Stabilizing Receding Horizon H_∞ Control for Linear Discrete Time-varying Systems

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Abstract: This paper presents sufficient conditions7 for monotonicity of the saddle point value for receding horizon H_∞ control (RHHC). The resulting monotonicity is used to prove the stability of the closed loop. Under these sufficient conditions, the well known terminal equality condition is handled as a special case and the condition on the state weighting matrix is weakened so as to include even the zero matrix. The whole procedure is much simpler than the previous results, and thus is expected to be easily extended for constrained, delayed, and/or nonlinear systems with the RHHC.

1. Introduction

Receding horizon control (RHC) is a closed-loop strategy, where an optimal decision of the control is taken at each sample, and thus finite horizons can be considered. This contrasts with the steady-state linear quadratic (LQ) control, where the control law is determined at the initial time and only infinite horizons can be taken into account. For these reasons, the RHC has been widely investigated as a successful feedback strategy [1]-[8].

For the closed loop stability of the RHC, it is well known that the condition on the terminal weighting matrix as well as the state weighting matrix is important. For finite horizons, one approach to achieve the stability is to impose infinite terminal weighting which is equivalent to setting a zero terminal weighting matrix for the inverse Riccati equation [1], [2]. This is referred to as the terminal equality condition. Since imposing infinite terminal weighting is demanding, use of finite terminal weighting matrices has been investigated in [6]-[9]. As an alternative approach to finite horizons, an infinite horizon formulation has been explored in [4]-[6]. However, infinite horizon formulations in [4]-[6], can also be treated as finite horizon formulations with appropriate finite terminal weighting matrices, as shown in [6], [8], [9].

In proofs of the closed loop stability of the RHC, the monotonicity of the optimal cost has been widely used since it does not only imply the monotonicity of the Riccati equations for linear systems but can also be used to show attractivity of nonlinear systems [4]-[6], [8], [9].

This RHC has been applied to H_∞ problems in order to combine the practical advantage of the RHC with the robustness of the H_∞ control. This receding horizon H_∞ control is referred to as RHHC here. For continuous-time systems, the terminal equality condition is proposed in [10], and the terminal inequality condition is in [11] for the monotonicity of the Riccati equations. A discrete-time result is also presented in [12]. However, the procedure of proving the stability using the resulting terminal inequality is rather demanding as monotonicity of the saddle point value is not clearly discussed. In addition, the results in [11], [12] do not include the terminal equality condition in [10] and require nonzero state weighting matrices.

In this paper, we show that such a terminal condition as in [12] is actually sufficient for the monotonicity of the saddle point value. One important consequence of this new theorem is that the stability of the RHHC follows from the theorem in quite a straightforward manner. In other words, the proof of stability via the proposed monotonicity is much simpler than that in [12], and thus is expected to be easily extended for nonlinear systems. The well known terminal equality condition is handled as a special case of a new proposed inequality condition. The condition on the state weighting matrix is weakened so as to include even the zero matrix, still guaranteeing the closed loop stability of the RHHC.

The rest of this paper is organized as follows. In Section 2, terminal inequality conditions are presented for linear discrete time-varying systems which guarantee the monotonicity of the saddle point value or the Riccati equations. In Section 3, under the presented terminal inequality conditions, the closed-loop stability of the RHHC is shown for linear discrete time-varying systems. Finally, conclusions are presented in Section 4.

3. Monotonicity of the saddle point value

Consider a linear discrete time-varying system

$$\begin{aligned} x(i+1) &= A(i)x(i)+B_1(i)w(i)+B_2(i)u(i) \\ z(i) &= C(i)x(i)+D(i)u(i) \end{aligned} \quad (1)$$

where $x(i)\in R^n$ is the state, $u(i)\in R^m$ the control, $w(i)\in R^l$ the disturbance, $z(i)\in R^p$ the controlled output, $D^T(i)D(i)=I$, and $C^T(i)D(i)=0$. For this system, consider the following dynamic game problem with a finite cost horizon

$$\min_u \max_w J(x(i_0), i_0, i_f) \quad (2)$$

where

$$J(x(i_0), i_0, i_f) = \sum_{i=i_0}^{i_f-1}[x^T(i)Q(i)x(i)+u^T(i)u(i)-\gamma^2 w^T(i)w(i)] + x^T(i_f)Q_f(i_f)x(i_f).$$

Here, $Q(i)=C^T(i)C(i)\geq 0, Q_f(i_f)\geq 0$, and γ is the disturbance attenuation level.

The matrices $A(i)$, $B_1(i)$, $B_2(i)$, $C(i)$, $D(i)$, and $Q_f(i_f)$ are assumed to be bounded. Define $B_\gamma(i)$ as $B_\gamma(i)=\gamma^{-1}B_1(i)$.

Then, the dynamic game theory described by (1) and (2) [13] admits a unique feedback saddle-point solution, if and only if

$$[I-B_\gamma^T(i)M(i+1, i_f)B_\gamma(i)]>0 \text{ for } i\in[i_0, i_f-1] \quad (3)$$

where the sequence of nonnegative definite matrix $M(i+1, i_f)$ over $i\in[i_0, i_f-1]$ is generated by the following equation

$$M(i, i_f) = A^T(i)[I+M(i+1, i_f)\hat{Q}(i)]^{-1}M(i+1, i_f)A(i)+Q(i) \quad (4)$$

with $\hat{Q}(i)=B_2(i)B_2^T(i)-B_\gamma(i)B_\gamma^T(i)$ and the boundary condition

$$M(i_f, i_f) = Q_f(i_f) \quad (5)$$

Then the unique saddle-point solution is given by

$$u^*(i) = -B_2^T(i)[I+M(i+1, i_f)\hat{Q}(i)]^{-1}M(i+1, i_f)A(i)x(i) \quad (6)$$

$$w^*(i) = \gamma^{-1}B_\gamma^T(i)[I+M(i+1, i_f)\hat{Q}(i)]^{-1}M(i+1, i_f)A(i)x(i), \quad i_0\leq i\leq i_f-1 \quad (7)$$

Also, the saddle point value of the dynamic game of (2) with (6) and (7) is given by

$$J^*(x(i), i, i_f) = x^T(i)M(i, i_f)x(i) \quad (8)$$

Note that the solvability condition (3) is assumed to be satisfied throughout the rest of the paper. We are now in a position to present our main theorem below.

Theorem 2.1 Assume that $Q_f(i_f)$ in (5) satisfies the following inequality:

$$Q_f(\sigma) \geq Q(\sigma)+A^T(\sigma)[I+Q_f(\sigma+1)\hat{Q}(\sigma)]^{-1}Q_f(\sigma+1)A(\sigma) \quad (9)$$

Then, for any τ and $\sigma(\geq\tau)$, we have

$$J^*(x(\tau), \tau, \sigma) \geq J^*(x(\tau), \tau, \sigma+1) \quad (10)$$

and thus $M(\tau, \sigma)\geq M(\tau, \sigma+1)$.

proof : Define

$\Delta J^*(x(\tau), \sigma)$ as $\Delta J^*(x(\tau), \sigma)=J^*(x(\tau), \tau, \sigma+1)-J^*(x(\tau), \tau, \sigma)$.

Then

$$\begin{aligned} \Delta J^*(x(\tau), \sigma) = &\sum_{i=\tau}^{\sigma}[x_1^T(i)Q(i)x_1(i)+u_1^T(i)u_1(i)-\gamma^2 w_1^T(i)w_1(i)] \\ &+J^*(x_1(\sigma), \sigma, \sigma+1) \\ &-\sum_{i=\tau}^{\sigma-1}[x_2^T(i)Q(i)x_2(i)+u_2^T(i)u_2(i)-\gamma^2 w_2^T(i)w_2(i)] \\ &-x_2^T(\sigma)Q_f(\sigma)x_2(\sigma) \end{aligned}$$

where the pair $(u_1(i), w_1(i))$ is a saddle point solution for $J(x(\tau), \tau, \sigma+1)$ and the pair $(u_2(i), w_2(i))$ is for $J(x(\tau), \tau, \sigma)$. If we replace $u_1(i)$ and $w_2(i)$ by $u_2(i)$ and $w_1(i)$ up to $\sigma-1$, then

$$\Delta J^*(x(\tau), \sigma) \leq J(x(\sigma), \sigma, \sigma+1)-x^T(\sigma)Q_f(\sigma)x(\sigma)$$

Since this inequality holds for any $u(\sigma)$, we have

$$\begin{aligned} \Delta J^*(x(\tau), \sigma) &\leq \max_{w(\sigma)}\min_{u(\sigma)}J(x(\sigma), \sigma, \sigma+1)-x^T(\sigma)Q_f(\sigma)x(\sigma) \\ &= J^*(x(\sigma), \sigma, \sigma+1)-x^T(\sigma)Q_f(\sigma)x(\sigma) \end{aligned} \quad (11)$$

where $x(\sigma)$ is the state at σ resulting from $u(i)=u_2(i)$ and $w(i)=w_1(i)$ for $i\in[\tau, \sigma-1]$.

It thus follows that

$$\Delta J^*(x(\tau), \sigma)\leq x^T(\sigma)[Q(\sigma)+A^T(\sigma)(I+Q_f(\sigma+1)\hat{Q}(\sigma))^{-1}Q_f(\sigma+1) A(\sigma)-Q_f(\sigma)]x(\sigma)\leq 0$$

where the first inequality is obtained using (4), (5) and (8) for $i_0=\sigma$ and $i_f=\sigma+1$. This completes the proof.

Remark 2.1 In [12], the closed-loop is shown to be stable if there exists a matrix $H(\sigma)$ of appropriate dimension such that

$$Q_f(\sigma) \geq Q(\sigma)+H^T(\sigma)H(\sigma)+F^T(\sigma)Q_f(\sigma+1)B_\gamma(\sigma) \quad (12)$$
$$[I-B_\gamma^T(\sigma)Q_f(\sigma+1)B_\gamma(\sigma)]B_\gamma^T(\sigma)Q_f(\sigma+1)F(\sigma)+F^T(\sigma)Q_f(\sigma+1)F(\sigma)$$

where $F(\sigma)=A(\sigma)-B_2(\sigma)H(\sigma)$.

This sufficient condition is obtained by the use of an upper bound for $J^*(x(\sigma),\sigma,\sigma+1)$ as in (11), which is obtained by fixing $u(\sigma)=-H(\sigma)x(\sigma)$ and maximizing $J(x(\sigma),\sigma,\sigma+1)$ with $w(\sigma)$. Hence, it is straightforward to conclude that the sufficient condition for monotonicity (and thus for stability) given in this letter is less conservative. In other words, the set of $Q_f(\sigma)$'s for stability guarantees is wider here.

Consider a cost function with an infinite horizon. If the pair $(A(i),B_2(i))$ is uniformly stabilizable and the system is uniformly asymptotically stable with $u(i)=-B_2^T(i)[I+M(i+1,\infty)\hat{Q}(i)]^{-1}M(i+1,\infty)A(i)x(i)$ and

$w(i)=\gamma^{-1}B_\gamma^T(i)[I+M(i+1,\infty)\hat{Q}(i)]^{-1}M(i+1,\infty)A(i)x(i)$

for $i\geq\sigma\geq\tau$, then

$$\min_{u(i)}\max_{w(i),i\in[\tau,\sigma-1]}\sum_{i=\tau}^{\infty}[x^T(i)Q(i)x(i)+u^T(i)u(i)-\gamma^2w^T(i)w(i)]$$
$$=\min_{u(i)}\max_{w(i),i\in[\tau,\sigma-1]}\sum_{i=\tau}^{\sigma-1}[x^T(i)Q(i)x(i)+u^T(i)u(i)$$
$$-\gamma^2w^T(i)w(i)]+x^T(\sigma)Q_f(\sigma)x(\sigma)$$

where $Q_f(\sigma)$ is bounded and satisfies

$$Q_f(\sigma) = Q(\sigma)+A^T(\sigma)[I+Q_f(\sigma+1)\hat{Q}(\sigma)]^{-1}Q_f(\sigma+1)A(\sigma) \quad (13)$$

with $M(\sigma+1,\infty)=Q_f(\sigma+1)$. Note that (13) is a special case of (9) and therefore the infinite-horizon RHC is stabilizing.

In the following, the nondecreasing monotonicity of the saddle point value is studied.

Theorem 2.2 Assume that $Q_f(i_f)$ in (5) satisfies the following inequality:

$$Q_f(\sigma) \leq A^T(\sigma)[I+Q_f(\sigma+1)\hat{Q}(\sigma)]^{-1}Q_f(\sigma+1)A(\sigma)+Q(\sigma) \quad (14)$$

Then, for any τ and $\sigma(\geq\tau)$, we have

$$J^*(x(\tau),\tau,\sigma+1) \geq J^*(x(\tau),\tau,\sigma), \quad (15)$$

and thus $M(\tau,\sigma+1)\geq M(\tau,\sigma)$.

Proof : It can proved in the same way as Theorem 2.1 by replacing $u_2(i)$ and $w_1(i)$ with $u_1(i)$ and $w_2(i)$ up to $\sigma-1$.

The free terminal condition, $Q_f(\cdot)=0$, satisfies (14). Thus, Theorem 2.2 includes the monotonicity of the saddle point value of the free terminal case.

In LQ problems, it is shown that the Riccati equations holds once, it holds for all prior times by using the optimality [8]. The following extends the result in [8] to H_∞ problems.

Corollary 2.1 (1) If $J^*(x(\tau'),\tau',\sigma+1)\leq J^*(x(\tau'),\tau',\sigma)$ (or $\geq J^*(x(\tau'),\tau',\sigma)$) for some τ', then $J^*(x(\tau''),\tau'',\sigma+1)\leq J^*(x(\tau''),\tau'',\sigma)$ (or $\geq J^*(x(\tau''),\tau'',\sigma)$) where $\tau_0\leq\tau''<\tau'$. (2) If $M(\tau',\sigma+1)\leq M(\tau',\sigma)$ (or $\geq M(\tau',\sigma)$) for some τ', then $M(\tau'',\sigma+1)\leq M(\tau'',\sigma)$ (or $\geq M(\tau'',\sigma)$) where $\tau_0\leq\tau''\leq\tau'$.

Proof : It can be easily proved in the same way as in Theorems 2.1 and 2.2.

Now, we introduce the inverse form of (4), which is used in the next section. We assume that $A(i)$ is nonsingular when the inverse of $M(i,i_f)$ is necessary. If $M(i,i_f)$ is nonsingular at $i\leq i_f$, there exists $P(i,i_f)=M^{-1}(i,i_f)+\hat{Q}(i-1)$ which satisfies

$$P(i,i_f) = A^{-1}(i)[I+P(i+1,i_f)A^{-T}(i)Q(i)A^{-1}(i)]^{-1}$$
$$P(i+1,i_f)A^{-T}(i)+\hat{Q}(i-1) \quad (16)$$

In order for $M(i,i_f)$ to be nonsingular for $i\in[i_0+1,i_f]$, we must have $P(i_f,i_f)=M^{-1}(i_f,i_f)+\hat{Q}(i_f-1)=Q_f^{-1}(i_f)+\hat{Q}(i_f-1)>0$ and thus, $Q_f(i_f)$ must be nonsingular.

In the next section, we also consider a little different approach. We assume that $P(i,i_f)$ in (16) is given from the beginning with a terminal constraint $P(i_f,i_f)=P_f(i_f)$ and some $\gamma>0$ rather than the one obtained from inverting (4). Note that $P_f(i_f)=0$ corresponds to the case where $Q_f(i_f)=\infty I$ $(i.e., Q_f^{-1}(i_f)=0)$ even if it is mathematically not rigorous. For the case $Q_f(i_f)=\infty I$, $P(i_f,i_f)=\hat{Q}(i_f-1)$ since $Q_f^{-1}(i_f)=0$. The Riccati equations $P(i,i_f)$ in (16) can be replaced by $P(i,i_f+1)$ with $P(i_f+1,i_f+1)=0$.

In fact, the Riccati equations (16) with the condition $P_f(i_f)\geq 0$ can be obtained from the following problem. Consider the following adjoint system of (1):

$$\hat{x}(i+1) = A^{-T}(i)\hat{x}(i)+A^{-T}(i)C^T(i)\hat{u}(i) \quad (17)$$

with the cost function

$$\hat{J}(\hat{x}(i_0), i_0, i_f) = \sum_{i=i_0}^{i_f-1}[\hat{x}^T(i)\hat{Q}(i-1)\hat{x}(i) + \hat{u}^T(i)\hat{u}(i)] + \hat{u}^T(i)\hat{u}(i)] + \hat{x}^T(i_f)P_f(i_f)\hat{x}(i_f) \quad (18)$$

where $\gamma > 0$ satisfies $\hat{Q}(i-1) \geq 0$ for $i \in [i_0, i_f-1]$. Then, the optimal cost is given by $\hat{J}^*(\hat{x}(i), i, i_f) = \hat{x}^T(i)P(i, i_f)\hat{x}(i)$. In the following, a terminal inequality condition is proposed by using Theorem 2.2 for the nondecreasing monotonicity of $P(i, i_f)$ with $P_f(i_f)$.

Corollary 2.2 Assume that $\hat{Q}(i) \geq 0$ for all i and $P_f(i_f)$ in (18) satisfies the following:

$$P_f(\sigma) \leq A^{-1}(\sigma)[I + P_f(\sigma+1)A^{-T}(\sigma)Q(\sigma)A^{-1}(\sigma)]^{-1} P_f(\sigma+1)A^{-T}(\sigma) + \hat{Q}(\sigma-1). \quad (19)$$

Then, for any τ and $\sigma(\geq \tau)$, we have $\hat{J}^*(\hat{x}(\tau), \tau, \sigma+1) \leq \hat{J}^*(\hat{x}(\tau), \tau, \sigma), \tau \leq \sigma$, and thus

$$P(\tau, \sigma+1) \geq P(\tau, \sigma). \quad (20)$$

The proof here follows that of Theorem 2.2. $P_f(i_f) = 0$ satisfies (19)), and thus (20) holds.

In the following section, stabilizing receding horizon H_∞ controls (RHHC) are proposed by using the monotonicity of the saddle point value or the Riccati equations.

4. Stability of the RHHC

The receding horizon H_∞ control is obtained by replacing i_f with $i+N$ for $1 \leq N < \infty$ in (6):

$$u^*(i) = -B_2^T(i)[I + M(i+1, i+N)\hat{Q}(i)]^{-1} M(i+1, i+N)A(i)x(i) \quad (21)$$

where $M(i+1, i+N)$ is computed from (4) with $M(i+N, i+N) = Q_f(i+N)$.

Assume that the selected $Q_f(i+N)$ is positive semidefinite and bounded for all i, and N is nonzero and finite throughout the rest of this paper.

Definitions of uniform controllability and uniform observability with positive integers l_c and l_o in [2] are used. Let $l = \max(l_c, l_o)$. Denote the transition matrix of the matrix $f(i)$ as $\Psi_{n,l}(f) = f(n-1)f(n-2)\cdots f(l-1)f(l),\ n \geq l$ with $\Psi_{l,l}(f) = I$.

The following results are needed to show the stabilizing properties of (21).

Lemma 3.1 Consider a linear discrete time-varying system

$$x(i+1) = A(i)x(i). \quad (22)$$

Assume that $A(i)$ is bounded. If the system (22) is uniformly attractive, then the system is uniformly asymptotically stable.

Proof : If the system is uniformly attractive over $[i_0, \infty)$, for $\forall \varepsilon > 0$ there exist positive constants γ and $T = T(\gamma, \varepsilon)$ such that $\|x(i)\| \leq \gamma$ implies $\|x(j)\| \leq \varepsilon$ for $\forall j \leq i+T$, independently of i. Let us define $\Gamma = \max_k \|\Psi_{i+k,i}(A)\|$ for $k \in [1, T-1]$, and $\delta = \min(\gamma, \frac{\varepsilon}{\Gamma})$. Since $A(i)$ is bounded, Γ is finite and thus, there exists a positive constant δ. For $j \geq i+T$, we have $\|x(j)\| \leq \varepsilon$. For $i \leq j < i+T$, we have $\|x(j)\|'' \leq \varepsilon$ since $\delta \leq \frac{\varepsilon}{\Gamma}$ and $\|x(j)\| \leq \|\Psi_{j,i}(A)\|\,\|x(i)\|$. Therefore, $\|x(i)\| \leq \delta$ implies $\|x(l)\| \leq \varepsilon$ for $\forall l \geq i$, independently of i.

Now, we are ready to show the closed-loop stability of the RHC.

Theorem 3.1 Assume that the pair $(A(i), C(i))$ is uniformly observable. If $J^*(x(i+1), i+1, i+N) \geq J^*(x(i+1), i+1, i+N+1)$, then the system (1) with the RHHC (21) is uniformly asymptotically stable for $1 \leq N < \infty$.

Proof : From assumption and

$$J^*(u(i), w^*(i)) \geq J^*(u^*(i), w^*(i)) \geq J^*(u^*(i), w(i)),$$

$$\begin{aligned} J^*(x(i), i, i+N) &= x^T(i)Q(i)x(i) + u^T(i)u(i) - \gamma^2 w^T(i)w(i) + J^*(x_1(i+1), i+1, i+N) \\ &\geq x^T(i)Q(i)x(i) + u^T(i)u(i) + J^*(x_2(i+1), i+1, i+N) \\ &\geq x^T(i)Q(i)x(i) + u^T(i)u(i) + J^*(x_2(i+1), i+1, i+N+1) \end{aligned}$$

where $x_2(i+1)$ is the state at $i+1$ due to $x_2(i) = x(i)$ and $w(i) = 0$. Note that $u(i)$ is the RHHC (21). Hence,

$$J^*(x(i), i, i+N) \geq \sum_{j=i}^{i+l-1}[x^T(j)Q(j)x(j) + u^T(j)u(j)] + J^*(x(i+l), i+l, i+N+l)$$

where $w(i) = 0$ at each i. since $J^*(x(i), i, i+N)$ is nonincreasing and $J^*(x(i), i, i+N) \geq 0$, $J^*(x(i), i, i+N) \to c$ for some nonnegative constant c as $i \to \infty$. Thus, as $i \to \infty$, $u(j) \to 0$ and

$$\sum_{j=i}^{i+l-1} x^T(j)Q(j)x(j) = x^T(i)\sum_{j=i}^{i+l-1}\Psi_{j,i}^T(A)\,Q(j)\,\Psi_{j,i}(A)\,x(i) = x^T(i)\,G_o(i, i+l)\,x(i) \to 0$$

. However, since the pair $(A(i), C(i))$ is uniformly observable, there exists a positive constant α_1 satisfying

$G_o(i, i+l) \geq \alpha_1 I$ for $l \geq l_o$. This means that $x(i) \rightarrow 0$ as $i \rightarrow \infty$ independently of i_0. Therefore, the closed-loop system is uniformly attractive when $w(i)=0$ for all i. Since $J^*(x(i), i, i+N)$ is bounded, the closed-loop system is uniformly asymptotically stable from Lemma 3.1

Next, we suggest a sufficient condition for Theorem 3.1.

Corollary 3.1 Assume that the pair $(A(i), C(i))$ is uniformly observable. If $Q_f(i+N)$ satisfies (9) for all i, then, the system (1) with (21) is uniformly asymptotically stable for $1 \leq N < \infty$.

The result in [12] requires both uniform detectability of the pair $(A(i), C(i))$ and $J^*(x(i), i, i+N) > 0$ for all $x(i) \neq 0$ while this paper requires only uniform observability.

In the following, we consider an extended boundary condition $P(i+N+1, i+N+1) = P_f(i+N+1) \geq 0$ which includes the well-known terminal equality condition $P_f(i+N+1)=0$ in [2]. Let us consider another receding horizon H_∞ control:

$$u(i) = -B_2^T(i) P^{-1}(i+1, i+N+1) A(i) x(i) \tag{23}$$

where $P(i+1, i+N+1)$ is computed from (16) with $P(i+N+1, i+N+1) = P_f(i+N+1)$.

From here, we assume that the selected $P_f(i+N+1)$ is positive semidefinite and bounded, and $\widehat{Q}(i) \geq 0$ for all i throughout the rest of this paper. The following results are needed to show the stabilizing properties of (23).

Lemma 3.2 There exists a positive constant α_2 satisfying $P(i+1, i+N+1) \geq \alpha_2 I$ for $N \geq l_c$ if the pair $(A(i), B(i))$ is uniformly controllable where $B(i)$ satisfies $B(i) B^T(i) = \widehat{Q}(i)$.

Proof : It is proved by showing $\widehat{J}^*(\hat{x}(i+1), i+1, i+N+1) \geq \alpha_2 \|\hat{x}(i+1)\|^2$. Consider the adjoint system (17) from which $P(i+1, i+N+1)$ stems. Assume that $\widehat{J}^*(\hat{x}(i+1), i+1, i+N+1) = 0$ for $\hat{x}(i+1) \neq 0$. Since $\widehat{x}^T(\tau) \widehat{Q}(\tau-1) \hat{x}(\tau) = \widehat{u}^{*T}(\tau) \widehat{u}^*(\tau) = \widehat{x}^T(i+N+1) P_f(i+N+1) \hat{x}(i+N+1) = 0$ for $\tau \in [i+1, i+N]$,

$$\begin{aligned}\widehat{J}^*(\hat{x}(i+1), i+1, i+N+1) &\geq \sum_{\tau=i+1}^{i+N} \widehat{x}^T(\tau) \widehat{Q}(\tau-1) \hat{x}(\tau) \\ &= \widehat{x}^T(i+1) \sum_{i+1}^{i+N} \Psi_{\tau, i+1}(A^{-1}) B(\tau-1) B^T(\tau-1) \Psi_{\tau, i+1}(A^{-T}) \hat{x}(i+1) \\ &= \widehat{x}^T(i+1) G_c(i+1, i+N+1) \hat{x}(i+1)\end{aligned}$$

where $G_c(i+1, i+N+1)$ is a controllability Grammian. Since the pair $(A(i), B(i))$ is uniformly controllable, there exists a positive constant α_2 satisfying

$$\widehat{J}^*(\hat{x}(i+1), i+1, i+N+1) \geq \alpha_2 \|\hat{x}(i+1)\|^2 \quad \text{for} \quad N \geq l_c$$

This contradicts $\widehat{J}^*(\hat{x}(i+1), i+1, i+N+1) = 0$. Note that $P_f(i+N+1) = 0$ is included.

Lemma 3.3 If the pair $(A(i), B(i))$ is uniformly controllable, then the pair $(A(i), B_2(i))$ is uniformly controllable.

Proof : It can be proved directly from [14].

Now, we are ready to state the following result.

Theorem 3.2 Assume that the pair $(A(i), B(i))$ is uniformly controllable and $A(i)$ is nonsingular. If $P(i+1, i+N+2) \geq P(i+1, i+N+1)$, then the system (1) with (23) is uniformly asymptotically stable for $l_c + 1 \leq N < \infty$.

Proof : Consider the adjoint system of (1) with the RHHC (23) when $w(i)=0$:

$$\hat{x}(i+1) = [A(i) - B_2(i) B_2^T(i) P^{-1}(i+1, i+N+1) A(i)]^{-T} \hat{x}(i). \tag{24}$$

Then, we define the associated scalar valued function $V(\hat{x}(i), i) = \widehat{x}^T(i) A^{-1}(i) P(i+1, i+N+1) A^{-T}(i) \hat{x}(i)$. As a consequence of uniform controllability, there exists a positive constant α_3 satisfying $\|A^{-1}(i)\| \leq \alpha_3$[15]. Here, $P(i+1, i+N+1)$ is bounded since the system matrices and $P_f(i+N+1)$are bounded, and N is finite. Therefore, from Lemma 3.2, there exist positive constants α_2 and α_4 satisfying $\alpha_2 \|\hat{x}(i)\|^2 \leq V(\hat{x}(i), i) \leq \alpha_4 \|\hat{x}(i)\|^2$.

$$\begin{aligned}V(\hat{x}(i), i) - V(\hat{x}(i+1), i+1) &= \widehat{x}^T(i) A^{-1}(i) P(i+1, i+N+1) A^{-T}(i) \hat{x}(i) \\ &\quad - \widehat{x}^T(i+1) A^{-1}(i+1) P(i+2, i+N+2) A^{-T}(i+1) \hat{x}(i+1) \\ &= -\widehat{x}^T(i+1)[B_2(i) B_2^T(i) - B_2(i) B_2^T(i) P^{-1}(i+1, i+N+1) \\ &\quad B_2(i) B_2^T(i)] \hat{x}(i+1) - \widehat{x}^T(i+1)[P(i+1, i+N+2) \\ &\quad + B_r(i) B_r^T(i) - P(i+1, i+N+1) + Z(i)] \hat{x}(i+1)\end{aligned}$$

where

$$\begin{aligned}Z(i) = {} & A^{-1}(i+1) P(i+2, i+N+2) A^{-T}(i+1) C^T(i+1)[I + C(i+1) A^{-1}(i+1) \\ & P(i+2, i+N+2) A^{-T}(i+1) C^T(i+1)]^{-1} C(i+1) A^{-1}(i+1) P(i+2, i+N+2) \\ & A^{-T}(i+1).\end{aligned}$$

Since $P(i+1, i+N+2) - P(i+1, i+N+1) \geq 0$, then

$$V(\hat{x}(i), i) - V(\hat{x}(i+1), i+1) \leq -\widehat{x}^T(i+1) \{B_2(i)[I + B_2^T(i) L^{-1}(i) B_2(i)]^{-1} B_2^T(i) + B_r(i) B_r^T(i)\} \hat{x}(i+1)$$

where

$L(i) = A^{-1}(i+1) P^{\frac{1}{2}}(i+2, i+N+1) S(i) P^{\frac{1}{2}}(i+2, i+N+1) A^{-T}(i+1)$ and

$S(i) = [I + P^{\frac{1}{2}}(i+2, i+N+1)A^{-T}(i+1)Q(i+1)A^{-1}(i+1)P^{\frac{1}{2}}(i+2, i+N+1)]^{-1}$.

Note that $L(i)$ is nonsingular since $P(i+2, i+N+1)$ is positive definite for $N \geq l_c+1$ from Lemma 3.2. Thus, we can have

$$V(\hat{x}(i+1; \hat{x}(i_0), i_0), i+1) - V(\hat{x}(i_0), i_0)$$
$$\geq \hat{x}^T(i_0)[\sum_{k=i_0}^{i} \Psi_{i_0,k+1}(A - B_2H)\alpha_5 B_2(k)B_2^T(k) + B_\gamma(k)B_\gamma^T(k)]\Psi^T_{i_0,k+1}(A - B_2H)]\hat{x}(i_0)$$

where $H(k) = B_2^T(k)P^{-1}(k+1, k+1+N)A(k)$ for some positive constant α_5. Since the pair $(A(i), B_2(i))$ is uniformly controllable from Lemma 3.3 and the uniform controllability of the system (24) with (23) is invariant [15], there exists a positive constant α_6 satisfying $V(\hat{x}(i+1; (\hat{x}(i_0), i_0)), i+1) - V(\hat{x}(i_0), i_0) \geq \alpha_6 \|\hat{x}(i_0)\|$. This implies that the closed-loop system (24) is exponentially increasing, i.e., the closed-loop system (1) with (23) is exponentially decreasing when $w(i)=0$ for all i.

Next, we suggest a sufficient condition for Theorem 3.2.

Corollary 3.2 Assume that the pair $(A(i), B(i))$ is uniformly controllable and $A(i)$ is nonsingular. If $P_f(i+N+1)$ satisfies (19) or if $P_f(i+N+1)=0$ for all i, then the system (1) with (23) is uniformly asymptotically stable for $l_c+1 \leq N < \infty$.

The results in Corollary 3.1 and 3.2 are different in that the former is applied to the condition (9), while the latter to (19). Note that if $P_f(i+N+1)=0$, Corollary 3.2 holds for arbitrary $Q(i) \geq 0$ including the zero matrix which is different from Corollary 3.1. When $Q(i)=0$, $P(i+1, i+N+1)$ can be represented as:

$$P(i+1, i+N+1) = \sum_{j=i+1}^{i+N} \Psi_{i+1,j}(A)\hat{Q}(j-1)\Psi^T_{i+1,j}(A) + \Psi_{i+1,i+N+1}(A)\Psi_{i+1,i+N+1}(A)P_f(i+N+1)\Psi^T_{i+1,i+N+1}(A).$$

where $A(i)$ is nonsingular. Note that there is a positive constant α_2 satisfying $P(i+1, i+N+1) \geq \alpha_2 I$ for $l_c \leq N < \infty$ if the pair $(A(i), B(i))$ is uniformly controllable. In the above equation, $P_f(i+N+1)$ can be zero. This may be the simplest RHHC.

In the following theorem, the state weighting matrix is also weakened so as to include even the zero matrix for guaranteeing the closed-loop stability of the RHHC.

Lemma 3.4 If there exists a positive constant α_7 satisfying $Q_f(i+N) \geq \alpha_7 I$, then there exists a positive constant α_8 satisfying $J^*(x(i+1), i+1, i+N) \geq \alpha_8\|x(i+1)\|^2$ when $\Psi_{i+N,i+1}(A)$ is nonsingular and thus, $M(i+1, i+N) \geq \alpha_8 I$ for all i and $N \geq 1$.

Proof : Assume that $J^*(x(i+1), i+1, i+N)=0$ for $x(i+1) \neq 0$. Since $J^*(u^*(i+1), w^*(i+1)) \geq J(u^*(i+1), w(i+1)=0)$, $x^T(\tau)Q(\tau)x(\tau) = u^{*T}(\tau)u^*(\tau) = x^T(i+N)Q_f(i+N)x(i+N)=0$ for $\tau \in [i+1, i+N-1]$. Since $x^T(i+N)Q_f(i+N)x(i+N) = x^T(i+1)\Psi^T_{i+N,i+1}(A)Q_f(i+N)\Psi_{i+N,i+1}(A)x(i+1)$ and $\Psi_{i+N,i+1}(A)$ is nonsingular, $x(i+1)=0$. This contradicts $x(i+1) \neq 0$. Note that N can be 1.

Now, we introduce our last main result.

Theorem 3.3 Assume that the pair $(A(i), B(i))$ is uniformly controllable, and $A(i)$ is nonsingular. If $Q_f(i+N) \geq \alpha_7 I$ for a positive constant α_7 and $M(i+1, i+N) \geq M(i+1, i+N+1)$ for all i, then the system (1) with (21) is uniformly asymptotically stable for $l_c+2 \leq N < \infty$.

Proof : Let $\widehat{M^{-1}}(i+1, i+N) = M^{-1}(i+1, i+N) + \hat{Q}(i)$. Then, the control (21) can be changed into $u^*(i) = -B_2^T(i)\hat{M}(i+1, i+N)A(i)x(i)$. From Lemma 3.4, there exists a positive constant α_9 satisfying $\alpha_9 I \leq \hat{M}(i+1, i+N)$. From Lemma 3.2, there exists a positive constant α_{10} satisfying $\hat{M}(i+1, i+N) \leq \alpha_{10} I$ for $N \geq l_c+1$ since $\widehat{M^{-1}}(i+1, i+N) = P(i+1, i+N)$. Therefore, it can be proved in the same way as in Theorem 3.2 by replacing $P(i+1, i+N+1)$ with $P(i+1, i+N+1)\widehat{M^{-1}}(i+1, i+N)$.

Now, we show that the stabilizing receding horizon controllers guarantee the H_∞ norm bound of the closed-loop system.

Theorem 3.4 Under the assumptions given in Theorem 3.1, Corollary 3.1, or Theorem 3.3, the H_∞ norm bound of the closed-loop system with (21) is guaranteed.

Proof : If $[I - B_\gamma^T(i)M(k+1, i+N)B_\gamma(i)] > 0$ for $k \in [i, i+N-1]$, $M(i+1, i+N) \geq 0$ from [13]. Therefore,

$$-x^T(i_0)M(i_0, i_0+N)x(i_0) \leq \sum_{i=i_0}^{\infty}[x^T(i+1)M(i+1, i+N+1)x(i+1) - x^T(i)M(i, i+N)x(i)] \quad (25)$$

The remaining procedure is the same as those in [12].

In the same way, under the assumptions given in Theorem 3.2 or Corollary 3.2, the H_∞ norm bound of the closed-loop system with (21) is guaranteed with $M(i+1, i+N+1)$ and $M(i, i+N)$ replaced by

$[P(i+1, i+N+1) - \hat{Q}(i)]^{-1}$ and $[P(i, i+N) - \hat{Q}(i-1)]^{-1}$, respectively. The inverse matrices exist from Lemma 3.2 for $N \geq l_c+1$ since $P(i, i_f) - \hat{Q}(i-1) = [A^T(i) P^{-1}(i+1, i_f) A(i) + Q(i)]^{-1}$ from (16).

It is obvious that the whole results obtained here apply to the case of time-invariant systems. In this case, the controllability and observability can be replaced by the stabilizability as in Theorem 2 of [4] and Proposition 4.1 of [15], and by the detectability as in Remark 8 of [8], respectively. The RHHC (21) becomes $u^*(i) = -B_2^T[I+M(N-1)\hat{Q}]^{-1}M(N-1)Ax(i)$ where $M(i)$ satisfies $M(i+1) = A^T[I+M(i)\hat{Q}]^{-1}M(i)A + Q$ with the boundary condition $M(0) = Q_f$. The RHHC (23) becomes $u(i) = -B_2^T P^{-1}(N)Ax(i)$ where $P(i)$ satisfies $P(i+1) = A^{-1}[I+P(i)A^{-T}QA^{-1}]^{-1}P(i)A^{-T} + \hat{Q}$ with $P(0) = P_f$.

5. Conclusion

In this paper, sufficient conditions are presented for monotonicity of the saddle point value for receding-horizon H_∞ control. The resulting monotonicity is used to prove the stability of the closed-loop. Under these sufficient conditions, many previous results for the stability of the RHC are obtained as special cases of the results in this paper.

The whole procedure is much simpler than the previous results, and thus is expected to be easily extended for constrained, delayed, and/or nonlinear systems with the RHHC.

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