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사용자수 제한과 상수 서비스시간을 갖는 개방형 대기행렬의 출력 프로세스에 관한 연구*

이 영**

Analysis of a Departure Process on the Population Constrained Tandem Queueing Network with Constant Service Times*

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■ Abstract ■

We consider an open tandem queueing network with population constraint and constant service times. The total number of customers that may be present in the network can not exceed a given value k. Customers arriving at the queueing network when there are more than k customers are forced to wait in an external queue. The arrival process to the queueing network is assumed to be arbitrary.

It is known that the queueing network with population constraint and constant service times can be transformed into a simple network involving only two nodes. In this paper, the departure process from the queueing network is examined using this simple network. An approximation can be calculated with accuracy. Finally, validations against simulation data establish the tightness of these.

1. Introduction

In this paper, we consider a stationary departure process from the open tandem queueing network with population constraint and constant service times. The total number of customers simultaneously present in the queueing network can not exceed a given value k. Customers arriving

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while the network has k or more customers are forced to wait in an external queue which is assumed to have an infinite capacity.

Morse [4] and Burke [2] first considered departures from a queueing system. The authors observed that the output of Poisson input queue with a single channel having exponential service time and in steady state must be Poisson with the same rate as the input and formally proved that the output of $M/M/1$ queue is again Poisson. This result has applications in problems of tandem queueing network, that is, the output process is Markovian with respect to the state of the system because of the randomness of the input and the exponential holding time distribution. The interdeparture time and the state of the system are shown the independence at equilibrium.

Reich [5] subsequently proved the observation that the Markov chain with a stationary birth and death process for any $M/M/s$ queue is reversible, which means the birth and the death processes have a same joint distribution. This argument involving reversibility can be applied with profit to other continuous time Markovian queues. Shanbhag [8] and Shanbhag and Tambouratzis [9] generalized the results to general service time in Boes [1]. King [3] showed the $M/D/1$ queue with a waiting room size of 1 has a renewal output process.

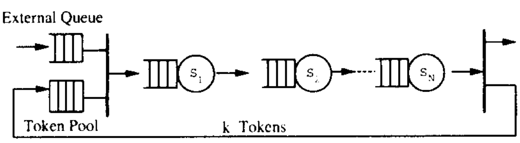
Whitt [10] examined the departure process of a $G/G/s$ system. The author showed that the departure process in a large class of stationary $G/G/s$ queue is approximately Poisson, when there are many busy slow server. However, the departure process tends to be even more complicate to the tandem queueing network with population constraint. Recently, Rhee and Perros [6, 7] studied the mean waiting time of the population constrained queueing network with constant service times. The motivation for studying the departure process arises chiefly in problems of tandem queueing network, which occur in a variety of applications. This is because the departure characteristics from each queueing network or each queue become the arrival characteristics to the following next queueing network or queue. Queueing networks with population constraint have been used to model flow control mechanisms in data communication systems, automatic assembly lines with a fixed number of pallets, and semaphore-controlled software in an operating system.

This paper is organized as follows. In section 2, we present some basic properties for the departure process from the queueing network. In section 3, we first obtain an upper and a lower bound on the variance of the interdeparture time from the semaphore controlled queueing network. Subsequently, we construct an approximation to the variance by appropriately combining these two bounds in section 4. Numerical examples are given in section 5. Finally, conclusions are given in section 6.

2. Some Basic Properties

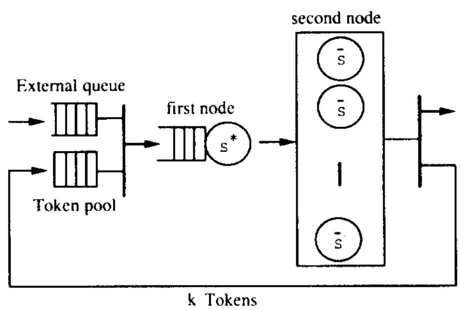
The main interest in the departure process of the queueing network is that such processes may themselves be input or arrival process to the following next queueing network. There are two obvious observations for the departure process to the queueing network with constant service times. First, if the service times are constant, then the departure process is just a shifted version of arrival process when there are infinitely many server at all nodes. Thus, the departure process and the arrival process are identical. Second, the open tandem queueing network with single server at each node, is equivalent with a single node queue with the longest service time as far as the departure process is concerned, when population constraint is not applied.

Let us consider an open tandem queueing network with a population constraint and constant service times. We assume that the queueing network consists of N nodes. The arrival process to the queueing network is assumed to be an arbitrary general distribution with rate λ, and the service time at each node i is constant equal to s_i, $i=1,2 \cdots N$. The population constraint of the queueing network is controlled by a semaphore as shown in [Figure 1]. That is, the management of a shared resource can be carried out using a semaphore queue. The semaphore is a mechanism that consists of a pool of k tokens and an external queue. An arriving customer takes a token and enters the queueing network. The customer holds this token until it leaves the network. At that time, the token is returned to the pool without delay. Customers that arrive during the time when the pool is empty are forced to wait in the external queue. The first customer in the external queue enters the queueing network as soon as a token is returned to the pool.



[Figure 1] An open queueing network with constant service times

For this queueing network, the waiting time of a customer remains the same even though the order of the service times is rearranged. In particular, let us rearrange the nodes of the open queueing network so that the node with the longest service time is placed at the beginning of the queueing network. Then, as discussed by Rhee and Perros [6] a customer's waiting time is the same in both the rearranged queueing network and the original queueing network. Here, a customer's waiting time is defined in this paper as the time a customer spends queueing up in the external queue and in the queues inside the network, service times not included. Since there is no queueing after the first node in the rearranged queueing network, the time a customer spends in the remaining nodes is the sum of the service times $\sum_{i=2}^{N} s_i$. In view of this, we can represent the queueing network by a simpler two-node queueing network as shown in [Figure 2].



[Figure 2] Two-node queueing network with constant service times

For presentation purposes we shall refer to these two nodes as the *first node* and the *second node*. s^* represents the longest service time in the network, and $\bar{s}$ is the sum of the remaining service times, i.e., $\bar{s}=\sum_{i=1}^{N} s_i - s^*$. The number of parallel servers at the second node is infinite. A customer's waiting time in the two-node queueing network is the same as in the rearranged queueing network, and consequently it is the same as in the original queueing network under study. In this paper, we focus on the simpler two-node queueing network.

In this section, we describe some basic properties for the variance of the interdeparture time on the semaphore controlled queueing network. Let us define a_i^0 and a_i^1 to be the interarrival time of the i^{th} customer to the external queue and to the first node in two-node queueing network. Also, let w_i^e be the waiting time of the i^{th} customer in the external queue. The interarrival time of the i^{th} customer to the first node a_i^1 becomes

$$a_i^1 = a_i^0 + w_i^e - w_{i-1}^e. \tag{1}$$

By taking expectations, we have

$$E[a_i^1] = E[a_i^0] + E[w_i^e] - E[w_{i-1}^e]. \tag{2}$$

In the steady state, since the queueing network is stable, we have that $E[w_i^e] = E[w_{i-1}^e]$. Therefore, $E[a_i^1] = E[a_i^0]$. That is, as anticipated, the mean interarrival time to the external queue is equal to the mean interarrival time to the first node. Now, let us consider the variance of the interarrival time to the first node. From (1), we have

$$\begin{aligned} V[a_i^1] = V[a_i^0] + 2\{&cov[a_i^1, w_i^e] \\ &- cov[a_i^0, w_{i-1}^e]\} \end{aligned} \tag{3}$$

Since the covariance between a_i^0 and w_{i-1}^e is zero, we have

$$V[a_i^1] = V[a_i^0] + 2cov[a_i^1, w_i^e] \tag{4}$$

Further, we observe that a_i^1 and w_i^e are positively correlated. This can be shown by noting that if a_i^1 increases by a small positive value ε, then w_i^e also increases by ε, and vice versa. Thus, we have $cov[a_i^1, w_i^e] \geq 0$ and

$$V[a_i^1] \geq V[a_i^0] \tag{5}$$

We note the variance of the interdeparture time from the queueing network is the same as that from the first node. This is because, the second node is an infinite server queue with constant service time. For the case $ks^* \geq T$, the interdeparture time from the queueing network is the same with that of a $G/D/1$ queue with service time s^* [6].

Let us consider the departure process from a $G/D/1$ queue with service time s^*. Let t_i and d_i be the arriving and departure time of the i^{th} customer respectively. Let τ_i be the interdeparture time between the i^{th} and the $(i+1)^{st}$ customer. We have

$$\begin{aligned} \tau_i &= d_{i+1} - d_i \\ &= (t_{i+1} - t_i) + (w_{i+1} - w_i) \end{aligned}$$

In steady state, we have that the expected interdeparture time from the queueing network is given by $E[\tau] = \frac{1}{\lambda}$ where λ is the arrival rate. When the $(i+1)^{st}$ customer arrives before the i^{th} customer's departure,

$$w_{i+1} = w_i + s^* - a_{i+1}$$

where a_{i+1} is the time between the i^{th} and $(i+1)^{st}$ customer's arrival to the queueing network and w_i is the i^{th} customer's waiting time in the queueing network. However, when the $(i+1)^{st}$ customer arrives after the i^{th} customer's departure,

$$-I_i = w_i + s - a_{i+1}$$

where I_i is the length of the idle time between the $(i+1)^{st}$ customer's arriving time to the queueing network and the i^{th} customer's departure time from the queueing network. Therefore, we have $w_{i+1} - I_i = w_i + s^* - a_{i+1}$.

Letting $E[w]$ be the expected waiting time, we note that the variance of the interdeparture time τ of a $G/D/1$ queue with service time s^* is given by the well-known expression,

$$V[\tau] = V[a] - 2(\frac{1}{\lambda} - s^*)E[w], \qquad (6)$$

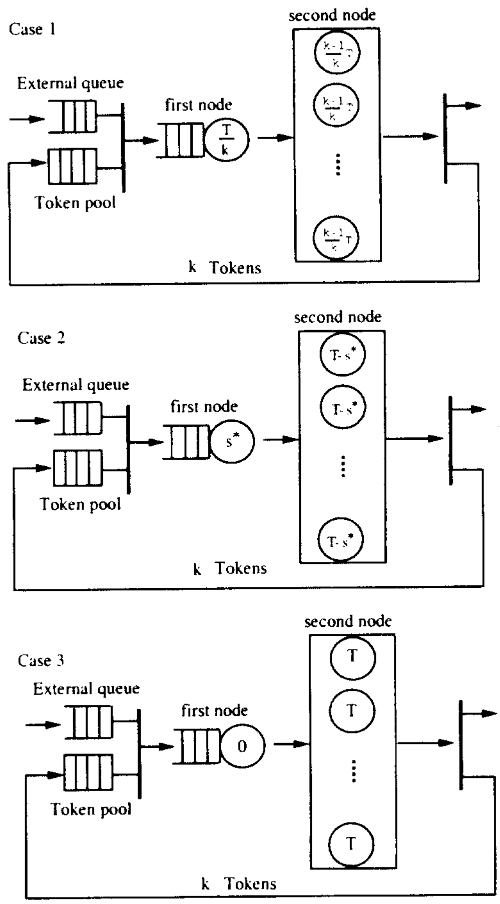
3. Bounds of the variance on the interdeparture times

We construct an upper and a lower bound of the variance of the interdeparture time using the lower and upper bound models for the mean waiting time presented in [6].

3.1 A Lower bound of variance on the interdeparture times

Let T represent the sum of the service times, $T = \sum_{i=1}^{N} s_i$. For the two-node queueing network, the upper bound on the mean waiting time was obtained using a $G/D/1$ queue with a service time $\frac{T}{k}$ [6]. This queueing is equivalent to the two-node queueing network shown in case 1, [Figure 3]. That is, the waiting time of the i^{th} customer is the same in either queueing systems. This can be easily shown by recalling that if $ks^* \geq T$, then the two-node queueing network is equivalent to a $G/D/1$ queue with a service time s^*. For this case, we have that $s^* = \frac{T}{k}$, and thus $\left(\frac{T}{k}\right)k \geq T$.

The variance of the interdeparture time from the queueing network in case 1, [Figure 3], is less than or equal to the variance of the interdeparture time of the two-node queueing network under



[Figure 3] Two-node queueing networks with constant service times

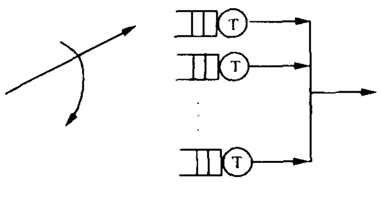
study, shown in case 2, [Figure 3]. This can be shown intuitively as follows. We have that the i^{th} customer's waiting time in case 1, is larger than that in case 2 [6]. Since the waiting time for each customer is larger, the length of the busy period in case 1 is longer than the busy period in case 2. In view of this, one can argue that the interdeparture times in case 1 are more regular than those in case 2. Therefore, the variance of the interdeparture time in case 1 is less than the variance of the interdeparture time in the two-node queueing network under study.

3.2 An Upper bound of the variance on the interdeparture times

The lower bound on the mean waiting time was obtained using a queueing model consisting of k queues in parallel as shown in [Figure 4]. The service time at each queue is equal to T. The arrival process is cyclic, so that every k^{th} customer joins the same queue. For the purpose of obtaining the lower bound of the mean waiting time it sufficed to study one of these k queues. Below, we show that the variance of the interdeparture time of the superposition of the departure processes from the k queues of the queueing network in [Figure 4], is an upper bound of the variance of the interdeparture time of the original queueing network under study. We first show that this superposition can be easily obtained.

Theorem 1. The superposition of the departure processes from the k queues of the queueing system shown in [Figure 4], is identical to that of a single queue served by k parallel servers as shown in [Figure 5]. The service time at each server is T and the arrival process is the same as in the queueing system in [Figure 4], i.e., identical to the arrival process in the two-node queueing network under study.

Proof. Let us consider the beginning of a busy period. For the first k arriving customers to the queueing system shown in [Figure 4], the departure



[Figure 4] k queues in parallel

time of the j^{th} customer from the queueing system d_j, is

$$\begin{aligned} d_j &= \sum_{i=1}^{j} a_i + w_j + T \\ &= \sum_{i=1}^{j} a_i + T \end{aligned} \tag{7}$$

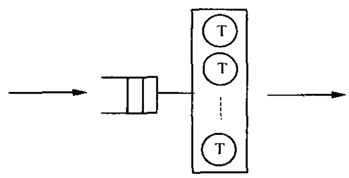
After the k^{th} arrival, the departure time of the j^{th} customer d_j, $j \geq k+1$, is

$$d_j = \sum_{i=1}^{j} a_i + w_j + T, \tag{8}$$

where $w_j = max\{0, T + w_{j-k} - \sum_{i=j-k+1}^{j} a_i\}$.

Now, let us consider the queueing system in [Figure 5]. For the first k arriving customers, we have the same departure times as above. For the j^{th} arriving customer ($j \geq k+1$), its arriving time is $\sum_{i=1}^{j} a_i$, and the earliest time it starts service is

$$d_{j-k} = \sum_{i=1}^{j-k} a_i + w_{j-k} + T. \tag{9}$$



[Figure 5] A single node with k servers

Hence, d_j for $j \geq k+1$ is

$$d_j = \sum_{i=1}^{j} a_i + w_j + T, \tag{10}$$

where $w_j = max\{0, \sum_{i=1}^{j-k} a_i + w_{j-k} + T - \sum_{i=1}^{j} a_i$, which is equal to $max\{0, T + w_{j-k} - \sum_{i=j-k+1}^{j} a_i\}$. Therefore, the two queueing models are equivalent as far as a customer's departure time is concerned. □

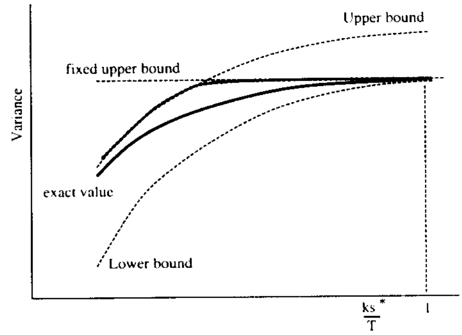
We now observe that the single node shown in [Figure 5] is equivalent to the queueing system shown in case 3 of [Figure 3]. Following similar arguments as in the lower bound case, we can show that the variance of the interdeparture time of the queueing network in case 3 is an upper bound of the variance of the interdeparture time of the queueing network shown in case 2. In particular, we have that the i^{th} customer's waiting time in case 3, is smaller than that in case 2 [6]. Since the waiting time for each customer decreases, the length of the busy period in case 3 is less than that in case 2. In view of this, the interdeparture times from the queueing network in case 2 are more regular than those in case 3. Therefore, the variance of interdeparture time in case 3 is larger than the variance of interdeparture time in the queueing network under study.

To summarize, case 1 of [Figure 3] gives a lower bound on the variance of the interdeparture time, and an upper bound on the mean waiting time. Case 3 also gives an upper bound on the variance of the interdeparture time, and a lower bound on the mean waiting time.

We note that the upper bound of the variance of the interdeparture time tends to be the variance of the interarrival time as the number of tokens k increases. This is because, as k increases, the number of servers in the queueing system in [Figure 5] increases as well and the waiting time tends to be zero. Also, the lower bound of the variance of the interdeparture time becomes equal to the variance of the interdeparture time of a $G/D/1$ queue with a service time s^*, when $\frac{ks^*}{T} = 1$. This is because, in this case we have that $k = \frac{T}{s^*}$, and therefore, the service time $\frac{T}{k}$ of the $G/D/1$ queue that gives the lower bound becomes equal to s^*. Finally, let us consider the original two-node queueing network under study. As the number of tokens k increases, the queueing network behaves like a $G/D/1$ queue with service time s^*. Therefore, the variance of the interdeparture time tends to be the same value as the lower bound, as $\frac{ks^*}{T}$ goes to 1.

The behavior of these bounds in relation to the exact variance, as a function of $\frac{ks^*}{T}$, is shown in [Figure 6]. Empirically, we have observed that the two bounds and the exact variance are concave functions of k, and that the exact variance tends to be the upper bounds as $\frac{ks^*}{T}$ becomes small, and it tends to be the lower bound as $\frac{ks^*}{T}$ goes to 1.

An alternative upper bound of the variance of the interdeparture time can be obtained using a $G/D/1$ queue with a service time s^*. We refer to this upper bound as the *fixed upper bound*. This is because the fixed upper bound is obtained by the independency of the number of tokens. Therefore, a better upper bound can be constructed by taking the minimum of the variance of the interdeparture time of a $G/D/1$ queue with service time s^*, and the variance of the interdeparture time from the queueing system shown in [Figure 6]. This upper bound is indicated by a thick dotted line in [Figure 6].



[Figure 6] Bounds of variance

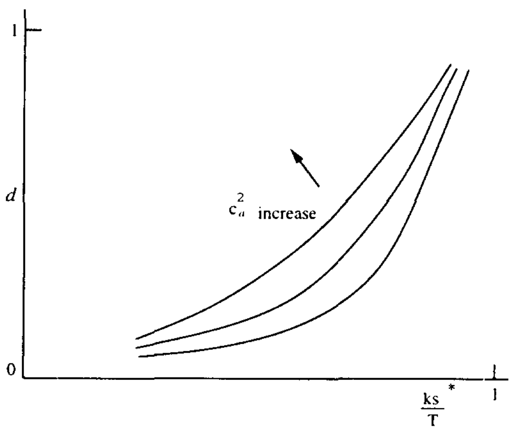
4. An approximation to the variance of the interdeparture time

In this section, we present an approximation to the variance of the interdeparture time of the semaphore controlled queueing network. It is known that the coefficient of variation of the interarrival time affects how close the exact mean waiting time is to either the lower bound or the upper bound [7].

Letting c_a^2 be the squared coefficient of variation of the interarrival time, the distance from the exact mean waiting time to the lower bound of mean waiting time is represented by the following expression :

$$d = \frac{\text{exact mean waiting time} - \text{lower bound}}{\text{upper bound} - \text{lower bound}} \quad (11)$$

We shall refer to d as the normalized distance. So that we have that $0 \le d \le 1$. [Figure 7] shows how this normalized distance d varies as a function of $\frac{ks^*}{T}$ for different values of c_a^2, the squared coefficient of variation of the interarrival time.



[Figure 7] Normalized distance vs. c_a^2 and $\frac{ks^*}{T}$

(These results were calculated by estimating the exact mean waiting time by simulation.) For a given value of c_a^2, d behaves in the same way as the exact mean waiting time. We note that d tends to become a linear as c_a^2 increases.

However, from (6), we observe that the variance of the interdeparture time behaves in the opposite way compare to the mean waiting time. That is, it tends to be the upper bound as $\frac{ks^*}{T}$ becomes very small, and it tends to be the lower bound as $\frac{ks^*}{T}$ goes to 1. That is, the coefficient of variation of the interarrival times affects how close the exact variance of interdeparture times is to either the lower bound or the upper bound. Let us define V_l and V_u to be the lower and upper bound of the variance of interdeparture time respectively. Based on the above empirical observations, an approximation to the variance of interdeparture time can be obtained by combining its upper and lower bounds of variance using the following weight function $f(c_a^2, s^*, T, k)$. The function $f(c_a^2, s^*, T, k)$ is simply chosen so that it tends to become linear as c_a^2 increases and it becomes to close to the lower bound of the variance on the interdeparture time as $\frac{ks^*}{T}$ goes to 1. An approximated variance V_a is,

$$V_a = V_u - f(c_a^2, s^*, T, k)(V_u - V_l) \quad (12)$$

$$f(c_a^2, s^*, T, k) = \left(\frac{ks^*}{T}\right)^{(1+\frac{1}{c_a^2})} \quad (13)$$

The accuracy of the approximation will be tested by comparing it against simulation.

5. Numerical examples

The tightness of the lower and upper bounds of the variance of the interdeparture time was checked by comparing them against simulation estimates of the variance of the interdeparture time in the two-node queueing network. For the lower bound, the variance is computed using (6). The mean waiting time is obtained, by the approximation presented. For the upper bound, the variance of the interdeparture time of the $G/D/k$ queue with service time T, can be calculated using the following expression due to Whitt [10] has suggested as follow

$$V[\tau] = (1-\rho^2)V[a] + (\frac{T}{k})^2(1-k^{-0.5}) \quad (14)$$

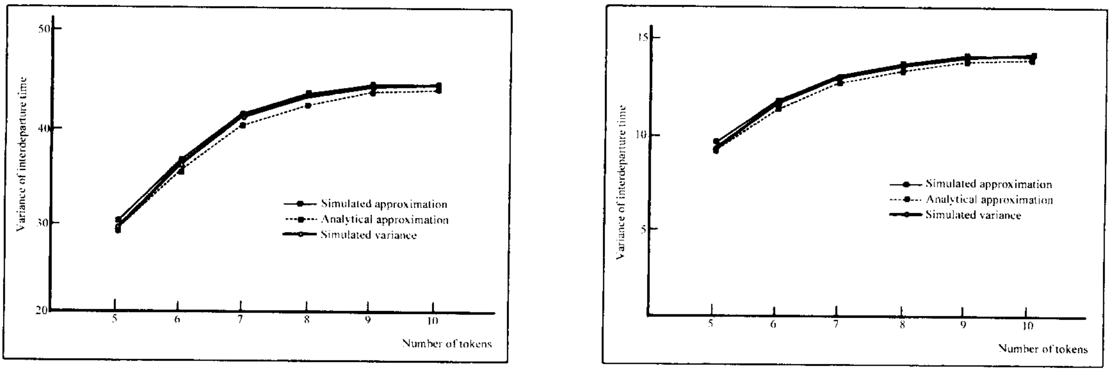
where τ is the variance of the interdeparture time.

For Poisson arrivals, we note that (14) agrees with (6) when $k=1$. However, Whitt's refinement does not seem to work well when the squared coefficient of variation of the arrival process, c_a^2 is large ($c_a^2 \gg 1$). And it does not seem to work well even in the case that c_a^2 is less than 1. This can be observed through the various simulations. That is, the numerical value from (14) is less than the value from the simulation when c_a^2 is greater than 1. On the other hand, the value from (14) is greater than the simulated value in the case that c_a^2 is less than 1. In view of this, to reflect the above observations that the squared coefficient of variation of the arrival process affects the variance of interdeparture time, we suggest a slight modification of Whitt's formula (14).

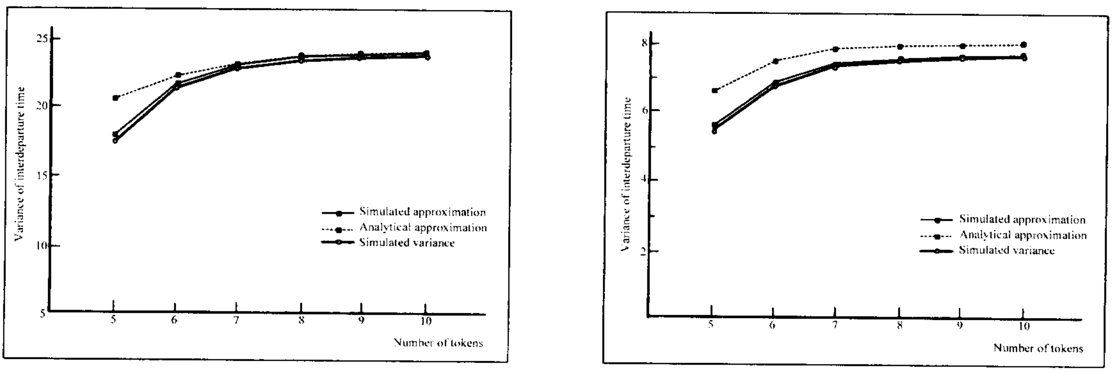
$$V[\tau] = (1-\rho^2)V[a] + \frac{1+c_a^2}{2}(\frac{T}{k})^2(1-k^{-0.5}) \quad (15)$$

The comparisons between (14) and (15) show that Whitt's approximation is greater than (15) in the case of $c_a^2 < 1$ and otherwise (15) is greater than or equal to Whitt's approximation. First, the lower and upper bounds are obtained by calculating (6) and (15) respectively. And later, the approximation can be obtained by combining these two values using the weighing function (12) (referred to as analytical approximation). In additional experiments we have also carried out simulations to get the variance of upper and lower bound. The queueing network in case 1 of [Figure 3] is simulated to get the lower bound of variance. And the queueing network in case 3 of [Figure 3] is also simulated to get the upper bound of variance. The approximation can be obtained by combining these two simulated upper and lower bounds using (12) (referred to as simulated approximation). The reason of executing the simulations to find the upper and lower bounds of variance is that even though (15) is relatively good approximation compare to Whitt's approximation, we are interested in a simulated approximation by weighing an accurate values of upper and lower bound of variance. Finally, we carried out one more simulation to obtain the variance of interdeparture time using the real system as shown in case 2 of [Figure 3] (referred to as simulated variance).

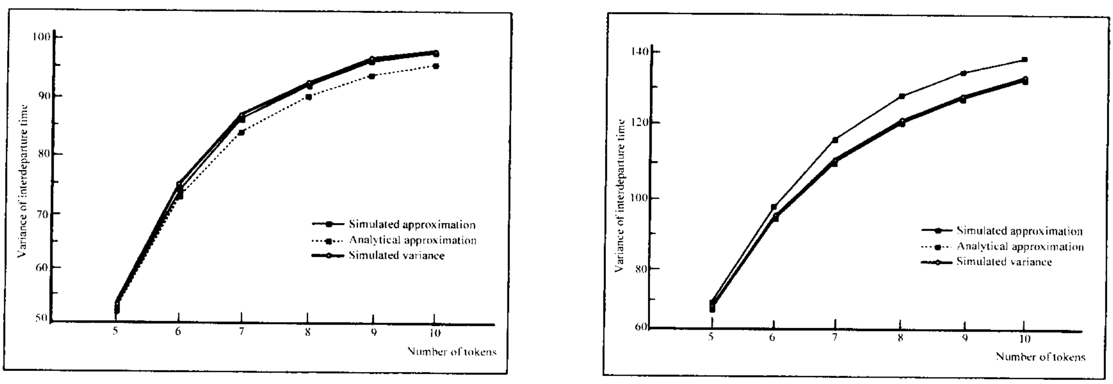
The experiments were carried out assuming a Poisson arrival process and phase type arrival processes, i.e. Erlang and hyper-exponential distributions. Simulations were executed using SLAM and C, 200 types of simulations for each distribution by changing parameters, 30 replications for each simulation, and 100 thousands entity generations for each replication. All simulations were also carried out as a function of the number of tokens. [Figure 8, 9 and 10] describe three types of the variance of the interdeparture with respect to the suggested distributions. The relative error as the absolute difference between the simulated



[Figure 8] An approximation for Poisson($\frac{1}{7}$,2.5,28) (left) and ($\frac{1}{4}$,1.5,16) (right)



[Figure 9] An approximation for Erlang (2, $\frac{1}{3.5}$,2.5,28) (left) and (2, $\frac{1}{2.0}$,1.5,16) (right)



[Figure 10] An approximation for Hyper ($\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{15}$, $\frac{1}{3}$, 2.5, 28, 2.31) (left) and ($\frac{1}{3}$, $\frac{2}{3}$, $\frac{4}{70}$, $\frac{4}{7}$, 2.5, 28, 3.25) (right) variance and the analytical approximation (or simulated approximation) divided by the simulated variance, is considered. Poisson arrival process is described as Poisson (λ, s^*, $\bar{s}$), Erlang arrival process is described as Erlang (number of phase, λ, s^*, $\bar{s}$) and hyper-exponential arrival process is also described as Hyper ($p_1, p_2, \lambda_1, \lambda_2, s^*$, $\bar{s}, c_a^2$).

As can be seen in [Figure 8, 9 and 10], the simulated approximation has a good accuracy compare to the analytical approximation. Therefore, we deduce that the better analytical approximation of the variance of interdeparture time can be achieved as long as providing the good bounds of the variance of interdeparture time. That is, the good analytical approximation depends on the accuracy of the bounds of variance presented in section 4. [Figure 9] shows that the analytical approximation is higher than simulated variance in case of Erlang arrival process. However, for the case of hyper-exponential arrival process, [Figure 10] shows that the simulated approximation is higher than the simulated variance. Furthermore we expect that the difference between the analytical approximation and the simulated variance increases by applying Whitt's approximation. The analytical approximations for the variance of the interdeparture time have a relative error less than 5 %.

6. Conclusions

We considered the departure process from the open tandem queueing network with population constraint and constant service times.

It was known that the queueing network can be transformed into a simple queueing network involving only two nodes. Using this simple queueing network, upper and lower bounds on the variance of the interdeparture time were obtained. We proved that the variance of the interdeparture time is bounded within these two bounds intuitively. The approximation to the variance of the interdeparture time was calculated by weighing these two bounds appropriately. The good analytical approximation depends on the accuracy of the bounds of variance. Validations against simulation data showed that the approximations to the variance of interdeparture time have a good accuracy.

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