0.9976(0.6)	0.9937(0.7)	0.9869(0.8)	0.9797(0.8)	0.9644(0.9)
$6^2 - 6^1$	0.9965(0.6)	0.9903(0.7)	0.9797(0.8)	0.9653(0.8)	0.9416(0.9)
$7^2 - 7^1$	0.9958(0.6)	0.9881(0.7)	0.9748(0.8)	0.9576(0.9)	0.9272(0.9)
$8^2 - 8^1$	0.9953(0.6)	0.9866(0.7)	0.9714(0.8)	0.9506(0.9)	0.9172(0.9)

percent of overlapping between months t and $t-3$ becomes larger. In Table 4.4. from $2^2 - 10^1$ to $3^4 - 3^3$ show better efficiency than $4^2 - 8^1$. As seen in Theorem 3.2, $Var(S_{t,3})$ increases when $Cov(y_t, y_{t-1})$ and $Cov(y_t, y_{t-2})$ increase. Hence the designs having bigger overlapping percentage between t and $t-1$, $t-2$ like $5^2 - 5^1$ to $8^2 - 8^1$ are less efficient than $4^2 - 8^1$.

Table 4.4: Efficiency of alternative design compared to $4^2 - 8^1$ design for $S_{t,3}$ with equal sample size under exponential correlation(optimum value of ω)

Designs	ρ				
	0.5	0.6	0.7	0.8	0.9
$2^2 - 10^1$	0.8527(0.3)	0.8300(0.4)	0.8112(0.4)	0.7962(0.5)	0.7834(0.6)
$2^3 - 4^2$	0.8527(0.3)	0.8316(0.3)	0.8148(0.4)	0.8082(0.4)	0.8256(0.5)
$2^4 - 2^3$	0.8633(0.3)	0.8517(0.3)	0.8518(0.4)	0.8697(0.5)	0.9062(0.8)
$2^4 - 4^3$	0.8529(0.3)	0.8316(0.3)	0.8153(0.4)	0.8094(0.4)	0.8298(0.5)
$2^5 - 2^4$	0.8644(0.2)	0.8534(0.3)	0.8547(0.3)	0.8768(0.5)	0.9253(0.8)
$2^6 - 2^5$	0.8646(0.2)	0.8546(0.3)	0.8570(0.3)	0.8815(0.4)	0.9399(0.8)
$3^2 - 9^1$	0.9436(0.3)	0.9330(0.3)	0.9224(0.4)	0.9126(0.5)	0.9018(0.7)
$3^3 - 3^2$	0.9458(0.2)	0.9371(0.2)	0.9343(0.2)	0.9458(0.3)	1.0003(0.4)
$3^4 - 3^3$	0.9453(0.2)	0.9371(0.2)	0.9348(0.2)	0.9478(0.2)	1.0077(0.3)
$5^2 - 5^1$	1.0374(0.4)	1.0463(0.5)	1.0563(0.5)	1.0723(0.6)	1.1061(0.7)
$5^2 - 10^1$	1.0374(0.4)	1.0463(0.5)	1.0547(0.6)	1.0633(0.7)	1.0691(0.7)
$6^2 - 6^1$	1.0647(0.4)	1.0799(0.5)	1.0970(0.6)	1.1185(0.6)	1.1553(0.7)
$7^2 - 7^1$	1.0858(0.4)	1.1064(0.5)	1.1282(0.6)	1.1560(0.7)	1.1991(0.8)
$8^2 - 8^1$	1.1014(0.4)	1.1265(0.5)	1.1545(0.6)	1.1875(0.7)	1.2338(0.8)

Table 4.5 shows the efficiency of the alternative designs for the yearly change of quarter sum. The same pattern persists as seen in Table 4.4. the covariance betwen S_t^α and $S_{t-t^*}^\alpha$ contributes to the reduction of the $Var(S_t^\alpha - S_{t-t^*}^\alpha)$ and it

Table 4.5: Efficiency of alternative design compared to 4^2-8^1 design for $S_t^3 - S_{t-12}^3$ with equal sample size under exponential correlation(optimum value of ω)

Designs	ρ				
	0.5	0.6	0.7	0.8	0.9
2^2-10^1	0.8527(0.3)	0.8304(0.4)	0.8126(0.4)	0.8030(0.5)	0.8154(0.6)
2^3-4^2	0.8530(0.3)	0.8319(0.3)	0.8175(0.4)	0.8214(0.4)	0.8800(0.6)
2^4-2^3	0.8634(0.3)	0.8522(0.3)	0.8544(0.4)	0.8780(0.7)	0.8967(0.9)
2^4-4^3	0.8530(0.3)	0.8317(0.3)	0.8160(0.4)	0.8133(0.4)	0.8469(0.5)
2^5-2^4	0.8641(0.2)	0.8535(0.3)	0.8558(0.3)	0.8793(0.6)	0.8911(0.9)
2^6-2^5	0.8645(0.2)	0.8543(0.3)	0.8565(0.3)	0.8788(0.5)	0.8882(0.9)
3^2-9^1	0.9437(0.3)	0.9331(0.3)	0.9239(0.4)	0.9202(0.6)	0.9282(0.7)
3^3-3^2	0.9451(0.2)	0.9374(0.2)	0.9369(0.3)	0.9584(0.3)	1.0326(0.9)
3^4-3^3	0.9452(0.2)	0.9373(0.2)	0.9356(0.2)	0.9502(0.3)	1.0147(0.5)
5^2-5^1	1.0377(0.4)	1.0464(0.5)	1.0575(0.6)	1.0771(0.7)	1.1072(0.9)
5^2-10^1	1.0377(0.4)	1.0463(0.5)	1.0560(0.7)	1.0642(0.8)	1.0721(0.9)
6^2-6^1	1.0647(0.4)	1.0796(0.5)	1.0952(0.6)	1.1108(0.8)	1.1165(0.9)
7^2-7^1	1.0853(0.4)	1.1056(0.6)	1.1258(0.7)	1.1442(0.8)	1.1503(0.9)
8^2-8^1	1.1013(0.4)	1.1261(0.6)	1.1515(0.7)	1.1747(0.8)	1.1832(0.9)

depends on the cumulative percent of overlapping between t and $t-t^*$, $1 \le t' \le t^*$ as well as ω^{t^*}. As the time lag t^* increases. the ω^{t^*} decrease rapidly, and thus, $Cov(S_t^\alpha, S_{t-t^*}^\alpha)$ becomes negligible. This explains Table 4.5. Because so many factors are involved in these examples, it is difficult to choose one superior design based only on the efficiency. The choice of design depends on which estimator meets our needs. As a rule of thumb, we choose r_1, r_2 and m which make less overlapping between months to have reliable measures of level and aggregate level, whereas for reliable measure of change's we choose r_1, r_2 and m making much overlapping between months.

Appendix A: Proof of Theorem 3.1

Observe that

$$\begin{aligned} Var(y_t) = Var\Big(\sum_{i=1}^{mr_1} a_i x_{t,i}\Big) + \omega^2 Var\Big(\sum_{i=1}^{mr_1} b_i x_{t-1,i}\Big) + \omega^2 Var\Big(y_{t-1}\Big) \\ - 2\omega Cov\Big(\sum_{i=1}^{mr_1} a_i x_{t,i}, \sum_{i=1}^{mr_1} b_i x_{t-1,i}\Big) + 2\omega Cov\Big(\sum_{i=1}^{mr_1} a_i x_{t,i}, y_{t-1}\Big) \\ - 2\omega^2 Cov\Big(\sum_{i=1}^{mr_1} b_i x_{t-1,i}, y_{t-1}\Big) \end{aligned}$$

By (3.4)

$$\begin{aligned} (1-\omega^2)Var(y_t) = \sigma^2\Big(\mathbf{a}'\mathbf{a} + \omega^2\mathbf{b}'\mathbf{b} - 2\omega\Big(\rho_1\mathbf{a}'\mathbf{RLR}'\mathbf{b}\Big)\Big) \\ + 2\omega Cov\Big(\sum_{i=1}^{mr_1} a_i x_{t,i}, y_{t-1}\Big) - 2\omega Cov\Big(\sum_{i=1}^{mr_1} b_i x_{t-1,i}, y_{t-1}\Big) \end{aligned} \quad (A.1)$$

On the other hand, observe that

$$\begin{aligned} Var(y_t - y_{t-t*}) &= 2Var(y_t) - 2Cov(y_t, y_{t-t^*}) \\ &= 2Var(y_t) - 2\Big[\sum_{k=0}^{t^*-1}\Big\{\omega^k Cov\Big(\sum_{i=1}^{mr_1} a_i x_{t-k,i}, y_{t-t^*}\Big) \\ &\quad - \omega^{k+1} Cov\Big(\sum_{i=1}^{mr_1} b_i x_{t-1-k,i}, y_{t-t^*}\Big)\Big\} + \omega^{t^*} Var(y_{t-t^*})\Big] \end{aligned} \quad (A.2)$$

where the second term is derived by recursively solving (3.1) as

$$y_t = \sum_{k=0}^{t^*-1}\omega^k \sum_{i=1}^{mr_1} a_i x_{t-k,i} - \sum_{k=0}^{t^*-1}\omega^{k+1}\sum_{i=1}^{mr_1} b_i x_{t-1-k,i} + \omega^{t^*} y_{t-t^*}.$$

For any coefficient c_i's, $0 \le k^* \le t^* - 1$ and $t_0 \ge 1$,

$$\begin{aligned} Cov\Big(\sum_{i=1}^{mr_1} c_i x_{t-k^*,i}, y_{t-t_0}\Big) \quad (A.3) \\ = Cov\Big(\sum_{i=1}^{mr_1} c_i x_{t-k^*,i}, \sum_{j=0}^{\infty}\Big(\omega^j \sum_{i=1}^{mr_1} a_i x_{t-t_0-j,i} - \omega^{j+1}\sum_{i=1}^{mr_1} b_i x_{t-t_0-1-j,i}\Big)\Big) \end{aligned}$$

Since there is no overlapping between $x_{t,i}$ and $x_{t_0,i'}$ if $|t - t_0| \ge T$ by the overlapping formula (2.1) given in Section 2, $Cov(x_{t,i}, x_{t_0,i'}) = 0$ when $|t - t_0| \ge T$ for any $i, i' =$

$1, 2, \cdots, mr_1$. Hence (A.3) is reduced as

$$Cov\Bigg(\sum_{i=1}^{mr_1} c_i x_{t-k^*,i}, \sum_{j=0}^{T-1+k^*-t_0} \omega^j \sum_{i=1}^{mr_1} a_i x_{t-t_0-j,i} - \sum_{j=0}^{T-2+k^*-t_0} \omega^{j+1} \sum_{i=1}^{mr_1} b_i x_{t-t_0-1-j,i}\Bigg).$$

By (3.4), this covariance can be rewritten as

$$\sigma^2\Bigg(\sum_{j=0}^{T-1+k^*-t_0} \omega^j \rho_{t_0+j-k^*} \mathbf{c}'\mathbf{R}\mathbf{L}^{t_0+j-k^*}\mathbf{R}'\mathbf{a} - \sum_{j=0}^{T-2+k^*-t_0} \omega^{j+1} \rho_{t_0+j+1-k^*} \mathbf{c}'\mathbf{R}\mathbf{L}^{t_0+j+1-k^*}\mathbf{R}'\mathbf{b}\Bigg)$$

$$= \sigma^2\Big(\omega^{k^*-t_0}\mathbf{c}'\mathbf{Q}_{t_0-k^*}\mathbf{a} - \omega^{k^*-t_0}\mathbf{c}'\mathbf{Q}_{t_0+1-k^*}\mathbf{b}\Big) \qquad \text{(A.4)}$$

where $\mathbf{c}$ is the same form of $\mathbf{a}$ or $\mathbf{b}$ given in (3.4). Set $t_0 = 1$ in (A.4). Then we have the desired $Var(y_t)$ by substituting the result (A.4) into the second and the third terms of (A.1) with $\mathbf{c}$ replaced by $\mathbf{a}$ and $\mathbf{b}$, and k^* replaced 0 and 1, respectively. Plugging $t_0 = t^*$, $k^* = k$ and $\mathbf{c} = \mathbf{a}$ in (A.4) for the first covariance of (A.2) produces

$$\sigma^2 \sum_{k=0}^{t^*-1} \Big(\omega^{2k-t^*}\mathbf{a}'\mathbf{Q}_{t^*-k}\mathbf{a} - \omega^{2k-t^*}\mathbf{a}'\mathbf{Q}_{t^*-k+1}\mathbf{b}\Big) \qquad \text{(A.5)}$$

For simplicity and efficiency in computation, by (3.5), (A.5) has the matrix form as

$$\sigma^2 \frac{1-\omega^{t^*}}{1-\omega^2}\Big(\mathbf{a}'\mathbf{Q}^*_{t^*,0}\mathbf{a} - \mathbf{a}'\mathbf{Q}^*_{t^*,1}\mathbf{b}\Big)$$

Similarly, plugging $t_0 = t^*$, $k^* = k-1$ and $\mathbf{c} = \mathbf{b}$ in (A.4) for the second covariance of (A.2) and following the same fashion, we get the second result in Theorem 3.1.

Appendix B: Proof of Theorem 3.2

$$\begin{aligned}
Var(S_t^\alpha) &= Var(y_t + y_{t-1} + y_{t-2} + \cdots + y_{t-\alpha+1}) \\
&= \sum_{i=0}^{\alpha-1} Var(y_{t-i}) + 2\sum_{i=0}^{\alpha-2}\sum_{i'=i+1}^{\alpha-1} Cov(y_{t-i}, y_{t-i'}) \\
&= \alpha Var(y_t) + 2\sum_{j=1}^{\alpha-1}(\alpha-j)Cov(y_t, y_{t-j}) \\
&= \alpha Var(y_t) + \sum_{j=1}^{\alpha-1}(\alpha-j)\Big(2Var(y_t) - Var(y_t - y_{t-j})\Big) \\
&= \alpha^2 Var(y_t) - \sum_{j=1}^{\alpha-1}(\alpha-j)Var(y_t - y_{t-j}).
\end{aligned}$$

$$
\begin{aligned}
Var(S_t^{\alpha} - S_{t-t^*}^{\alpha}) &= Var(y_t + \cdots + y_{t-\alpha+1} - (y_{t-t^*} + \cdots + y_{t-t^*-\alpha+1})) \\
&= \sum_{i=0}^{\alpha-1} \Big(Var(y_{t-i}) + Var(y_{t-t^*-i}) \Big) + 2 \Bigg[\sum_{i=0}^{\alpha-2} \sum_{i'=i+1}^{\alpha-1} \Big(Cov(y_{t-i}, y_{t-i'}) \\
&\qquad + Cov(y_{t-t^*-i}, y_{t-t^*-i'}) \Big) - \sum_{i=0}^{\alpha-1} \sum_{i'=0}^{\alpha-1} Cov(y_{t-i}, y_{t-t^*-i'}) \Bigg] \\
&= 2\alpha Var(y_t) + \sum_{j=1}^{\alpha-1} \Big[2\{2Var(y_t) - Var(y_t - y_{t-j})\} \\
&\qquad - \{2Var(y_t) - Var(y_t - y_{t-t^*+j})\} - \{2Var(y_t) - Var(y_t - y_{t-t^*-j})\} \Big] \\
&\qquad - \alpha\{2Var(y_t) - Var(y_t - y_{t-t^*})\} \\
&= \alpha Var(y_t - y_{t-t^*}) + \sum_{j=1}^{\alpha-1} (\alpha - j) \Big[Var(y_t - y_{t-t^*+j}) \\
&\qquad + Var(y_t - y_{t-t^*-j}) - 2Var(y_t - y_{t-j}) \Big] \\
&= \sum_{j=-\alpha+1}^{\alpha-1} (\alpha - |j|) Var(y_t - y_{t-t^*+j}) - 2 \sum_{j=1}^{\alpha-1} Var(y_t - y_{t-j})
\end{aligned}
$$

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