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Inverse Heat Conduction Problem in One-Dimensional Time-Dependent Medium with Modified Newton-Raphson Method

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Abstract — An inverse problem is solved to determine the space-dependent thermal conductivity in one-dimensional time-dependent heat conduction medium with the data imposed and measured at the two end-points. The thermal conductivity is approximated as a linear combination of known functions with unknown coefficients and the unknowns are obtained by the governing and sensitivity equations using modified Newton-Raphson method. The estimated results are compared with exact thermal conductivities and it shows good agreements. This approach is expected to be used to estimate spatial composition of heat conduction medium.

1. Introduction

The estimation of the thermal conductivity is an interesting subject in the engineering application as well as in the inverse problem. For inhomogeneous medium, the thermal conductivity varies with position and an accurate estimation of the spatially varying thermal conductivity is necessary in many thermal management system. Also. this kind of problem may be encountered in geological waste disposal and also have applications in petroleum engineering.

Many engineering and mathematics researchers have considered the inverse heat conduction problem (IHCP). The IHCP may be classified into two categories; the determination of thermal properties like thermal conductivity and heat capacity, and the estimation of boundary/initial conditions or heat sources from a knowledge of the temperature and/or heat flux measurements taken at the interior and/or boundary of the heat conduction medium. Regarding the former, some researches focused on the temperature-dependent thermal properties[1]-[6] and others dealt with the space-dependent ones[7]-[8]. They used the data measured at interior medium in their inverse algorithms[7]-[8] In this study. the space-dependent thermal conductivity in one-dimensional time-dependent heat conduction medium is obtained based on sensitivity equation approach. Especially, the algorithm in this work does not require any interior measurements. In practical application such as tomographic technique, the boundary measurements are more common and practical than interior measurements.

Also, in order to obtain the inverse solution, modified Newton-Raphson method is introduced. Many numerical experiments are performed for various spatially varying thermal conductivities to compare the estimated ones with the exact ones. In the comparison, the proposed inverse algorithm shows reasonably good predictability.

2. Mathematical Model

In order to estimate the spatially varying thermal conductivity in one-dimensional domain k(x), the following linear form of thermal conductivity is assumed:

$$k(x)=\sum_{n=1}^{N}k_n\phi_n(x), \tag{1}$$

where $\phi_n(x)$ can be any first order derivative continuous function in the problem domain and k_n are the unknown coefficients. N is an integer. Thus, the present inverse problem is converted into the problem to identify the coefficient k_n.

Consider an infinite slab with thickness $\bar{L}$ and suppose that this slab is a time-dependent heat conduction medium. Initially, the temperature distribution in the slab is uniform and $\bar{T}_0$. At a specific time, $\bar{t}=0$, a heat flux $\bar{q}_1$ is applied to the front surface at $\bar{x}=0$

and another heat flux $\bar{q}_2$ is applied to the back surface at $\bar{x}=\bar{L}$. The heat conduction equation can be non-dimensionalized with $\bar{L}$, $\bar{T}_0$, reference thermal conductivity $\bar{k}_r$, and heat capacity per unit volume $\bar{\rho}\bar{c}_p$. The dimensionless governing equation becomes

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)=\frac{\partial T}{\partial t},\ x\in(0,1),\ t>0 \tag{2}$$

with an initial condition

$$T(x,0)=1,\ x\in[0,1],\ t=0 \tag{3}$$

and boundary conditions

$$-k\frac{\partial T}{\partial x}=q_1,\ x=0,\ t>0,$$
$$-k\frac{\partial T}{\partial x}=q_2,\ x=1,\ t>0. \tag{4}$$

The dimensionless variables are defined as

$$x=\frac{\bar{x}}{\bar{L}},\ T=\frac{\bar{T}}{\bar{T}_0},\ k=\frac{\bar{k}}{\bar{k}_r},\ q=\frac{\bar{L}\bar{q}}{\bar{k}_r\bar{T}_0},\ t=\frac{\bar{k}_r\ \bar{t}}{\bar{\rho}\bar{c}_p\bar{L}^2}. \tag{5}$$

The present paper suggests an iterative algorithm to find the space-dependent thermal conductivity. The algorithm consists of two phases; the forward problem phase and the inverse problem phase. In the forward problem phase, the thermal conductivity k(x) is assumed as a known value and the temperature distribution can be easily obtained. The numerically obtained temperature distribution will be compared with measured temperatures at specified locations. In the inverse problem phase, a linearization method is used to guide the assumed k(x) to approach the actual value. Then an intermediate value of thermal conductivity is replaced with the assumed one in the forward problem phase. This procedure is repeated until the convergence criteria are satisfied.

In order to decide the search step in each iteration, the sensitivity equations are introduced[6]. The sensitivity equations are defined as the derivative $\partial/\partial k_n$ of Eqs. (2)~(4) and it can be expressed as

$$\frac{\partial}{\partial x}\left(k\frac{\partial X_n}{\partial x}\right)+\frac{\partial}{\partial x}\left[\phi_n(x)\frac{\partial T}{\partial x}\right]=\frac{\partial X_n}{\partial t},\ x\in(0,1),\ t>0 \tag{6}$$

with an initial condition

$$X_n(x,0)=0,\ x\in[0,1],\ t=0 \tag{7}$$

and boundary conditions

$$-k\frac{\partial X_n}{\partial x}=-\frac{\phi_n(0)}{k}q_1,\ x=0,\ t>0,$$
$$k\frac{\partial X_n}{\partial x}=-\frac{\phi_n(1)}{k}q_2,\ x=1,\ t>0 \tag{8}$$

where the sensitivity $X_n=\partial T/\partial k_n$. These sensitivities can be obtained if k(x) and T are given.

The forward and sensitivity equations are spatially discretized using the finite element method and the temporal discretization is achieved by Crank-Nicholson representation.

The search step is determined by the following procedure. At first, the discrepancy function is defined as

$$T_D(l,\ m)=T_{cal}(l,\ m)-T_{meas}(l,\ m) \tag{9}$$

where T_{cal} and T_{meas} are calculated and measured temperature at spatial coordinate l (l=1...L) and temporal coordinate m (m=1...M). The discrepancy vector is defined as a column vector of dimension LM whose components are the discrepancies between the calculated and the measured temperatures at both end points (that is L=2),

$$\{T_D\}=\{T_{cal}\}-\{T_{meas}\}$$

and the column vector $\{k\}=\{k_1,\ k_2,\ \cdots,\ k_N\}^T$ should satisfy

$$\{T_D\}=\{T_{cal}\}-\{T_{meas}\}=0. \tag{10}$$

This will be expressed as a least square problem to find the coefficient vector {k} minimizing the objective function

$$f(k)=\frac{1}{2}\{T_D\}^T(T_D). \tag{11}$$

Many researches constructed the object function by using the data measured at interior medium in their inverse algorithms[7][8]. However, in practical application, it is better to use the data which measured at the boundaries.

The modified Newton-Raphson method is implemented to solve the above nonlinear least square problem and the subroutine DUMIDH of the IMSL[9] is used. The Jacobian of the objective function is calculated from the sensitivity solution:

$$\frac{\partial f}{\partial k_n}=\{T_D\}^T\left(\frac{\partial T_{cal}(l,m)}{\partial k_n}\right)=\{T_D\}^T\{X_n(l,m)\} \quad (12)$$

In this, $\{X_n(l, m)\}$ is a column vector whose components are the sensitivity value at spatial coordinate l and temporal coordinate m, and these are obtained from the sensitivity equations, Eqs. (6)-(8).

3. Examples

For the evaluation of the proposed algorithm, three examples are introduced. The model function to approximate the thermal conductivity is assumed to be a fourth-order polynomial (N=5).

The boundary heat fluxes are taken as $q_1=10$ and $q_2=10$, that is, at both sides the heat is added to the medium. The initial values of thermal conductivity coefficients are set to unity, $k_n=1$.

The followings are the description of the illustrative examples:

Example 1. $k(x)=1+2x+3x^2$

Example 2. $k(x)=1+\sqrt{x}$

Example 3. $k(x)=1+\dfrac{1}{2x+1}$

For all examples, the spatial domain is equally discretized into 10 quadratic elements and the calculation is terminated at t=1. The measurement is assumed to be made 20 times (i.e. M=20) only at the end points (i.e. L=2).

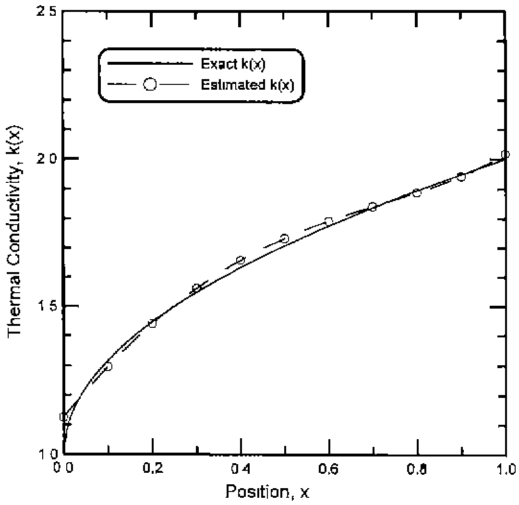

Fig. 2. The estimated thermal conductivity in Example 2.

The predicted and real thermal conductivities for Examples 1~3 are compared in Figs. 1~3, respectively. In all examples, the fourth-order polynomial can reproduce the exact thermal conductivity distribution satisfactorily. As shown in these figures, the positions at which maximum local errors occur are dependent on the real conductivity profiles.

The overall estimated error is defined to investigate

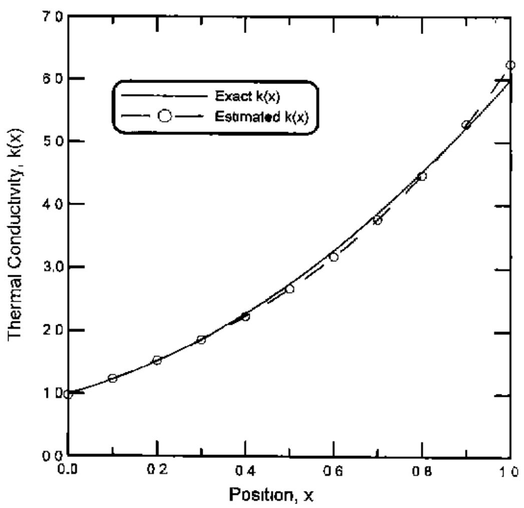

Fig. 1. The estimated thermal conductivity in Example 1.

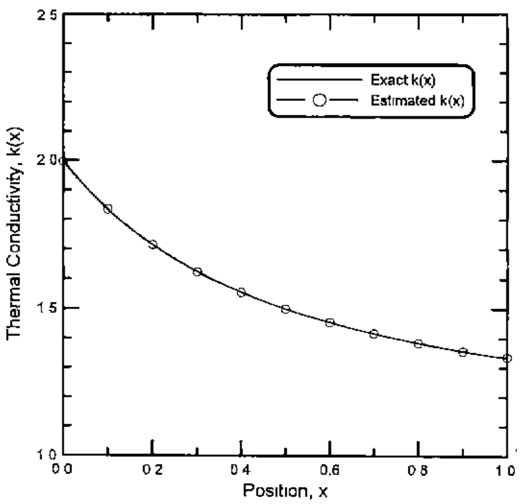

Fig. 3. The estimated thermal conductivity in Example 3.

Table 1. The estimated results and the estimated errors.

No.	Exact k(x)	Estimated thermal conductivity $\sum_{n=1}^{N} k_n \phi_n(x)$	Estimated error
1	$1+2x+3x$	$0.9769498+2.491069x+1.194975x^2+0.8326064x^3+0.7504693x^4$	0.018519
2	$1+\sqrt{x}$	$1.125001+1.820492x-1.129985x^2-0.5216538x^3+0.7237171x^4$	0.008281
3	$1+\frac{1}{2x+1}$	$1.995705-1.845189x+2.590048x^2-2.129298x^3+0.7248000x^4$	0.000537

the deviation of the estimated thermal conductivity from the exact value:

$$E=\frac{\int_0^1 \left| k(x)-\sum_{n=1}^{N} k_n \phi_n(x) \right| dx}{\int_0^1 k(x)dx}. \quad (13)$$

Table 1 shows the estimated result and the error for each example. The estimated thermal conductivities are obtained within 2% relative error for all of 3 examples. From this, it is shown that our algorithm can predict the thermal conductivity profiles which show monotonic increase or decrease in space.

4. Conclusion and Further Studies

A numerical algorithm is presented to estimate the space-dependent thermal conductivity in one-dimensional time-dependent heat conduction medium without interior measurements. The algorithm is composed of two phases, namely the forward problem phase and the inverse problem phase, and they are iteratively applied. In each phase, the heat conduction equation and the sensitivity equations are solved, respectively. The forward and inverse solutions are used to find the search step by modified Newton-Raphson method. Several examples with thermal conductivities are introduced to demonstrate the usefulness of the proposed algorithm and the estimated thermal conductivities are successfully compared with the exact ones. The proposed method may be applicable to the estimation of the spatial composition in heat conduction medium.

Acknowledgement

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