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Generalized One-Level Rotation Designs with Finite Rotation Groups Part I : Generation of Designs †

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Abstract

In this paper, we consider one-level rotation designs with finite rotation groups such that the design satisfies two basic requirements: all rotation groups are included in any given survey period, and overlapping rates depend only on the time lag. First we present the necessary number of rotation groups and a rule for the length of time the sample units are to be in or out of the sample to satisfy the requirements. Second, an algorithm is presented to put rotation groups to proper positions in a panel in order to include all finite rotation groups for any survey period. Third, we define an one-level rotation pattern which is invariant in the survey period and has useful properties in practical sense.

Key Words: One-level rotation design; Finite rotation groups; Rotation groups in panel; Allocation; Overlapping.

1. Introduction

In rotation sampling design, the sample units are systematically rotated in or out, completely or partially, according to specific rules in the successive time periods. Rotation is used to avoid undue burden of reporting from survey respondents and to obtain information of changes from overlapping units. Such rotation design has been used in sample surveys since the 1950's, and certain rules are followed to satisfy the basic requirements of the design (Hansen, 1955; Woodruff, 1963; Rao and Graham, 1964; Cochran,1977; Wolter, 1979).

In this paper, we assume that the entire sample for the life of a survey is divided into a finite number of groups which are called as rotation groups; for the

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survey period sample, each rotation group is again divided into an appropriate of subsamples in such a way that all subsamples in each rotation group are surveyed for the life of survey. For each survey period, one subsample from each rotation group is selected to be surveyed and hence the survey period sample is comprised of subsamples as many as the number of rotation groups.

In one-level rotation sampling designs, subsamples from some of the rotation groups at a specific survey period are replaced with new subsamples in the same rotation groups at the next survey period and subsamples from the remaining rotation groups are retained in the sample at the next survey period. At each survey period, a subsample reports only one period of data(Hansen, 1955; Rao and Graham, 1964; Cantwell, 1990). This type of survey is used for the Canadian Labor Force Survey(LFS) conducted by Statistics Canada, the Current Population Survey(CPS) at the U.S. Bureau of Census and the Labor Force Survey in Japan.

For instance, CPS includes 8 rotation groups in a psu, each rotation group includes a certain number of subsamples, and each subsample includes about four households. One subsample from each of eight rotation groups is rotated in and out. The subsample stays in the sample for 4 months, leaves the sample for the next 8 months, then returns to the sample for the following 4 months. We call this 4-8-4 design to which alternative designs are compared. Here, the length of time staying in and leaving out of the sample, and the number of repetitions are restricted to 4 months, 8 months, and 2 times, respectively. Freeing from such restrictions, we present the rules of creating alternative one-level rotation designs. From here on, one month is a survey period; during this period, a monthly sample includes one subsample from each rotation group, which stays, drops out, or returns to the sample simultaneously. We assume that the rotation scheme is conducted in a psu without loss of generality since all discussions below can be applied exactly by the same way for any psu.

The one-level rotation sampling design with finite rotation groups expressed as $r_1^m - r_2^{m-1}$ is defined as follows : a subsample is interviewed for consecutive r_1 periods, drops out for the next r_2 succeeding periods, and return to the sample for the another r_1 periods. This process is repeated a finite m times before the subsample drops out of the sample completely. We set $r_2 = 0$ if $m = 1$ in the expression $r_1^m - r_2^{m-1}$.

Our findings are presented in the following three sections. Section 2 discusses two basic rules regulating the number of months for a subsample to stay, leave, or return again to the sample to satisfy the basic requirements. In Section 3, define the algorithm to allocate rotation groups and a class of one-level rotation design with conditions to follow the rotation pattern which depends on only the rotation group a sample unit belongs to. In Section 4 a formula to calculate overlapping ratio between any two survey months is presented.

2. Generalized One-Level Rotation Design

One-level rotation design including $r_1^m - r_2^{m-1}$ design is a compromise between a permanent sample and a completely new sample at each month. Determination of r_1, r_2 and m in $r_1^m - r_2^{m-1}$ design depends on the objectives of the survey, since r_1, r_2 and m regulate the how long a subsample is to be in, out of the sample and how many time it returns to the sample, respectively. The $r_1^m - r_2^{m-1}$ design accomplishes the following objectives: acquires for sample to be representative of the population for each survey month by including all rotation groups in the sample for each survey month; improves the reliability of measures of change by replacing the outgoing subsample with highly correlated subsample, which is achieved by assigning the same rotation group to these two subsamples; gives the same effect on the correlation structure between measures from different two months by adding an assumption for the proportion of sample in common between two months.
For these objectives, we assume in $r_1^m - r_2^{m-1}$ design that

(i) All rotation groups are included in the sample for each month, and the sample size is the same for all survey months.

(ii) Subsample in a particular rotation group is replaced the subsample in the same rotation group.

(iii) The overlapping percentage between the present month t and $t+t^*$ depends on only time lag t^*, for $t^* = 0, \pm 1, \pm 2, \cdots$.

LFS and CPS satisfy these assumptions. Let $D(K_1, K_2)$ be the class of $r_1^m - r_2^{m-1}$ design, $1 \leq m \leq \infty$ which have the overlapping at most between months t and $t + K_1 \cdot K_2$, for $K_1, K_2 \geq 1$. $D(K_1, K_2)$ contains the $r_1^m - r_2^{m-1}$ designs with at most $K_1 \cdot K_2$ months overlapping ; $D(3, K_2)$ are the $r_1^m - r_2^{m-1}$ designs with at most K_2 quarters overlapping ; $D(12, K_2)$ consists of the $r_1^m - r_2^{m-1}$ designs with at most K_2 years overlapping. For example, LFS is in $D(1,5)$ and $D(5,1)$ with $m = 1$, and the CPS is in $D(1,15)$, $D(3,5)$, $D(5,3)$ and $D(12,1)$.

Theorem 2.1. *Suppose that a $r_1^m - r_2^{m-1}$ design in $D(K_1, K_2)$ for each fixed $K_1, K_2 \geq 1$. IF for $r_1 \geq 1$, $r_2 \geq 0$, and $1 \leq m < \infty$, the $r_1^m - r_2^{m-1}$ design satisfies the assumptions (i), (ii) and (iii), then*

(a) *For each survey month and each $i = 1, 2, \ldots, mr_1$, there is a subsample appearing in the sample for the ith time is presented, and for the life of survey, mr_1 rotation groups are necessary, and*

(b) *$r_2 = lr_1$ where $l = 0, 1, \ldots$. For each $K_1, K_2 = 1, 2, \ldots$, and l, m and r_1 are determined with the constraint $K_1 \cdot K_2 - r_1 + 1 \leq r_1(m-1)(1+l) \leq K_1(K_2+1) - r_1$ except for the cases of r_1 satisfying $K_1 \cdot K_2 - r_2 - 1 < r_1(m_0^* - 1)(1+l) < K_1 \cdot K_2 - r_1 + 1$ where $m_0^* = \{m^*; r_1(m^*-1)(1+l) \leq K_1 \cdot K_2 \leq m^* r_1(1+l), m^* = 1, 2, \ldots, m\}$.*

The proof of the Theorem 2.1 is given in Appendix. It is lengthy, but shows the insight of the Theorem. When $m = 1$, the number of rotation groups is r_1 by (a) of Theorem 2.1. Then we can always define $r_1^1 - 0^0$ design to satisfy the assumptions given in Theorem 2.1 by replacing the subsample that is retiring permanently from the sample with the new subsample in the same rotation group. Since we have mr_1 rotation groups and each rotation group contains exactly one subsample at any survey month, we have mr_1 pairs of subsamples for any different two months in which two subsample in each pair comes from the same rotation group. Among these mr_1 pairs, each two subsamples in some pairs are the same subsamples because the subsample return to the sample later months, and each two subsamples in the remaining pairs are different because they are replaced in the same rotation group as time advances. In order to identify which two subsamples are the same among the mr_1 pairs, we introduce the αth panel P_α, $\alpha = 1, 2, \cdots$ in which each panel consists of mr_1 rotation groups and the index α indicates the αth subsample in each rotation group. By this panel together with rotation groups, any subsample can be uniquely represented by the affiliation index (γ, P_α) where γ is the γth rotation group, and hence each two subsamples at any two months can be identified; if two subsamples with the same rotation group belong to the same panel, the two subsamples are the same; if not, they are different subsamples. In CPS, this panel is defined as sample designation. The panel P_α will be further discussed in Section 3.

Example 2.1. Figure 2.1 illustrates Theorem 2.1 with $2^4 - 2^3$ design where 'o' denotes which subsamples are being in surveyed at a given month. Since $m = 4$ and $r_1 = r_2 = 2$, $mr_1 = 8$ rotation groups by (a) in each of the 4 panels, P_α,

Figure 2.1: $2^4 - 2^3$ system.

		Panels and rotation groups																															
		P_1								P_2								P_3								P_4							
YEAR	MON	1	2	3	4	5	6	7	8	3	4	5	6	7	8	1	2	1	2	3	4	5	6	7	8	3	4	5	6	7	8	1	2
Y1	JAN	o	o			o	o			o	o			o	o																		
	FEB		o	o			o	o			o	o			o	o																	
	MAR			o	o			o	o			o	o			o	o																
	APR				o	o			o	o			o	o			o	o															
	MAY					o	o			o	o			o	o			o	o														
	JUN						o	o			o	o			o	o			o	o													
	JUL							o	o			o	o			o	o			o	o												
	AUG								o	o			o	o			o	o			o	o											
	SEP									o	o			o	o			o	o			o	o										
	OCT										o	o			o	o			o	o			o	o									
	NOV											o	o			o	o			o	o			o	o								
	DEC												o	o			o	o			o	o			o	o							
Y2	JAN													o	o			o	o			o	o			o	o						
	FEB														o	o			o	o			o	o			o	o					

$\alpha = 1, 2, 3, 4$. Each panel consists of 8 rotation groups with different arrangement of rotation groups. The arrangement is discussed in Section 3. The sample of Jan.Year 1 consists of the group numbers $(8, 7, 4, 3, 6, 5, 2, 1)$, counting from the right to the left. The rotation groups 8, 7, 4, 3 in the panel P_2 appear in the sample for the 1st, 2nd, 3rd, and 4th time, respectively and the rotation groups 6, 5, 2, 1 in the panel P_1 appear in the sample for the 5th, 6th, 7th, and 8th time, respectively. The sample of Feb.Year1 is determined by simply moving one step to the right from the first sample. The second sample also consists of the eight rotation groups that is $(1, 8, 5, 4, 7, 6, 3, 2)$, again counting from the right to the left, and the order indicates the return time to the sample, the group 1 first time, group 8 second time, $\cdots$, group 2 eighth time. At months Jan. and Feb.Year1, we have 8 pairs of subsamples, each pair from each of 8 rotation groups. For instance, two subsamples from the rotation group 5, one from P_1 and the other one from P_2. Thus these two subsamples are different. Similarly, two subsamples from each of rotation groups, 2, 6, 4 and 8 are the same subsample and two subsamples from each of the other rotation groups, 1, 5, 3 and 7 are different. The sample groups at month t are taken by moving exactly $t - 1$ steps to the right from the first sample. This procedure ensures the consistent overlapping for a subsample to be in the sample for $r_1 (= 2)$ months and to drop out of the sample for next $r_2 (= 2)$ months. The $2^4 - 2^3$ design satisfies the condition that

$r_2 = 2$ is a multiple of $r_1 = 2$ given in Theorem 2.1. Each month includes 8 subsamples, one from each rotation group, with the overlapping that depends on only time lag. However, in Figure 2.1, if we choose the rotation groups $1,2,3,4$ instead of $1,2,5,6$ in P_1, and rotation groups $5,6,7,8$ instead of $3,4,7,8$ in P_2 as the first sample, the 4th sample of April.Year 1 would not include the rotation group number 3. This violates the assumption that all rotation groups are in sample for any survey month. Reordering P_2 from $P_2 = (3,4,5,6,7,8,1,2)$ to $P_2 = (1,2,5,6,7,8,3,4)$ also violates the assumption. Therefore, an algorithm is needed to have the rotation groups properly ordered within a panel, and further discussed in Section 3.

The condition (b) of Theorem 2.1 is not easy to understand; thus, a numerical example is given below to understand it better.

Example 2.2. We use the $3^3 - 6^2$ design for $K_1 = 12$ and $K_2 = 1$ to illustrate the condition (b) of Theorem 2.1. Since $r_1 = 3, r_2 = 6, m = 3$ and $l = 2$, the $3^3 - 6^2$ design satisfies $r_2 = lr_1$ and $K_1 \cdot K_2 - r_1 + 1 = 10 \leq r_1(m-1)(1+l) = 18 \leq K_1(1+K_2) - r_1 = 21$. The number $m^* = 2$ satisfies $r_1(m^* - 1)(1+l) \leq 12 \leq m^* r_1(1+l)$. But, the $3^3 - 6^2$ design also satisfies the exception case of $K_1 \cdot K_2 - r_2 - 1 < r_1(m_0^* - 1)(1+l) < K_1 \cdot K_2 - r_1 + 1$. Therefore, the $3^3 - 6^2$ design can not be a member of $D(12,1)$. Now take another design $5^2 - 10^1$ for $K_1 = 12$ and $K_2 = 1$. This design satisfies the conditions $r_2 = lr_1$ and $K_1 \cdot K_2 - r_1 + 1 = 8 \leq r_1(m-1)(1+l) = 15 \leq K_1(1+K_2) - r_1 = 19$ for $r_1 = 5$, $r_2 = 10$, $l = 2$, $m = 2$, and $K = 1$. But $m_0^* = 1$ in the $5^2 - 10^1$ design, it does not satisfy the exception condition of $K_1 \cdot K_2 - r_2 - 1 < r_1(m_0^* - 1)(1+l) < K_1 \cdot K_2 - r_1 + 1$. Hence, the $5^2 - 10^1$ design is a member of our $D(12,1)$. The following designs did satisfy the conditions of Theorem 1: 43 designs for $K_1 = 12$ and $K_2 = 1$ are $13^1 - 0^0$, $14^1 - 0^0$, $15^1 - 0^0$, $16^1 - 0^0$, $17^1 - 0^0$, $18^1 - 0^0$, $19^1 - 0^0$, $20^1 - 0^0$, $21^1 - 0^0$, $22^1 - 0^0$, $23^1 - 0^0$, $24^1 - 0^0$, $1^2 - 11^1$, $2^2 - 10^1$, $3^2 - 9^1$, $4^2 - 8^1$, $5^2 - 5^1$, $5^2 - 10^1$, $6^2 - 6^1$, $7^2 - 7^1$, $8^2 - 8^1$, $1^3 - 5^2$, $2^3 - 4^2$, $3^3 - 3^2$, $1^4 - 3^3$, $1^4 - 5^3$, $2^4 - 2^3$, $2^4 - 4^3$, $3^4 - 3^3$, $1^5 - 2^4$, $1^5 - 3^4$, $2^5 - 2^4$, $1^6 - 2^5$, $1^6 - 3^5$, $2^6 - 2^5$, $1^7 - 1^6$, $1^7 - 2^6$, $1^8 - 1^7$, $1^8 - 2^7$, $1^9 - 1^8$, $1^{10} - 1^9$, $1^{11} - 1^{10}$, and $1^{12} - 1^{11}$; 63 designs for $D(12,2)$ including $1^{18} - 1^{17}, 2^9 - 2^8, 3^6 - 3^5, 4^3 - 8^2$ and $6^3 - 6^2$; 69 designs for $D(12,3)$ including $1^{14} - 2^{13}, 2^7 - 4^6, 3^2 - 15^2, 9^3 - 9^2$ and $16^2 - 16^1$; 75 designs for $D(12,4)$ including $2^{14} - 2^{13}, 4^8 - 4^7, 5^6 - 5^5, 8^3 - 16^2$ and $12^3 - 12^2$; 82 designs for $D(12,5)$ including $3^6 - 9^5, 10^4 - 10^3, 14^3 - 14^2, 17^2 - 34^1$ and $24^2 - 24^1$.

3. Allocation of Rotation Groups

This Section discusses the rules to allocate rotation groups systematically to a proper position in a panel, and yet satisfy the Theorem 2.1. The mr_1 rotation groups are identified by the number $1, 2, \ldots, mr_1$. The αth panel P_α includes the mr_1 rotation groups as $P_\alpha = \{1, 2, \ldots, mr_1\}$ and the index α indicates the αth subsample in each rotation group. The algorithm is summarized as follows: (i) In an initial survey month, decide an appropriate number of panels, and the mr_1 rotation groups in each panel are ordered by a proper arrangement. Then different mr_1 rotation groups are chosen from the panels; the subsamples in each of the selected rotation groups are randomly ordered, and one subsample in a rotation group is taken; These subsamples are designated as the first sample. For example, in Figure 2.1, P_1 and P_2 are the necessary panels, and P_1 is ordered as (12345678) and P_2 is ordered as (34567812); then rotation group numbers 1, 2, 5 and 6 are selected in P_1, and 3, 4, 7 and 8 are from P_2. These (12563478) in combination with the corresponding panels are the first sample in $2^4 - 2^3$ design. In the first sample, the order counting from right to the left indicates the return time to sample; the rotation group, 8 located at the last place on the right side provides its second subsample appearing in the sample for the first time, similarly, the rotation group, 7 located at the second to the last place provides its second subsample appearing in the sample for the second time, $\cdots$, and finally, the rotation group, 1 located at the first place provides its first subsample appearing in the sample for the last or the mr_1th time. (ii) After the first sample is taken for the initial survey month t, the groups, which provide the subsamples, are determined systematically for the succeeding months, $t+1, t+2, \ldots$; each monthly sample includes different mr_1 subsamples. The mr_1 subsamples are uniquely represented by a arrangement of rotation group numbers 1 to mr_1 and panels for any survey month. The order of each monthly sample indicates the appearance time to sample for the 1st to the mr_1th time from the right to the left as in (i). For example, in Figure 2.1, we have (34781256) at Sep.Year1. The rotation group 1 in the panel P_3 provides a subsample appearing to the sample for the 4th time.

Above procedures are illustrated by the algorithm in the following 6 steps with the $2^4 - 2^3$ design. To arrange rotation groups properly in the $r_1^m - r_2^{m-1}$ design, the assumptions and properties of Theorem 2.1 should be satisfied. The basic panels $P_1, P_2, \ldots, P_h$, each with mr_1 rotation groups, should also meet the conditions that $h = l$ if $mr_1 = (m-1)r_2$ and $h = l + 1$ if otherwise, where l comes from the equation $r_2 = lr_1$, $l = 0, 1, \ldots$.

Step 1 For the first panel P_1, allocate the rotation group numbers by increasing order.
Namely, $P_1 = (1, 2, 3, \ldots, mr_1)$. For example, in Figure 2.1, $P_1 = (1, 2, \ldots, 8)$.

Step 2 Write out the basic panels $P_1, P_2, \ldots, P_h$, each occupying mr_1 positions. The αth panel P_α includes positions from the position $(\alpha - 1)mr_1 + 1$ to αmr_1. In Figure 2.1, we have $hmr_1 = 16$ positions for $h = 2$, and the second panel P_2 has the positions from the 9th to the 16th.

Step 3 Indicate the first r_1 positions by the symbol 'o'. After the next r_2 positions with no symbol, indicate the second r_1 positions with the same symbol 'o'. And so on until the mth r_1 positions are marked. See the row of Jan.Year1 in Figure 2.1.

P_1	P_2
1 2 3 4 5 6 7 8	□ □ □ □ □ □ □ □
o o o o	o o o o

Step 4 Fill the checked positions in $P_2, P_3, \cdots, P_h$ in turn with the numbers of rotation groups that were not checked in P_1. The rotation group numbers checked up to the Step 4 provide the first sample.
In Step 3, the first and second positions, of P_2 are filled by group numbers, 3 and 4, respectively. Other two positions checked by 'o' in P_2 are the 5th and 6th positions, filled by the group numbers, 7 and 8. These 3, 4, 7 and 8 in P_2 were not checked in P_1 as seen below.

P_1	P_2
1 2 3 4 5 6 7 8	3 4 □ □ 7 8 □ □
o o o o	o o o o

Step 5 Fill the remaining rotation group numbers by circular order in the empty positions of P_α, $\alpha = 2, \ldots, h$ that are partially occupied from Step 4. In Step 4, the 4 empty positions in P_2 are the 3rd, 4th, and 7th and 8th places. The first 2 empty positions and the second 2 empty positions in P_2 are filled by group numbers 5 and 6, 1 and 2, respectively because the number 8 is followed by 1 by circular ordering as seen in the figure below.
When the panels P_α are totally unchecked for $\alpha \leq h$, we copy the previous panels $P_{\alpha'}$, $\alpha' < \alpha$.

Now, we have the arrangement of h panels, P_1, $P_2, \cdots, P_h$. Copy these panels to the next h panels, $P_{h+1}, \cdots, P_{2h}$ and so on. For instance the panels P_3 and P_4 in the algorithm are the copy of the panels P_1 and P_2 as seen in Figure 2.1.

P_1	P_2
1 2 3 4 5 6 7 8	3 4 5 6 7 8 1 2
○ ○ ○ ○	○ ○ ○ ○

Step 6 With the first sample defined in Step 4 and P_α given in Step 5, the sample of month t for $t \geq 2$ is determined by going exactly $t-1$ steps to the right from the first sample as shown in Figure 2.1.

In $r_1^m - r_2^{m-1}$ design, m subsamples are replaced for each survey month : one subsample is replaced by a new subsample and the other $m-1$ subsamples are replaced by old subsamples which are in the sample r_2 months before. For each span of mr_1 survey months, unless each of mr_1 rotation groups contains exactly one new subsample, some rotation groups contain more than one new subsamples while the others do not contain a new subsample. This may cause inaccurate estimators for characteristics of interest. For instance, $2^4 - 2^3$ design in Figure 2.1, from Jan.Year 1 to Aug.Year 1, for $mr_1 = 8$ survey months 8 new subsamples are replaced, but these subsamples are from only 6 rotation groups. The new subsamples from rotation group 1 and 2 are rotated in twice during 8 survey months whereas subsamples from rotation group 6 and 7 do not rotated in. Define that a $r_1^m - r_2^{m-1}$ design has mr_1 span if each of mr_1 rotation groups contains exactly one new subsamples during each span of mr_1 survey months. Therefore $2^4 - 2^3$ design dose not have mr_1 span. We further restrict our attention on the $r_1^m - r_2^{m-1}$ design which has mr_1 span. Let $\gamma_t(i)$ be the rotation group including the subsample which appears in the sample for the ith time at month t. Suppose that a $r_1^m - r_2^{m-1}$ design satisfies the following rotation pattern: for appropriately given m', $1 \leq m' \leq m$,

$$\gamma_t[i] = \begin{cases} \gamma_{t+1}[mod_m(m+m'-1-k)r_1+1] & \text{if } i = (2m'-1-k)r_1, \\ \gamma_{t+1}[i+1] & \text{otherwise} \end{cases}$$
$$k = -(m-2m'+1), \cdots, 0, \cdots, 2m'-2 \qquad (3.1)$$

This rotation pattern depends on the appearance time, m and m'. In particular, if we choose $m' = m$ in (3.1) the rotation pattern is $\gamma_t(i) = \gamma_{t+1}[i+1]$, $i =$

$1, \ldots, mr_1 - 1$ and $\gamma_t[mr_1] = \gamma_{t+1}[1]$. LFS and CPS follow this rotation pattern. Whether a $r_1^m - r_2^{m-1}$ design has the rotation pattern given in (3.1) depend on r_1, r_2 and m. And an appropriate m' can be also defined through r_1, r_2 and m as shown in the following Theorem.

Theorem 3.1. *Suppose that $r_1^m - r_2^{m-1}$ designs satisfy Theorem 2.1 and follows the rotation pattern given in (3.1). Then, for each given r_1, r_2 and m, there is a positive integer m', $1 \leq m' \leq m$ such that $mod_m(m'r_1 + (m'-1)r_2) = 0$.*

The proof of Theorem 3.1 is given in Appendix. Theorem 3.1 is a necessary condition to have the rotation pattern given as (3.1). Conversely, the algorithm in this Section provides that the necessary condition in Theorem 3.1 is in fact a sufficient condition. From the algorithm in this Section our finding is that whenever there is a positive integer m', $1 \leq m' \leq m$ satisfying $mod_m(m'r_1 + (m'-1)r_2) = 0$ for each given r_1, r_2 and m, the arrangement of rotation groups in all panels P_α, $\alpha \geq 1$ is $P_\alpha = (1, 2, \cdots, mr_1)$. That is, if there are nonnegative integer, $l \geq 0$ and positive integer m', $1 \leq m' \leq m$ such that $r_2 = lr_1$ and $mod_m(m'r_1 + (m'-1)r_2) = 0$ for given r_1, r_2 and m, then the corresponding $r_1^m - r_2^{m-1}$ design follows the rotation pattern given in (3,1) and includes all mr_1 rotation groups for each survey month, and $P_\alpha = (1, 2, \cdots, mr_1)$ for all $\alpha \geq 1$. More over since $P_\alpha = (1, 2, \cdots, mr_1)$ for all $\alpha \geq 1$ the $r_1^m - r_2^{m-1}$ design having the rotation pattern of (3.1) has mr_1 span.

4. Overlapping

An overlapping between any two months occurs only when the same subsamples are in sample at both of the two months or equivalently, only when subsamples in the same rotation group and panel are in sample. The proof of Theorem 2.1 in the Appendix leads us to the following overlapping rule between months t and $t + t^*$ in the $r_1^m - r_2^{m-1}$ design : For any K_1, K_2 in $D(K_1, K_2)$,

$$O(t, t^*) = \begin{cases} \dfrac{r_1 - t^*}{r_1} & \text{if } \ 1 \leq t^* \leq r_1 - 1 \\ \dfrac{(m-i)(r_1 - |j|)}{mr_1} & \text{if } \ t^* = i(r_1 + r_2) - j, \\ & \quad i = 1, \ldots, m-1, \\ & \quad j = r_1 - 1, r_1 - 2, \ldots, 1 - r_1 \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

For fixed K_1 and K_2 in $D(K_1, K_2)$, the biggest overlapping from month to month (i.e. $t^* = 1$) occurs when $m = 2$. The reason is that the smaller m, the larger r_1 by (b) of Theorem 2.1. However, from (4.1), the overlapping between month t and $t+(r_1+r_2)$ is $(m-1)/m$. This implies that the overlapping between month t and $t+(r_1+r_2)$ becomes larger as m is larger.

Table 4.1: Overlapping percentages by designs when $K_1 = 12, K_2 = 1$, and $r_1 \geq 2$

Designs	t^*											
	1	2	3	4	5	6	7	8	9	10	11	12
$2^2 - 10^1$	50.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	25.0	50.0
$2^3 - 4^2$	50.0	0.0	0.0	0.0	33.3	66.7	33.3	0.0	0.0	0.0	16.7	33.3
$2^4 - 2^3$	50.0	0.0	37.5	75.0	37.5	0.0	25.0	50.0	25.0	0.0	12.5	25.0
$2^4 - 4^3$	50.0	0.0	0.0	0.0	37.5	75.0	37.5	0.0	0.0	0.0	25.0	50.0
$2^5 - 2^4$	50.0	0.0	40.0	80.0	40.0	0.0	30.0	60.0	30.0	0.0	20.0	40.0
$2^6 - 2^5$	50.0	0.0	41.7	83.3	41.7	0.0	33.3	66.7	33.3	0.0	25.0	50.0
$3^2 - 9^1$	66.7	33.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	16.7	33.3	50.0
$3^3 - 3^2$	66.7	33.3	0.0	22.2	44.4	66.7	44.4	22.2	0.0	11.1	22.2	33.3
$3^4 - 3^3$	66.7	33.3	0.0	25.0	50.0	75.0	50.0	25.0	0.0	16.7	33.3	50.0
$4^2 - 8^1$	75.0	50.0	25.0	0.0	0.0	0.0	0.0	0.0	12.5	25.0	37.5	50.0
$5^2 - 5^1$	80.0	60.0	40.0	20.0	0.0	10.0	20.0	30.0	40.0	50.0	40.0	30.0
$5^2 - 10^1$	80.0	60.0	40.0	20.0	0.0	0.0	0.0	0.0	0.0	0.0	10.0	20.0
$6^2 - 6^1$	83.3	66.7	50.0	33.3	16.7	0.0	8.3	16.7	25.0	33.3	41.7	50.0
$7^2 - 7^1$	85.7	71.4	57.1	42.9	28.6	14.3	0.0	7.1	14.3	21.4	28.6	35.7
$8^2 - 8^1$	87.5	75.0	62.5	50.0	37.5	25.0	12.5	0.0	6.2	12.5	18.8	25.0

Table 4.1 shows the overlapping percentages for the $r_1^m - r_2^{m-1}$ designs when $K_1 = 12, K_2 = 1$ and $r_1 \geq 2$. When the time $t^* = 1$, the overlapping increases as r_1 increases. The highest overlapping of quarter change ($t^* = 3$) is attained in the $8^2 - 8^1$ design with 62.5%. $2^2 - 10^1$, $2^4 - 4^3$, $2^6 - 2^5$, $3^2 - 9^1$, $4^2 - 8^1$ and $6^2 - 6^1$ designs attain the highest overlapping of year change($t^* = 12$) with 50%. The rate of overlapping is useful information to reduce the variance of differences, and to obtain better information of changes.

APPENDIX : Proof of Theorem 2.1

(a) By the definition of $r_1^m - r_2^{m-1}$ design, since a subsample can be interviewed for the maximum of mr_1 months, we can partition the subsamples into mr_1 individual subsets by the number of appearance in the sample at any survey month. In the following $\mathcal{G}_t$, the number of returns to the sample is indexed by i, and the number of months staying in sample is indexed by j. Define $\mathcal{G}_t = \{\gamma_{t,i,j}, 1 \leq i \leq$

$m, 1 \leq j \leq r_1\}$ where $\gamma_{t,i,j}$ is a subset of subsamples which appear in the sample $(i-1)r_1 + j$ times at a survey month t, $t = 1, 2, \ldots$. Assume that the number of subsamples in $\gamma_{t,i,j}$ is $n_{t,i,j}$. Since a subsample from a particular rotation group is rotated in, out and returns to the sample simultaneously, subsamples having different appearance times to the sample at a specific month come from different rotation groups. Moreover, only one subsample from each rotation group is selected for the monthly sample. Therefore $n_{t,i,j} = 0$ or 1 for all t, i and j. Consider two sets of $\mathcal{G}_t$ and $\mathcal{G}_{t+1}$. The followings can be shown easily by the definition of $r_1^m - r_2^{m-1}$ design :

(I) For each $J_1 = 1, 2, \ldots, r_1 - 1$, only $\{\gamma_{t,i,j}\}$ and $\{\gamma_{t+1,i,j}\}$ for $1 \leq i \leq m$ and $1 \leq j \leq r_1 - J_1$ return to the sample at month $t + J_1$ and $t + 1 + J_1$, respectively. Here define $J_1 = 0$ when $r_1 = 1$.

(II) For each $I = 1, 2, \ldots, m-1$ and $J_2 = r_1 - 1, r_1 - 2, \ldots, 0$, only $\{\gamma_{t,i,j}\}$ and $\{\gamma_{t+1,i,j}\}$ for $1 \leq i \leq I$ and $J_2 + 1 \leq j \leq r_1$ come back to the sample at month $t + (m-I)(r_1+r_2) - J_2$ and $t + 1 + (m-I)(r_1+r_2) - J_2$, respectively.

(III) For each $I = 1, 2, \ldots, m-1$ and $J_3 = 1, 2, \ldots, r_1 - 1$, only $\{\gamma_{t,i,j}\}$ and $\{\gamma_{t+1,i,j}\}$ for $1 \leq i \leq I$ and $1 \leq j \leq r_1 - J_3$ return to the sample at month $t + (m-I)(r_1+r_2) + J_3$ and $t + 1 + (m-I)(r_1+r_2) + J_3$, respectively. Define $J_3 = 0$ when $r_1 = 1$.

Therefore, for each J_1 the respective proportion of overlapping between two months t and $t + J_1$, and $t+1$ and $t + J_1 + 1$ are

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{r_1-J_1} n_{t,i,j}}{\sum_{i=1}^{m}\sum_{j=1}^{r_1} n_{t,i,j}} \quad \text{and} \quad \frac{\sum_{i=1}^{m}\sum_{j=1}^{r_1-J_1} n_{t+1,i,j}}{\sum_{i=1}^{m}\sum_{j=1}^{r_1} n_{t+1,i,j}} \tag{A.1}$$

For each I and J_2 the overlapping proportion between t and $t+(m-I)(r_1+r_2)-J_2$, and $t+1$ and $t + (m-I)(r_1+r_2) - J_2 + 1$ are

$$\frac{\sum_{i=1}^{I}\sum_{j=J_2+1}^{r_1} n_{t,i,j}}{\sum_{i=1}^{m}\sum_{j=1}^{r_1} n_{t,i,j}} \quad \text{and} \quad \frac{\sum_{i=1}^{I}\sum_{j=J_2+1}^{r_1} n_{t+1,i,j}}{\sum_{i=1}^{m}\sum_{j=1}^{r_1} n_{t+1,i,j}} \tag{A.2}$$

respectively; for each I and J_3 the overlapping proportion between t and $t+(m-I)(r_1+r_2)+J_3$, and $t+1$ and $t + (m-I)(r_1+r_2) + J_3 + 1$ are

$$\frac{\sum_{i=1}^{I}\sum_{j=1}^{r_1-J_3} n_{t,i,j}}{\sum_{i=1}^{m}\sum_{j=1}^{r_1} n_{t,i,j}} \quad \text{and} \quad \frac{\sum_{i=1}^{I}\sum_{j=1}^{r_1-J_3} n_{t+1,i,j}}{\sum_{i=1}^{m}\sum_{j=1}^{r_1} n_{t+1,i,j}} \tag{A.3}$$

respectively. Since $\sum_{i=1}^{m}\sum_{j=1}^{r_1} n_{t,i,j} = \sum_{i=1}^{m}\sum_{j=1}^{r_1} n_{t+1,i,j}$ and the overlapping percentage depends on only time lag by the assumptions (i) and (iii), recursively solving (A.3) from $J_3 = r_1 - 1$ to 1 for each I, we have

$$n_{t,i,j} = n_{t+1,i,j} \text{ for } 1 \le i \le m-1 \text{ and } 1 \le j \le r_1 - 1. \tag{A.4}$$

Similarly, from (A.2) together with (A.4), we have

$$n_{t,i,r_1} = n_{t+1,i,r_1} \text{ for } 1 \le i \le m-1. \tag{A.5}$$

Finally, the equation (A.1) with (A.4) and (A.5)

$$n_{t,m,j} = n_{t+1,m,j} \text{ for } 1 \le j \le r_1 - 1. \tag{A.6}$$

(A.4)-(A.6) and $\sum_{i=1}^{m}\sum_{j=1}^{r_1} n_{t,i,j} = \sum_{i=1}^{m}\sum_{j=1}^{r_1} n_{t+1,i,j}$ yield

$$n_{t,i,j} = n_{t+1,i,j}, \text{ for all } i, j. \tag{A.7}$$

Note that since $\gamma_{t,i,j-1} = \gamma_{t+1,i,j}$, $1 \le i \le m, 2 \le j \le r_1$, we have

$$n_{t,i,j-1} = n_{t+1,i,j} \text{ for } 1 \le i \le m \text{ and } 2 \le j \le r_1 \tag{A.8}$$

These (A.7) and (A.8) show that $n_{t,i,j}$'s and $n_{t+1,i,j}$'s are all the same. Since this is true for any two consecutive survey months, $n_{t,i,j}$ is 1 for all t, i, j since there is no sample when $n_{t,i,j} = 0$. This implies that the necessary number of rotation groups is mr_1 for each survey month t. Observe that $\{\gamma_{t,i,j}\}$ and $\{\gamma_{t+1,i,j+1}\}$ for $j \ne r_1$ are the same subsamples from the definition definition of $r_1^m - r_2^{m-1}$ design. Hence $\{\gamma_{t,i,j}; i = 1, \cdots, m\}$ and $\{\gamma_{t+1,i,1}\}$ are different subsamples. But for each i, $\gamma_{t+1,i,1}$ can be corresponded to a γ_{t,i',r_1}, $i = 1, \cdots, m$ in the sense that $\gamma_{t+1,i,1}$ and γ_{t,i',r_1} comes from the same rotation group since the rotation occurs in the same rotation group by (ii). This shows that the mr_1 subsamples at the month t and $t+1$ come from the same mr_1 rotation group, and hence mr_1 rotation groups are necessary for the life of the survey since any two successive the same months requires mr_1 rotation groups.

(b) Suppose a subsample is off for r_2 months. Then, during these r_2 months, the subsample will be replaced by new subsample, and the new subsample remains in sample for r_1 successive months. If such replacement happens l times, $l = 0, 1, \ldots,$ during these r_2 months, we have $r_2 = lr_1$ since each replaced subsample has to be surveyed for exactly r_1 successive months.

In order to get an estimate of change between month t and $t+K_1 \cdot K_2, t = 1, 2, \ldots,$ we need overlapping over $K_1 \cdot K_2$ months. Whether or not the overlapping occurs depends on m, r_1, and K_1, K_2. From (III) in (a), the last appearance of $\mathcal{G}_t$ occurs at month $t + mr_1 + (m-1)r_2 - 1$. Hence, when $mr_1 + (m-1)r_2 - 1 < K_1 \cdot K_2$, we have no overlapping for $K_1 \cdot K_2$ months. If $mr_1 + (m-1)r_2 - 1 \geq K_1(K_2+1)$, then we may have overlapping for $K_1(1 + K_2)$ months. This gives $K_1 \cdot K_2 - r_1 + 1 \leq r_1(m-1)(1+l) \leq K_1(K_2+1) - r_1$ for overlapping to occur.
The subsamples only in $\{\gamma_{t,1,1}, \cdots, \gamma_{t,m-m_0^*+1,1}\}$ return to the sample at month $t + r_1(m_0^* - 1)(1+l) + r_1 - 1$, and all subsamples in $\mathcal{G}_t$ do not return to the sample from the month $t + r_1(m_0^* - 1)(1+l) + r_1$ to the month $t + m_0^* r_1(1+l) - r_1$ by (III) in (a). And then only $\gamma_{1r_1}, \gamma_{2r_1}, \cdots, \gamma_{m-m_0^*, r_1}$ appear to the sample at month $t + m_0^*(r_1 + r_2) - r_1 + 1$. Therefore, by the definition of m_0^*, there is no overlapping between month t and $t + K_1 \cdot K_2$ when $r_1(m_0^* - 1)(1+l) + r_1 - 1 < K_1 \cdot K_2$ and $m_0^*(r_1 + r_2) - r_1 + 1 > K_1 \cdot K_2$.

APPENDIX : Proof of Theorem 3.1

By the definition of $r_1^m - r_2^{m-1}$ design, for each given $i = 1, 2, \ldots, m-1$ the subsample appearing in the sample for the ir_1th time at time t returns to the sample with appearance time $ir_1 + 1$ at the month $t + r_2 + 1$. Thus $\gamma_t[ir_1] = \gamma_{t+r_2+1}[ir_1+1]$, $i = 1, 2, \ldots, m-1$. Since $\gamma_t[ir_1]$, $1 \leq i \leq m-1$ are simultaneously rotated out at the month $t+1$ and rotated in the sample at the month $t + r_2 + 1$, it suffices to consider the subsample with the rotation group $\gamma_t[r_1]$ to find the condition satisfying $\gamma_t[r_1] = \gamma_{t+r_2+1}[r_1 + 1]$. We claim that the rotation group $\gamma_t[r_1]$ contains the subsample appearing in the sample for the $[mod_m\{(j+1)m - (j+1)m' + (j+1)\}r_1 + 1]$th time at month $t + jr_1 + 1$ by rotation pattern given (3.1). Suppose this is true. Since there is l such that $r_2 = lr_1$, $l \geq 0$, $t + r_2 + 1$ equals to $t + lr_1 + 1$. Thus at the month $t + lr_1 + 1$, $mod_m\{(l+1)m - (l+1)m' + (l+1)\}r_1 + 1 = r_1 + 1$. This means that

$$\begin{aligned} & mod_m\{(l+1)m - (l+1)m' + l\}r_1 \\ & \quad = mod_m\{(l+1)mr_1 - m'r_2 - m'r_1 + r_2\} = 0 \end{aligned} \tag{B.1}$$

Because $(l+1)mr_1$ is multiple of m, the (B.1) is the same as $mod_m\{-m'r_2 - m'r_1 + r_2\} = 0$. Hence, this implies that $mod_m\{m'r_2 + m'r_1 - r_2\} = mod_m\{m'r_1 + (m'-1)r_2\} = 0$. This shows Theorem 3.1.
To end proof of Theorem, we need the following Lemma.

Lemma B.1. *Let a, b and λ be positive integer, then*

$$mod_a(b + mod_a(\lambda b)) = mod_a((1+\lambda)b).$$

Proof: In is well known that $mod_a(b) = b - [a/b] \cdot b$ where $[\cdot]$ is largest integer less than equal to b/a. Therefore

$$\begin{aligned} mod_a(b + mod_a(\lambda b)) &= mod_a((1+\lambda)b - [\lambda b/a] \cdot a) \\ &= (1+\lambda)b - [\lambda b/a] \cdot a - \Big[(1+\lambda)/a - [\lambda b/a]\Big] \cdot a \\ &= (1+\lambda)b - [(1+\lambda)b/a] \cdot a \\ &= mod_a((1+\lambda)b). \end{aligned}$$

□

By the rotation pattern given in (3.1), we have $\gamma_t[r_1] = \gamma_{t+1}[mod_m\ (m - m' + 1)r_1 + 1]$. Since this rotation group retains in the sample until the month $t + r_1$, $\gamma_t[r_1] = \gamma_{t+r_1}[mod_m(m - m' + 1)r_1 + r_1]$. Again now by (3.1), we can acquire the position of rotation group at the month $t + r_1 + 1$ by solving the equation $(2m' - 2 - k)r_1 + r_1 = mod_m(m - m' + 1)r_1 + r_1$. Then $k = 2m' - 2 - mod_m(m - m' + 1)$ and by Lemma B.1,

$$\begin{aligned} &\gamma_{t+r_1+1}\big[mod_m\{m + m' - 1 - 2m' + 2 + mod_m(m - m' + 1)\}r_1 + 1\big] \\ &= \gamma_{t+r_1+1}\big[mod_m\{m - m' + 1 + mod_m(m - m' + 1)\}r_1 + 1\big] \\ &= \gamma_{t+r_1+1}\big[mod_m(2m - 2m' + 2)r_1 + 1\big] \end{aligned}$$

Following the same fashion, we get $\gamma_{t+2r_1+1}\big[mod_m(3m - 3m' + 3)r_1 + 1\big]$ when $k = 2m' - 2 - mod_m(2m - 2m' + 2)$ in which k is obtained by equating $(2m' - 2 - k)r_1 + r_1 = mod_m(2m - 2m' + 2)r_1 + r_1$, $\gamma_{t+3r_1+1}\big[mod_m(4m - 4m' + 4)r_1 + 1\big]$ when $k = 2m' - 2 - mod_m(3m - 3m' + 3)$, $\cdots$, and $\gamma_{t+jr_1+1}\big[mod_m\{(j+1)m - (j+1)m' + (j+1)\}r_1 + 1\big]$ when $k = 2m' - 2 - mod_m(jm - jm' + j)$. Therefore we arrived at

$$\gamma_t[r_1] = \gamma_{t+jr_1+1}[mod_m\{(j+1)m - (j+1)m' + (j+1)\}r_1 + 1]$$

where $j \geq 0$.

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