Robust and Reliable H_∞ State-Feedback Control : A Linear Matrix Inequality Approach

Seong-Woo Kim, Byung Kook Kim, and Chang-Jun Seo

Abstract : We present a robust and reliable H_∞ state-feedback controller design for linear uncertain systems, which have norm-bounded time-varying uncertainty in the state matrix, and their prespecified sets of actuators are susceptible to failure. These controllers should guarantee robust stability of the systems and H_∞ norm bound against parameter uncertainty and/or actuator failures. Based on the linear matrix inequality (LMI) approach, two state-feedback controller design methods are constructed by formulating to a set of LMIs corresponding to all failure cases or a single LMI that covers all failure cases, with an additional costraint. Effectiveness and geometrical property of these controllers are validated via several numerical examples Furthermore, the proposed LMI frameworks can be applied to multiobjective problems with additional constraints.

Keywords : robust and reliable H_∞ control, state-feedback control, linear matrix inequality, actuator failures

I. Introduction

The reliable control problem for dynamic systems is one of crucially important topics for practical situations, where various component failures and outages occur. Many design methods were suggested to tolerate actuator and sensor failures with various reliable control goals [11,12,13]. Specifically, Veillette *et al.* developed a systematic control methodology to provide guaranteed stability and H_∞ performance even in the event of susceptible actuator or sensor failures[13]. In order to get such a reliable controller, an appropriate solution is needed for a pair of modified algebraic Riccati equations. On the other hand, the problem of robust H_∞ control for linear uncertain systems has got a spotlight by many researchers. For instance, Xie and de Souza designed a controller that stabilizes uncertain systems while satisfying an H_∞ norm bound constraint on disturbance attenuation for all admissible time-varying norm-bounded uncertainties[14].

To tackle both reliable and robust problems, recently Seo developed an algebraic Riccati equation (ARE)-based robust and reliable H_∞ control methodology via state-feedback for linear uncertain systems with parameter uncertainties in the state matrix and possible actuator failures, and also its output feedback version for sensor failures[10]. However, the ARE approach faces nonconvex problem, having some difficulties in analytically solving various multiobjective control problems. On the other hand, the linear matrix inequality(LMI) approach, as discussed in [2,9], has advantages that it is a convex problem and can be solved using numerically efficient algorithm.

The main objective of this paper is to present design methods for robust and reliable H_∞ state-feedback control problem using the LMI formulation, which enables us to synthesize a multiobjective problem with additional performances such as H_2 performance, regional pole constraints, and so on. In this paper, we first show that the reliable H_∞ state-feedback control system can be rendered to LMI conditions. Then, we extend the method to design a robust and reliable H_∞ controller for linear uncertain systems with actuator failure. Thus, this result provides a LMI version of previous researches[13,14].

The rest of this paper is organized as follows: Section II presents the problem definition and basic lemmas. In Section III, we give mathematical formulation of actuator failure, and two robust and reliable H_∞ controller designs. Also, numerical algorithms to solve the LMIs and levels of disturbance attenuation in several failure cases are discussed. Simulation results are illustrated in Section IV to reveal the performance of proposed controllers. Finally, concluding remarks and further research area are presented in Section V.

II. Problem formulation

Consider a linear uncertain system with control input (or actuators) $u \in R^m$.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Gw(t) \\ z(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

where $x \in R^n$ is the state, $z \in R^q$ is the controlled output, $w \in R^p$ is the disturbance input, and A is a time-varying or constant matrix. $G \in R^{n \times p}$, $B \in R^{n \times m}$, $C \in R^{q \times n}$, and $D \in R^{q \times m}$ are constant matrices. We reasonably assume that $C^T D = 0$, $D^T D = R > 0$. The unreliable actuators among all actuators u herein can be outaged occasionally. Let $\Omega \subset \{0, \cdots, m\}$ denote the subset of actuators susceptible to failure: $\overline{\Omega}$ denotes the complementary subset of actuators, which are immune to fail. Then, the control input can be decomposed as (after a suitable ordering)

$$u = \begin{bmatrix} u_{\overline{\Omega}} \\ u_{\Omega} \end{bmatrix}$$

Moreover, the output of a faulty actuator is assumed to be any arbitrary energy-bounded signal belonging to $L_2[0, \infty)$ like the exogenous input $w(t)$. According to this

Manuscript Received : Sep 14, 1999., Accepted : Jan. 24, 2000.
Seong-Woo Kim Real-Time Computing Department, ETRI
Byung Kook Kim Department of Electrical Engineering in KAIST
Chang-Jun Seo . School of Electronic and Telecommunication Engineering in Inje University

classification, the input matrices B and D, the state feedback gain matrix $K\in R^{m\times n}$, and the weighting matrix $R=D^TD>0$ can also be decomposed similarly into

$$B = [\, B_{\bar{\Omega}} \;\; B_{\Omega}], \quad D=[\, D_{\bar{\Omega}} \;\; D_{\Omega}]$$
$$K = \begin{bmatrix} K_{\bar{\Omega}} \\ K_{\Omega} \end{bmatrix}, \quad R=\begin{bmatrix} R_{\bar{\Omega}} & 0 \\ 0 & R_{\Omega} \end{bmatrix}$$

where $B_{\bar{\Omega}}\in R^{n\times j}$, $B_{\Omega}\in R^{n\times k}$, $D_{\bar{\Omega}}\in R^{q\times j}$, $D_{\Omega}\in R^{q\times k}$, $R_{\bar{\Omega}}\in R^{j\times j}$, and $R_{\Omega}\in R^{k\times k}$. Here, $m=j+k$ where j and k are numbers of reliable and unreliable actuators respectively. We can decompose further as:

$$\begin{aligned} B_{\bar{\Omega}} &= [\, B_1 \;\; B_2 \;\; \cdots \;\; B_j] \\ B_{\Omega} &= [\, B_{j+1} \;\; B_{j+2} \;\; \cdots \;\; B_m] \\ D_{\bar{\Omega}} &= [\, D_1 \;\; D_2 \;\; \cdots \;\; D_j] \\ D_{\Omega} &= [\, D_{j+1} \;\; D_{j+2} \;\; \cdots \;\; D_m] \\ K_{\bar{\Omega}} &= \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_j \end{bmatrix}, \; K_{\Omega}= \begin{bmatrix} K_{j-1} \\ K_{j+2} \\ \vdots \\ K_m \end{bmatrix} \end{aligned}$$

where $B_i\in R^n$, $D_i\in R^q$, and $K_i\in R^{1\times n}$ for $i=1,2,\cdots,m$. Since failure cases are determined by any binary combinations among the set of k unreliable actuators, the number of all cases in terms of actuator failure is $N=2^k$ (including no failure case). With this system definition, two control problem under consideration are as follows.

Problem 1 : (Reliable H_∞ State-Feedback Control Problem): When a constant $\gamma>0$ given, design a fixed linear state-feedback controller to stabilize the linear system without parameter uncertainty(A), and to guarantee the given H_∞ *norm constraint* $\gamma>0$ *against augmented dis*turbances including any susceptible actuator failure signals.

Problem 2 : (Robust and Reliable H_∞ State-Feedback Control Problem): When a constant $\gamma>0$ given, design a fixed linear state-feedback controller to stabilize the linear uncertain system and to guarantee the given H_∞ norm constraint $\gamma>0$ against augmented disturbances including any susceptible actuator failure signals as well as all admissible time-varying uncertainties in the state matrix.

Before proceeding further, we need to the following lemmas additionally.

Lemma 1 : (Schur Complement) The linear matrix inequality

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} < 0 \tag{2}$$

where $Q(x)=Q(x)^T$, $R(x)=R(x)^T$ and $S(x)$ depends affinely on x, is equivalent to

$$R(x)<0, \quad Q(x)-S(x)R(x)^{-1}S(x)^T<0$$

or

$$Q(x)<0, \quad R(x)-S(x)^TQ(x)^{-1}S(x)<0$$

Proof : This is called as Schur complement. See Boyd *et al.* (1994) [1]. ■

Lemma 2 : Let $A\in R^{n\times n}$, $\Phi=\Phi^T\in R^{n\times n}$, $H\in R^{n\times i}$, and $E\in R^{j\times n}$ be given matrices. Suppose that there exist a scalar $\delta>0$ and a symmetric matrix $X>0$ such that the *following linear matrix inequality is satisfied.*

$$\begin{bmatrix} AX+XA^T+\Phi & \delta H & XE^T \\ \delta H^T & -\delta I & 0 \\ EX & 0 & -\delta I \end{bmatrix} < 0 \tag{3}$$

Then,

$$[A+HF(t)E]X+X[A+HF(t)E]^T+\Phi<0 \tag{4}$$

for all $F(t)\in R^{i\times j}$ satisfying $F^T(t)F(t)\le I$, $t>0$.

Proof : Using Lemma 1, (3) is equivalent to

$$AX+XA^T+\delta HH^T+\frac{1}{\delta}XE^TEX+\Phi<0 \tag{5}$$

The above equation and the following relation

$$HF(t)EX+XE^TF^T(t)H^T\le\delta HH^T+\frac{1}{\delta}XE^TEX$$

yield (4). ■

III. Main results

In this section, we apply the LMI approach to get reliable controllers for actuator failure. Our goal is to compute static state-feedback controllers $u(t)=Kx(t)$ that meet H_∞ norm-bound on the closed-loop behavior.

1. Reliable control framework

Consider an actuator failure case. Let $f\subseteq\Omega$ be particular subset of indices whose actuators are actually failed and $\bar{f}\supseteq\bar{\Omega}$ be the complementary subset of f. Without loss of generality, $B_{\bar{f}}$ and B_f, $D_{\bar{f}}$ and D_f, and $K_{\bar{f}}$ and K_f can be defined in terms of the failed actuators and normally operating actuators, respectively. Considered as a continuous-time system with constant A matrix, the resultant system is described as the following post-fault model.

$$\begin{aligned} \dot{x}(t) &= Ax(t)+B_{\bar{f}}u_{\bar{f}}(t)+B_fu_f(t)+Gw(t) \\ z_{\bar{f}}(t) &= Cx(t)+D_{\bar{f}}u_{\bar{f}}(t)=z(t)-D_fu_f(t) \end{aligned} \tag{6}$$

Since the failed actuator u_f is considered as a disturbance in addition to w, the disturbance vector is extended to $w_f=\begin{bmatrix} w \\ u_f \end{bmatrix}$. Closed-loop system with a state feedback (actuated by normally operating actuators only) *can be depicted as Figure 1 with transfer function* $T_{\bar{f}}$ *from* w_f to $z_{\bar{f}}$ given by

$$T_{\bar{f}}=\left[\begin{array}{c|c} \hat{A}_{\bar{f}} & \hat{B}_{\bar{f}} \\ \hline \hat{C}_{\bar{f}} & \hat{D}_{\bar{f}} \end{array}\right]=\left[\begin{array}{c|c} A+B_{\bar{f}}K_{\bar{f}} & [G \;\; B_f] \\ \hline C+D_{\bar{f}}K_{\bar{f}} & 0 \end{array}\right] \tag{7}$$

where the realization matrices of the closed-loop system are as follows:

$$\begin{aligned} \hat{A}_{\bar{f}} &= A+B_{\bar{f}}K_{\bar{f}} \\ \hat{B}_{\bar{f}} &= [\, G \;\; B_f] \\ \hat{C}_{\bar{f}} &= C+D_{\bar{f}}K_{\bar{f}} \end{aligned}$$

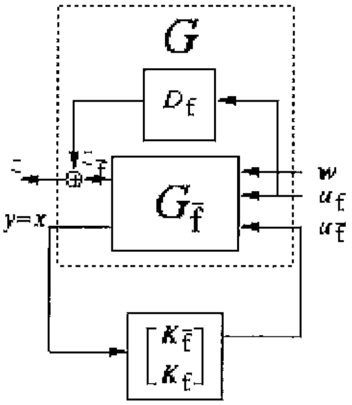

Fig. 1. The state feedback control system with actuator failure.

Now, we try to modify the Bounded Real Lemma to this case of actuator failure. The following lemma is presented as a basis to change the H_∞ constraint to matrix inequality for failure cases.

Lemma 3 : (Bounded Real Lemma) Consider the failed system (6) with closed-loop transfer function $T_{\bar f}$. Then, the following statements are equivalent:

(1) $\| T_{\bar f}\|_\infty = \| \widehat{D}_{\bar f} + \widehat{C}_{\bar f}(sI - \widehat{A}_{\bar f})^{-1}\widehat{B}_{\bar f}\|_\infty < \gamma$ and $A_{\bar f}$ is stable in the continuous-time sense ($Re(\lambda_i(\widehat{A}_{\bar f})) < 0$).

(2) There exists a symmetric positive definite solution $P_{\bar f}$ to the LMI :

$$\begin{bmatrix} \widehat{A}_{\bar f}^T P_{\bar f} + P_{\bar f}\widehat{A}_{\bar f} & P_{\bar f}\widehat{B}_{\bar f} & \widehat{C}_{\bar f}^T \\ \widehat{B}_{\bar f}^T P_{\bar f} & -\gamma^2 I & \widehat{D}_{\bar f}^T \\ \widehat{C}_{\bar f} & \widehat{D}_{\bar f} & -I \end{bmatrix} < 0 \qquad (8)$$

or, equivalently there exists a symmetric positive definite solution $Q_{\bar f}$ to the LMI:

$$\begin{bmatrix} Q_{\bar f}\widehat{A}_{\bar f}^T + \widehat{A}_{\bar f}Q_{\bar f} & \widehat{B}_{\bar f} & \widehat{C}_{\bar f}^T Q_{\bar f} \\ \widehat{B}_{\bar f}^T & -\gamma^2 I & \widehat{D}_{\bar f}^T Q_{\bar f} \\ Q_{\bar f}\widehat{C}_{\bar f} & Q_{\bar f}\widehat{D}_{\bar f} & -I \end{bmatrix} < 0 \qquad (9)$$

Proof : By direct use of the Bounded Real Lemma, the proof is naturally satisfied. See [2,8]. ■

At this time, our main interest is to design a state-feedback controller in order not only to robustly stabilize the system against any susceptible actuator failure but also to satisfy H_∞ constraint $\| T_{\bar f}\|_\infty < \gamma$. To derive main results, define a new matrix variable $V \in R^{m\times n}$ for the state-feedback controller[5]. Without loss of generality, the following relation is satisfied.

$$V = [\,V_{\bar\Omega}^T \;\; V_\Omega^T]^T = [\,V_1^T \;\; V_2^T \;\cdots V_m^T]^T$$

where each row vector V_i for $i = 1,2,\cdots,m$ corresponds to column vector B_i of the input matrix. We state the following main theorem

Theorem 1. Consider the linear system (6) with unreliable actuators u_Ω, and $(A, B_{\bar\Omega})$ is a stabilizable fair. The system is robustly stabilizable against any susceptible actuator failure f, and also the H_∞ constraint $\| T_{\bar f}\|_\infty < \gamma$ is satisfied if there exists a symmetric matrix X and V such that the following ($N+1$) LMIs are all satisfied

$$\begin{bmatrix} \begin{matrix} XA^T + AX \\ + B_{\bar f_i}V_{\bar f_i} + V_{\bar f_i}^T B_{\bar f_i}^T \end{matrix} & [G \;\; B_{f_i}] & XC^T + V_{\bar f_i}^T D_{\bar f_i}^T \\ [G \;\; B_{f_i}]^T & -\gamma^2 I & 0 \\ CX + D_{\bar f_i}V_{\bar f_i} & 0 & -I \end{bmatrix} < 0 \qquad (10)$$

for $i = 1,2,\cdots,N$, and

$$X > 0 \qquad (11)$$

where f_i is an i-th particular subset among $N = 2^k$ possible cases of actuator failure, $\bar f_i$ is the complementary set of f_i, and $B_{\bar f_i}$, B_{f_i} , $D_{\bar f_i}$, and $V_{\bar f_i}$ with proper dimensions consist of the corresponding row or column vectors of B, D, and V, respectively. Moreover, the control input is as follows.

$$u(t) = Kx(t), \quad KX = V = V_N \qquad (12)$$

Proof : By virtue of Lemma 3, the system is robustly stable against the actuator failure f and the H_∞ constraint $\| T_{\bar f}\|_\infty < \gamma$ is satisfied if and only if there exists a symmetric $Q_{\bar f} > 0$ with the following LMIs

$$\begin{bmatrix} \begin{matrix} Q_{\bar f}A^T + AQ_{\bar f} \\ + B_{\bar f}K_{\bar f}Q_{\bar f} + Q_{\bar f}K_{\bar f}^T B_{\bar f}^T \end{matrix} & [G \;\; B_f] & Q_{\bar f}C^T + Q_{\bar f}K_{\bar f}^T D_{\bar f}^T \\ [G \;\; B_f]^T & -\gamma^2 I & 0 \\ CQ_{\bar f} + D_{\bar f}K_{\bar f}Q_{\bar f} & 0 & -I \end{bmatrix} < 0 \quad (13)$$

However, Eq. (13) is yet nonlinear because of terms like $B_{\bar f}K_{\bar f}Q_{\bar f}$. To eliminate nonlinearity, new controller variable $V_{\bar f} = K_{\bar f}Q_{\bar f}$ is introduced. Then, in case of any one failure $f = f_i$, the sufficient conditions for reliable H_∞ control problem are reduced to one of the equations (10)

Now, the LMI formulation involves N Lyapunov matrices $Q_{\bar f_1}$, $\cdots$, $Q_{\bar f_N}$ for each failure case. As a multiobjective problem about all failures $f_1,\cdots,f_N$, our reliable control problem requires single controller related with $X > 0$. To recover convexity in the formulation, this is restricted by the following constraint.

$$Q_{\bar f_1} = \cdots = Q_{\bar f_N} = X \qquad (14)$$

Then, we have the final form in (10). Moreover, since $K_i = V_i X$ for $i = 1,\cdots,m$, the proposed state-feedback controller forms (12). ■

Apparently, notice that this condition includes two limited cases of actuator failure, which are the maximum failure with $B_{\bar f_1} = B_{\bar\Omega}$ and no failure cases with $B_{\bar f_N} = B$.

Remark 1 . This framework can be analyzed through a convex polytopic domain, similarly described in [4]. Consider

$$\widehat{G} = \begin{bmatrix} A & B & G \\ C & D & 0 \end{bmatrix} \in \{\widehat{G}_1, \cdots, \widehat{G}_N\} \qquad (15)$$

where $\{\widehat{G}_1, \cdots, \widehat{G}_N\}$ denotes the discrete set, in which each element corresponds to a failure among total N failure cases. This definition can be formulized as $\widehat{G}=\sum_{i=1}^{N}\alpha_i\widehat{G}_i$, $\sum_{i=1}^{N}\alpha_i=1$ where $\alpha_i=0$ or 1, and, moreover, it forms a kind of convex hull that is the set of such convex combinations of the given elements[4]. Fortunately, being different from output feedback case, the state feedback case does not need to use such as cross decomposition algorithm in order to compensate for the nonlinear terms in the LMIs and construct single controller for all convex failure conditions. ■

Nonetheless, this framework may have disadvantage of the computational load, which is caused by at least $N=2^k$ LMIs (10) with respect to k unreliable actuator. To reduce this burden, the LMI constraints of this problem should be reduced as possible. Hence, we suggest the following theorem that leads to single LMI instead of N equations (10).

Theorem 2 : Consider the linear system (6) with unreliable actuators u_Ω, and $(A, B_{\bar\Omega})$ is a stabilizable fair. The system is robustly stabilizable against any susceptible actuator failure, and also the H_∞ constraint $\|T_{\bar\gamma}\|_\infty<\gamma$ is satisfied if there exist a symmetric X, $V_{\bar\Omega}$, and V_Ω such that the following LMIs are satisfied

$$\begin{bmatrix} XA^T+AX+B_{\bar\Omega}V_{\bar\Omega}+V_{\bar\Omega}^TB_{\bar\Omega}^T & [G\ B_\Omega] & XC^T+V_{\bar\Omega}^TD_{\bar\Omega}^T & B_\Omega+V_\Omega^TR_\Omega \\ [G\ B_\Omega]^T & -\gamma^2I & 0 & 0 \\ CX+D_{\bar\Omega}V_{\bar\Omega} & 0 & -I & 0 \\ B_\Omega^T+R_\Omega V_\Omega & 0 & 0 & -R_\Omega \end{bmatrix}<0 \tag{16}$$

$$X>0 \tag{17}$$

where the diagonal matrix $R_\Omega=R_\Omega^T=D_\Omega^TD_\Omega>0$. Moreover, the control input is as follows.

$$u(t)=Kx(t),\quad KX=\begin{bmatrix} V_{\bar\Omega} \\ V_\Omega \end{bmatrix} \tag{18}$$

Proof : It suffices to show that (16) implies the equation set (10) of Theorem 1. After simple calculation by using Lemma 1, the LMI condition (16) is equivalent to

$$\begin{aligned}\Theta = {} & XA^T+AX+B_{\bar\Omega}V_{\bar\Omega}+V_{\bar\Omega}^TB_{\bar\Omega}^T+\frac{1}{\gamma^2}(GG^T+B_\Omega B_\Omega^T) \\ & + XC^TCX+V_{\bar\Omega}^TR_{\bar\Omega}V_{\bar\Omega}+(B_\Omega^T+R_\Omega V_\Omega)R_\Omega^{-1}(B_\Omega+V_\Omega^TR_\Omega)^T<0\end{aligned} \tag{19}$$

Now consider the system (6) that has an actual failure f. It follows from (13) that one of the LMI (10) of the previous theorem must be satisfied, corresponding to the failure. From

$$\begin{aligned} B_{\bar f}V_{\bar f} &= [\,B_{\bar\Omega}\ B_{\Omega-f}]\begin{bmatrix} V_{\bar\Omega} \\ V_{\Omega-f}\end{bmatrix}, \\ V_{\bar f}^TR_{\bar f}V_{\bar f} &= [\,V_{\bar\Omega}^T\ V_{\Omega-f}^T]\begin{bmatrix} R_{\bar\Omega} & 0 \\ 0 & R_{\Omega-f}\end{bmatrix}\begin{bmatrix} V_{\bar\Omega} \\ V_{\Omega-f}\end{bmatrix}\end{aligned}$$

equation. (10) is reduced to the following equation through similar process of (19).

$$\begin{aligned} & XA^T+AX+B_{\bar\Omega}V_{\bar\Omega}+V_{\bar\Omega}^TB_{\bar\Omega}^T+B_{\Omega-f}V_{\Omega-f}+V_{\Omega-f}^TB_{\Omega-f}^T \\ & +\frac{1}{\gamma^2}(GG^T+B_fB_f^T)+XC^TCX+V_{\bar\Omega}^TR_{\bar\Omega}V_{\bar\Omega}+V_{\Omega-f}^TR_{\Omega-f}V_{\Omega-f}^T<0\end{aligned} \tag{20}$$

where $R_{\bar\Omega}=D_{\bar\Omega}^TD_{\bar\Omega}$ and $R_{\Omega-f}=D_{\Omega-f}^TD_{\Omega-f}$. By completing the square in terms of elements associated with $\Omega-f$, the equation is rearranged to

$$\begin{aligned} & XA^T+AX+B_{\bar\Omega}V_{\bar\Omega}+V_{\bar\Omega}^TB_{\bar\Omega}^T+\frac{1}{\gamma^2}(GG^T+B_fB_f^T) \\ & +XC^TCX+V_{\bar\Omega}^TR_{\bar\Omega}V_{\bar\Omega}-B_{\Omega-f}R_{\Omega-f}^{-1}B_{\Omega-f}^T \\ & +(B_{\Omega-f}+V_{\Omega-f}^TR_{\Omega-f})R_{\Omega-f}^{-1}(B_{\Omega-f}+V_{\Omega-f}^TR_{\Omega-f})^T<0\end{aligned} \tag{21}$$

Noting that the relationship

$$\begin{aligned} & (B_\Omega+V_\Omega^TR_\Omega)R_\Omega^{-1}(B_\Omega+V_\Omega^TR_\Omega)^T \\ = {} & (B_{\Omega-f}+V_{\Omega-f}^TR_{\Omega-f})R_{\Omega-f}^{-1}(B_{\Omega-f}+V_{\Omega-f}^TR_{\Omega-f})^T \\ & +(B_f+V_f^TR_f)R_f^{-1}(B_f+V_f^TR_f)^T\end{aligned}$$

and the controller gain (18), subtracting (21) from (19) produces

$$(B_f+V_f^TR_f)R_f^{-1}(B_f+V_f^TR_f)^T+B_{\Omega-f}(\frac{1}{\gamma^2}I+R_{\Omega-f}^{-1})B_{\Omega-f}^T\geq 0 \tag{22}$$

Thus, it follows that (19) implies (20). ■

Let Σ denote a set of $n\times n$ real symmetric matrices, and define constraint sets from Theorem 1 and Theorem 2 as

$$\Lambda_1\equiv\{(V,X)\in R^{m\times n}\times\Sigma \quad X>0 \text{ and } Eqs.(10)\, hold\} \tag{23}$$

$$\Lambda_2\equiv\{(V,X)\in R^{m\times n}\times\Sigma \cdot X>0 \text{ and } Eq.(16)\, holds\} \tag{24}$$

The constraint sets Λ_1 and Λ_2 are both polytope and open convex hull, having bounded by each set of linear inequalities. Significantly, since (10) implies (16) but not converse, all V and X in Λ_2 are also in Λ_1 and it is followed by a corollary. Notice that Λ_2 can be considered as almost an ellipsoid contained in the polytope Λ_1.

Corollary 1 : Suppose that both Λ_1 and Λ_2 are not empty, having each stabilizing solution to the control system. Then, $\Lambda_1\supseteq\Lambda_2$. ■

Remark 2. Interestingly, this LMI approach has a connection to ARE-based H_∞ control. Define $V_{min}=\min_V\Theta$ After the completion of square with respect to $V_{\bar\Omega}$, we get $V_{min}=-R^{-1}B^T$. Hence, confine the controller gain to have $K=-R^{-1}B^TP$ where $P=X^{-1}$, $R=D^TD$ is invertible.

Multiplying (20) by P, the equation is equivalent to

$$A^TP+PA-PB_{\bar f}R_{\bar f}^{-1}B_{\bar f}^TP+\frac{1}{\gamma^2}P(GG^T+B_fB_f^T)P+C^TC<0 \tag{25}$$

Since this equation is dominated by the case of maximum failure ($f=\Omega$), we only need a symmetric matrix P such that the matrix inequality holds:

$$\begin{aligned} & A^TP+PA-PB_{\bar\Omega}R_{\bar\Omega}^{-1}B_{\bar\Omega}^TP \\ & +\frac{1}{\gamma^2}P(GG^T+B_\Omega B_\Omega^T)P+C^TC<0\end{aligned} \tag{26}$$

Hence, we can get a stabilizing solution P from the following linear matrix inequalities

$$\begin{bmatrix} XA^T+AX-B_{\bar{\Omega}}R_{\bar{\Omega}}^{-1}B_{\bar{\Omega}}^T & [G\ B_{\Omega}] & XC^T \\ [G\ B_{\Omega}]^T & -\gamma^2 I & 0 \\ CX & 0 & -I \end{bmatrix} < 0 \tag{27}$$

$X > 0$

with controller gain $K=-R^{-1}B^TP$ where $P=X^{-1}$.

Moreover, with the positive definite condition $P>0$, this inequality can be reduced to be equality condition that has a stabilizing solution P. Notice that this is a slightly modified version of [12]. While the ARE approach has computationally more beneficial, the convexity of LMI approach provides more flexible solution set. ■

2. Robust and reliable control framework

We extend the solution to robust and reliable H_∞ control problem for a class of linear uncertain systems. Consider a uncertain system with parameter uncertainty in the state matrix.

$$\begin{aligned} \dot{x}_{cl}(t) &= Ax_{cl}(t)+Bw(t) \\ z(t) &= Cx_{cl}(t) \end{aligned} \tag{28}$$

We use the concept of quadratic stability with disturbance attenuation in order to guarantee an H_∞ performance $\|z\|_2 < \gamma\|w\|_2$ for the given system subject to the norm-bounded uncertainties. We refine the quadratic stability from the work of Xie and de Souza [14] to suit our need.

Definition 1 : Given a scalar $\gamma>0$, the uncertain system (28) is said to be quadratically stable with an H_∞ norm bound γ if there exists a symmetric positive-definite matrix $P\in R^{n\times n}$ to the matrix inequality:

$$\begin{bmatrix} A^T(t)P+PA(t) & PB & C^T \\ B^TP & -\gamma^2 I & 0 \\ C & 0 & -I \end{bmatrix} < 0 \tag{29}$$

or, equivalently if there exists a symmetric positive definite solution Q to the matrix inequality:

$$\begin{bmatrix} QA^T(t)+A(t)Q & B & QC^T \\ B^T & -\gamma^2 I & 0 \\ CQ & 0 & -I \end{bmatrix} < 0 \tag{30}$$

This definition implies the following fact. This definition is similar to that of Lemma 2.1 in [14].

Lemma 4 : Suppose that the uncertain system (28) is quadratically stabilized with an H_∞-norm bound $\gamma>0$ via linear feedback. Then for any admissible parameter uncertainty in $A(t)$, the closed-loop system is uniformly asymptotically stable. Moreover, with the assumption of the zero initial condition, the controlled output z satisfied

$$\|z\|_2 < \gamma\|w\|_2 \tag{31}$$

for any admissible parameter uncertainty $\Delta A(\cdot)$ and all nonzero $w\in L_2[0,\infty)$. ■

When there are failures in unreliable actuators, the class of linear uncertain system is given by the following post-fault model:

$$\begin{aligned} \dot{x}(t) &= [A+\Delta A(t)]x(t)+B_{\bar{f}}u_{\bar{f}}(t)+B_f u_f(t)+Gw(t) \\ z_{\bar{f}}(t) &= Cx(t)+D_{\bar{f}}u_{\bar{f}}(t) \end{aligned} \tag{32}$$

where $\Delta A(t)=HF(t)E$ and the time-varying uncertainty matrix $F(t)\in R^{i\times j}$ satisfies $F^TF\le I$. By using Lemma 2 and 4, we get the following result.

Theorem 3 : Consider the linear uncertain system (32) with unreliable actuators u_Ω, and $(A, B_{\bar{\Omega}})$ is a stabilizable fair. The system is quadratically stabilizable with the H_∞ norm bound γ against any susceptible actuator failure if there exist a symmetric X and V such that the following LMIs are satisfied

$$\begin{bmatrix} XA^T+AX+B_{\bar{f}_i}V_{\bar{f}_i}+V_{\bar{f}_i}^TB_{\bar{f}_i}^T & [G\ B_{f_i}] & \delta H & XC^T+V_{\bar{f}_i}^TD_{\bar{f}_i}^T & XE^T \\ [G\ B_{f_i}]^T & -\gamma^2 I & 0 & 0 & 0 \\ \delta H^T & 0 & -\delta I & 0 & 0 \\ CX+D_{\bar{f}_i}V_{\bar{f}_i} & 0 & 0 & -I & 0 \\ EX & 0 & 0 & 0 & -\delta I \end{bmatrix} < 0 \tag{33}$$

for $i=1,2,\cdots,N$ and

$$X > 0 \tag{34}$$

where $V_{\bar{f}_i}$ is the i-th case among the set of linear combinations of unreliable actuators and δ is a positive scalar. Moreover, the control input is as follows.

$$u(t)=Kx(t), \quad KX=V=V_N \tag{35}$$

Proof : By Lemma 1, (33) is reduced to

$$\begin{bmatrix} XA^T+AX+B_{\bar{f}_i}V_{\bar{f}_i}+V_{\bar{f}_i}^TB_{\bar{f}_i}^T + \frac{1}{\gamma^2}(GG^T+B_{f_i}B_{f_i}^T) + XC^CX+V_{\bar{f}_i}^TR_{\bar{f}_i}V_{\bar{f}_i} & \delta H & XE^T \\ \delta H^T & -\delta I & 0 \\ EX & 0 & -\delta I \end{bmatrix} < 0 \tag{36}$$

for $i=1,2,\cdots,N$. Define

$$\begin{aligned} \Phi = {} & B_{\bar{f}_i}V_{\bar{f}_i}+V_{\bar{f}_i}^TB_{\bar{f}_i}^T+\frac{1}{\gamma^2}(GG^T+B_{f_i}B_{f_i}^T) \\ & +XC^TCX+V_{\bar{f}_i}^TR_{\bar{f}_i}V_{\bar{f}_i} \end{aligned} \tag{37}$$

Then, using Lemma 2 and similar procedure of the proof of Theorem 1, the proof is completed. ■

Remark 3 . This is an extension of Theorem 1 in case of having norm-bounded parameter uncertainties. After simple calculation, the LMI conditions (33)-(34) can be converted to

$$\begin{aligned} & XA^T+AX+B_{\bar{f}_i}V_{\bar{f}_i}+V_{\bar{f}_i}^TB_{\bar{f}_i}^T+\frac{1}{\gamma^2}(GG^T+B_{f_i}B_{f_i}^T) \\ & +\delta HH^T+XC^TCX+V_{\bar{f}_i}^TD_{\bar{f}_i}^TD_{\bar{f}_i}V_{\bar{f}_i}+\frac{1}{\delta}XE^TEX<0 \end{aligned} \tag{38}$$

for $i=1,2,\cdots,N$. Now, let $K=-R^{-1}B^TP$ where $P=X^{-1}$ and $R=D^TD$, and multiply both sides of (38) by P. Then, the equation is equivalent to the Riccati counterpart.

$$\begin{aligned} & A^TP+PA-PB_{\bar{f}_i}R_{\bar{f}_i}^{-1}B_{\bar{f}_i}^TP+\frac{1}{\gamma^2}P(GG^T+B_{f_i}B_{f_i}^T)P \\ & +\delta PHH^TP+C^TC+\frac{1}{\delta}E^TE<0 \end{aligned} \tag{39}$$

for $i=1,2,\cdots,N$. The equations herein are dominated by the case of maximum failure ($i=1$ or $f=\Omega$) and represented by

$$\begin{aligned} & A^TP+PA-PB_{\bar{\Omega}}R_{\bar{\Omega}}^{-1}B_{\bar{\Omega}}^TP+\frac{1}{\gamma^2}P(GG^T+B_{\Omega}B_{\Omega}^T)P \\ & +\delta PHH^TP+C^TC+\frac{1}{\delta}E^TE<0 \end{aligned} \tag{40}$$

where $B_{\bar{\Omega}} = B_{\bar{f}}$. This result is a modified version of Seo and Kim [10]. ■

Corollary 2 : The linear uncertain system (32) is quadratically stable with an H_∞-norm bound γ if there exists a $\delta > 0$ such that the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_{\bar{f}} u_{\bar{f}}(t) + [\,G \;\; B_f \;\; \gamma\sqrt{\delta} H\,] \begin{bmatrix} w(t) \\ u_{f(t)} \\ \hat{w}(t) \end{bmatrix} \\ z_{\bar{f}}(t) &= \begin{bmatrix} C \\ \frac{1}{\sqrt{\delta}} E \end{bmatrix} x(t) + \begin{bmatrix} D_{\bar{f}} \\ 0 \end{bmatrix} u_{\bar{f}}(t) \end{aligned} \tag{41}$$

is quadratically stable against any susceptible actuator failure f with H_∞ norm less than γ.

Proof : Applying Corollary 1 in the work of Gu[6] to aforementioned theorem completes the proof. ■

The following theorem is the counterpart with norm-bounded parameter uncertainties to Theorem 2.

Theorem 4 : Consider the linear system (32) with unreliable actuators u_Ω, and $(A, B_{\bar{\Omega}})$ is a stabilizable fair. The system is quadratically stabilizable with the H_∞ norm bound γ against any susceptible actuator failure if there exist a symmetric X, $V_{\bar{\Omega}}$ and V_Ω such that the following LMIs are satisfied

$$\begin{bmatrix} XA^T + AX + B_{\bar{\Omega}} V_{\bar{\Omega}} + V_{\bar{\Omega}}^T B_{\bar{\Omega}}^T & [G \;\; B_\Omega] & \delta H & XC^T + V_{\bar{\Omega}}^T D_{\bar{\Omega}}^T & XE^T & B_\Omega + V_\Omega^T R_\Omega \\ [G \;\; B_\Omega]^T & -\gamma^2 I & 0 & 0 & 0 & 0 \\ \delta H^T & 0 & -\delta I & 0 & 0 & 0 \\ CX + D_{\bar{\Omega}} V_{\bar{\Omega}} & 0 & 0 & -I & 0 & 0 \\ EX & 0 & 0 & 0 & -\delta I & 0 \\ B_\Omega^T + R_\Omega V_\Omega & 0 & 0 & 0 & 0 & -R_\Omega \end{bmatrix} < 0 \tag{42}$$

$$X > 0 \tag{43}$$

where $R_\Omega = R_\Omega^T = D_\Omega^T D_\Omega > 0$. Moreover, the control input is as follows.

$$u(t) = Kx(t), \quad KX = \begin{bmatrix} V_{\bar{\Omega}} \\ V_\Omega \end{bmatrix} \tag{44}$$

■

IV. Example and simulation

Three examples are given in this section to illustrate usefulness of this approach against the others. The first example presents some geometric features of various reliable controls, giving basic insights on their approaches. The second example concerns about more complex system that has time-varying parameter uncertainty and partial actuator failures. All LMI-related computation was performed from the LMI Control Toolbox [3].

Example 1 : Consider the following simple linear system with two inputs

$$\begin{aligned} \dot{x}(t) &= -x + \frac{1}{\sqrt{2}} w + u_1 + \frac{1}{\sqrt{2}} u_2 \\ z &= \begin{bmatrix} x \\ u_1 \\ \frac{1}{\sqrt{2}} u_2 \end{bmatrix} \end{aligned} \tag{45}$$

We assume that the reliable actuator and the unreliable actuator correspond to $u_{\bar{\Omega}} = u_1$ and $u_\Omega = u_2$, respectively. Moreover, the parameter γ is assumed to be 1. Now, we compare our LMI approach to other conventional ones: Standard LQ control, reliable LQ control, and ARE approach of reliable H_∞ control. To fit this example to conventional LQ approach, we consider the auxiliary regulated output $z_0 = z$ and the performance index J

$$J := \int_0^\infty z_0^T z_0 dt = \int_0^\infty (x^2 + u_1^2 + \frac{1}{2} u_2^2) dt \tag{46}$$

For the given performance index, it is well known that the standard LQ-optimal state-feedback controller is computed from the solution of the algebraic Riccati equation

$$A^T P + PA - PBR^{-1} B^T P + Q = 0 \tag{47}$$

and the controller gain $u = Kx$ with $K = -R^{-1} B^T P$. Likewise, from (5) in Veillette[12], the required ARE for reliable LQ state-feedback control is formulated as

$$A^T P + PA - PB_{\bar{\Omega}} R_{\bar{\Omega}}^{-1} B_{\bar{\Omega}}^T P + Q = 0 \tag{48}$$

and the controller gain $u = Kx$ with $K = -R^{-1} B^T P$. From the algebraic relations $Q = C^T C$, $R = D^T D$, and $K = VX^{-1}$ where $X = P^{-1}$ and $V = -R^{-1} B^T$ in the previous section, the solutions (V_1, V_2, X) of above two AREs obtained are shown in Table 1.

Table 1. Reliable control designs and their solutions.

Approach	Solution
LQ control	$P_1 = (-1, -\sqrt{2}, 1+\sqrt{3})$
Reliable LQ control	$P_2 = (-1, -\sqrt{2}, 1+\sqrt{2})$
Reliable H_∞ control (ARE approach)	$P_3 = (-1, -\sqrt{2}, 2)$
Reliable H_∞ control (LMI approach: Theorem 1)	$\Gamma_1 \cap \Gamma_2$: Eqs. (49), (50)
Reliable H_∞ control (LMI approach: Theorem 2)	Γ_3 : Eq. (51)

Finally, in the case of reliable H_∞ control, three cases considered are the ARE approach in Remark 2 and two LMI approaches of Theorems 1 and 2. First, the ARE approach can yield the solution by solving a Riccati equation which is the equality version of (26) in Remark 2.

Next, two reliable H_∞ state-feedback controllers are bounded in the sets Λ_1 of (23) and Λ_2 of (24). Now, define a constraint set Γ_1, which describes LMIs in case of u_2 failure ($u_{\bar{f}} = u_1$) as

$$\begin{aligned} \Gamma_1 = \{ (V_1, V_2, X) \in R \times R \times R \\ X > 0 \text{ and } (X-1)^2 + (V_1+1)^2 < 1 \} \end{aligned} \tag{49}$$

which denotes inside a cylinder in (V_1, V_2, X) space. Γ_2 describes LMIs in no failure ($u_{\bar{f}} = u$) as

$$\begin{aligned} \Gamma_2 = \{ (V_1, V_2, X) \in R \times R \times R \;.\; X > 0 \text{ and} \\ (X-1)^2 + (V_1+1)^2 + \frac{1}{2}(V_2 + \sqrt{2})^2 < \frac{5}{2} \} \end{aligned} \tag{50}$$

which denotes inside an ellipsoid in (V_1, V_2, X) space. Then, the controller gains using Theorem 1 can be determined by $\Lambda_1 \equiv \Gamma_1 \cap \Gamma_2$. On the other hand, the controller gains using Theorem 2 is bounded in LMIs (16) as the following set

$$\Gamma_3 = \{(V_1, V_2, X) \in R \times R \times R \cdot X > 0 \text{ and } (X-1)^2 + (V_1+1)^2 + \frac{1}{2}(V_2+\sqrt{2})^2 < 1\} \quad (51)$$

which denotes inside an ellipsoid in (V_1, V_2, X) space (smaller than (50)). Moreover, Γ_3 herein is equivalent to Λ_2. All reliable control designs are depicted geometrically in Fig. 2. It shows how much convex bound the reliable H_∞ controllers can attain in a geometrical sense. The fact that P_1 and P_2 are away from $\Gamma_1 \cap \Gamma_2$ or Γ_3 means that the conventional LQ approach does not meet the H_∞ norm bound. Moreover, Corollary 1 coincides with the fact that $\Gamma_1 \cap \Gamma_2$ (a cylinder with ellipsoidal edge) includes the ellipsoid Γ_3. This simple example validates the LMI approaches of reliable H_∞ state-feedback control design. With this framework, More complex multiobjective problems such as mixed H_2/H_∞ control problem can be solved by intersecting the constraint set or Λ_2 with level sets of additional design objectives (See, for instance, [7]).

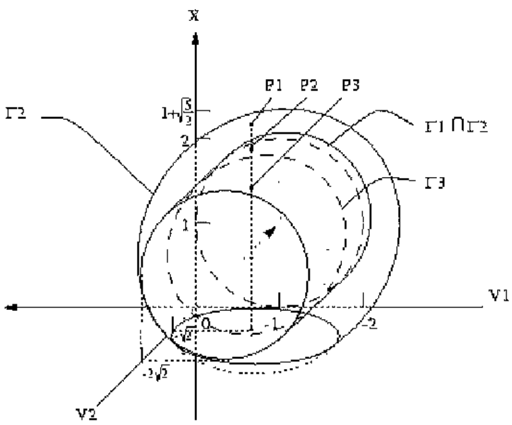

Fig. 2. Geometric representation of Λ_1 reliable control designs.

Example 2 : Consider the linear uncertain system with three inputs

$$\begin{aligned} \dot{x}(t) &= \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix} + \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \\ 0.51 & 0 \end{bmatrix} F(t) \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \right) x(t) \\ &+ \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t) + \begin{bmatrix} 0.001 \\ 0.001 \\ 0.001 \end{bmatrix} w(t) \\ z(t) &= \begin{bmatrix} 0.9 & 0.6 & 0.1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} u(t) \end{aligned} \quad (52)$$

where the uncertain matrix $F(t)$ is zero or time-varying as

$$F(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ or } F(t) = \begin{bmatrix} 0 & 1 \\ \sin(2t) & 0 \end{bmatrix}$$

The spectrum of A is given by $\Lambda(A) = \{0, -2, -3\}$. Note that the nominal system has an unstable mode. We compare a reliable state-feedback controller for nominal case ($F(t)=0$) with a robust and reliable state-feedback controller for time-varying uncertainty. For any given γ unreliable actuator set Ω, these can be obtained by each solving LMIs from Theorem 1/Theorem 2 and Theorem 3/Theorem 4. Through iterative process to me $Trace(P)$ from the MATLAB LMI solver, minimum H_∞-norm bound γ_{min} values according to the set of unreliable actuator failures are obtained as Table 2.

Table 2. Minimum H_∞ norms(γ_{min}) for robust and reliable controls.

Approach	no failure (no u_Ω)	u_1 failure ($u_\Omega = u_1$)	u_2 failure ($u_\Omega = u_2$)	u_1, u_2 failure ($u_\Omega = [u_1\ u_2]^T$)
Reliable H_∞ Control	0.0020	0.2557	0.1921	0.4696
Robust&Reliable H_∞ Control	0.0023	0.4395	0.2415	9.5315

Example 3 : Consider $u_\Omega$the linear uncertain system with three inputs

$$\begin{aligned} \dot{x}(t) &= \left(\begin{bmatrix} -2 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 \\ -1 & 0 & -2 & -3 \\ -2 & -1 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} F(t) \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \right) x(t) \\ &+ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} w(t) \\ z(t) &= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \sqrt{0.1} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t) \end{aligned} \quad (53)$$

where the uncertain matrix $F(t)$ is time-varying as

$$F(t) = \begin{bmatrix} 0 & 1 \\ \sin(2t) & 0 \end{bmatrix}$$

Assume that the reliable actuator and the unreliable actuator correspond to $u_{\bar{\Omega}} = u_3$ and $u_\Omega = [u_1\ u_2]^T$, respectively. Moreover, we let the parameter γ to be 3. The spectrum of A is given by inimiz $\Lambda(A) = \{-1.3160 \pm 2.9194i, 0.1906, -2.5585\}$. Note that the nominal system has an unstable mode. We design two robust and reliable state-feedback controllers for which the closed-loop systems satisfy H_∞ norm bound γ. These can be obtained each by solving LMIs from Theorem 3 and Theorem 4. Their controller gains K_1 and K_2, whose design objective is to minimize $Trace(P)$ for the LMI solver, are

$$K_1 = \begin{bmatrix} 68.338 & 99.558 & 42.145 & 30.721 \\ 58.084 & 83.770 & 34.693 & 26.865 \\ 66.985 & 88.238 & 36.158 & 31.146 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 182.629 & 275.280 & 119.513 & 82.049 \\ 132.858 & 182.629 & 74.626 & 58.544 \\ 55.245 & 71.353 & 29.562 & 25.785 \end{bmatrix}$$

For simulation, the exogenous input $w(t)$ is assumed as

$$w(t) = \begin{cases} 2 & where\ 5 \le t \le 10 \\ 0 & otherwise \end{cases}$$

Since this uniform function $w(t)$ in time-domain corresponds to a sinc function in frequency-domain, we can say the exogenous input $w \in L_2[0, \infty)$. For actuator failure, $u_1(t)$ and/or $u_2(t)$ are assumed to be failed in the interval $5 \le t \le 15$.

We simulated the designed control system with plant having various failure types as described by the following cases.

Case 1 Hard failure (hard-zero) :

$$\begin{cases} u_1 = 0 \;\; where \; 5 \le t \le 15, \\ u_2 = 0 \;\; where \; 8 \le t \le 15 \end{cases}$$

Case 2 Soft failure (soft-bias) :

$$\begin{cases} u_1 = u_1 + 1 \\ u_2 = u_2 + 1 \end{cases} \text{where} \;\; 5 \le t \le 15$$

The simulation results are given in Figures 3 and 4 which show time responses of the controlled output z_1(solid line) and the control input u_1(solid line), u_2(dot-dashed line), and u_3(dotted line) in cases of using K_1 and K_2, respectively. Note that, in these figures, both H_∞ controllers meet the disturbance attenuation bound γ against the additive disturbance including actuator failure signal,

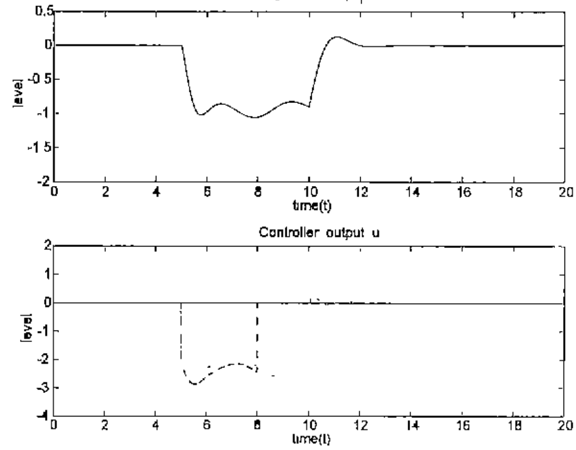

Fig. 3. System responses for Case 1 with LMI Controller 1 ($-$ z_1, u_1 ; $-\cdot-$ u_2; $\cdots$ u_3).

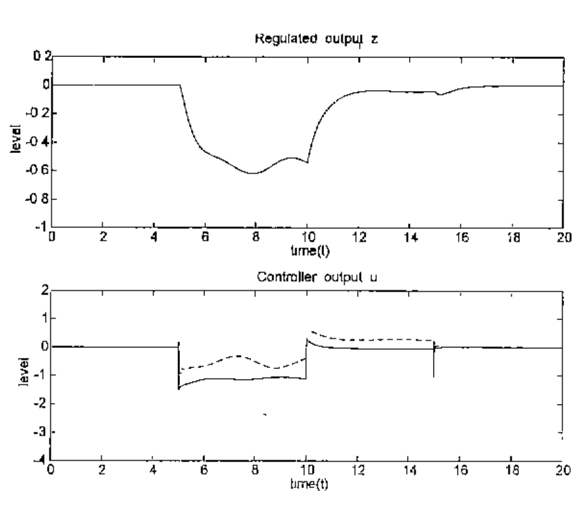

Fig. 4. System responses for Case 2 with LMI Controller 2 ($-$ z_1, u_1; $-\cdot-$ u_2, $\cdots$ u_3).

although K_1 optimized from broader solution set is smaller (more suboptimal) than K_2. Fig. 3 shows that the proposed controller stabilizes the uncertain system against maximum as well as partial hard failures of unreliable actuators; The controlled output does not diverge in spite of u_2 failure at $t=8$. On the other hand, Fig. 4 exemplifies the guaranteed stability with soft failure which makes the unreliable actuators to bias, not to stick a value. Concludingly, the proposed controllers guarantee the closed-loop stability with the given H_∞ attenuation specification and at the same time tolerate various actuator failures irrespective of those types.

V. Concluding remarks

We present control algorithms using LMI approach for robust and reliable H_∞ state-feedback control of linear uncertain systems with actuator failure. This results is altenative to previous Riccati solutions. The controller gains are obtained by numerically solving the LMIs as a convex optimization problem. The resultant system is reliable in the sense that it has robust stability and H_∞ disturbance attenuation peifoimance not only when all actuators are operating normally, but also when some actuators among a prespecified set of unreliable ones become faulty. This result is also validated via several examples Through a comparative study, the convexity of the two proposed controllers is shown geometrically. Another example shows that these H_∞ controllers is confined to different solution sets, i.e., single or multiple LMI constraint(s), but both meet the given specification about H_∞ disturbance attenuation and reliability irrespective of hard or soft failure.

References

[1] S. Boyd, L. ElGhaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System Control Theory* (SIAM Studies in Applied Mathematics), vol 15. SIAM, Philadelphia, 1994.

[2] P. Gahinet and P. Aparian, "A linear matrix inequality approach to H_∞ control," *Int J of Robust and Nonlinear Contr.*, vol 4, pp. 421-448, 1994.

[3] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox*. The MathWorks, Natick, MA., 1995.

[4] J. C. Geromel, J. M. Bernussou, and M. C. de Oliviera. " H_2-norm optimization with constrained dynamic output feedback controllers: decentralized and reliable control," *IEEE Trans. Automat Contr.*, vol. 44, no. 7, pp. 1449-1454, 1999.

[5] J. C. Geromel, P. L. D. Peres, and J. Bernussou, "On a convex parameter space method for linear control design of uncertain systems," *SIAM J. Contr. Optimiz.*, vol 29. no. 2, pp. 381-402, 1991.

[6] K. Gu, " H_∞ control of systems under norm bounded uncertainties in all system matrices," *IEEE Trans Automat. Contr.*, vol. 39. no. 6, pp. 1320-1322, 1994.

[7] P. P. Khargonekar and M. A. Rotea, "Mixed H_2/H_∞ control: A convex optimization approach," *IEEE Trans.*

Automat. Contr., vol. 36 no. 7, pp. 824-837, 1991.

[8] C. Scherer, *The Riccati inequality and the state-space H_∞-optimal control. Ph. D Thesis*, Univ. Wurzburg, Germany, 1990.

[9] C. Scherer, P. Gahinet, and M. Chilali, "Multiobjective output-feedback control via LMI optimization," *IEEE Trans. Automat. Contr.*, vol. 42 no. 7, pp. 896-911, 1997.

[10] C. J. Seo. *A study on robust and reliable H_∞ control for uncertain linear systems. Ph.. D. Thesis*, KAIST, 1995.

[11] D. D. Siljak, "Reliable control using multiple control systems," *Int. J. Control*, vol. 31 no. 2, 1980

[12] R. J. Veillette, "Reliable linear-quadratic state-feedback control," *Automatica*, vol. 31 no. 1, pp. 137-143, 1995.

[13] R. J. Veillette, J. V. Medanic, and W R. Perkins, "Design of reliable control systems," *IEEE Trans. Automat. Contr.*, vol. 37 no. 3, pp. 290-304, 1992.

[14] L. Xie and C. E. de Souza, "Robust H_∞ control for linear systems with norm-bounded time-varying uncertainty," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1188-1191, 1992.

Seong-Woo Kim

Seong-Woo Kim was born in 1969, Korea. He received B.S., M.S. and Ph.D. degrees in Electrical Engineering at KAIST (Korea Advanced Institute of Science and Technology) in 1991, 1993, and 1999. He is currently a senior researcher at ETRI (Korea Electronics and Telecommunications Research Institute). He is interested in real-time system, robust and reliable control, and intelligent control.

Byung Kook Kim

Byung Kook Kim received the B.S. degree in Electronics Engineering from Seoul National University in 1975 and the M.S. and Ph.D. degree from the KAIST in 1977 and 1981 respectively. Dr. Kim was a manager and founder of the Calibration Laboratory, Woojin Instrument Co.Ltd in 1981. His visiting researches include Electrotechnical Laboratory, Tsukuba, Japan in 1993, and the Univ. of Michigan, Ann Arbor, Michigan in 1996. He is currently a Professor of the Department of Electrical Engineering at KAIST. His research interests include reliable process control, real-time systems, mobile robot sensing and navigation, and manipulator control.

Chang-Jun Seo

Chang-Jun Seo was born in 1965 Korea. He received B.S degrees in Electronic Engineering at Kyungpook National University in 1989 and M.S. and Ph.D. degrees in Electrical Engineering at KAIST in 1991 and 1996, respectively. He is currently an associate professor of the School of Electronic and Telecommunication Engineering at Inje University. His current research interests include real-time control system, robust and reliable control, and process control.