Input Constrained Receding Horizon H_∞ Control : Quadratic Programming Approach

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Abstract: A receding horizon H_∞ predictive control method is derived by solving a min-max problem in non-recursive forms. The min-max cost index is converted to a quadratic form which, for systems with input saturation, can be minimized using QP. Through the use of closed-loop prediction, the prediction of states in the presence of disturbances are made non-conservative and it become possible to get a tighter H_∞ norm bound. Stability conditions and H_∞ norm bounds on disturbance rejection are obtained in infinite horizon sence. Polyhedral types of feasible sets for states and disturbances are adopted to deal with the input constraints. The weight selection procedures are given in terms of LMIs and the algorithm is formulated so that it can be solved via QP. This work is a modified version of an earlier work which was based on ellipsoidal type feasible sets[15].

1. Introduction

Model based predictive control (MPC) can handle saturation on manipulated variables in a systematic way. In the presence of input saturation, the stability of MPC can be proved using Lyapunov arguments on the monotonic behaviour of the cost, provided that there exist feasible inputs which bring the current state e.g. [1-4] or the unstable system modes [5] to zero in finite time. The existence of feasible inputs is a key issue because equality constraints can be overly stringent. To overcome this weakness, several methods have been proposed [6-9], which do not deploy end point equality constraints. In [7], a RHC with finite terminal weights (RHC-FW) removes the terminal constraints of [1] and a procedure for obtaining stabilizing terminal weights is given in linear matrix inequality (LMI) form. The feasibility of RHC-FW requires the existence of inputs, which bring in finite time the current state into a positively invariant ellipsoidal set [7-9].

Here a constrained receding horizon H_∞ predictive control (RHHC) is derived which not only guarantees stability but also provides an induced l_2 norm bound from disturbance to state. This norm bounding property will be referred to as the 'disturbance boundedness property' and the induced l_2 norm as 'disturbance H_∞ norm'. We adopt the RHC-FW framework, which uses a finite output and finite control horizon with finite terminal weights. The preference of RHC-FW is due to the fact that the requirement for the stabilizing constant feedback law to be feasible is not imposed at current time [9], but rather is postponed to the end of the output horizon; this allows for more degrees of freedom with which to enlarge the set of current feasible states. Use will be made of a min-max formulation, which is known to be an effective way of synthesizing robust controllers using H_∞ concepts [10-14]. However the concern here is to extend this approach to the case of input saturation and to derive conditions which guarantee the existence of stabilizing solutions. Furthermore, a closed-loop prediction strategy will be deployed to reduce the effect of disturbances in state predictions. First the worst case disturbances are computed as a function of current state and future control inputs and then they are substituted in the min-max formulation, to yield a quadratic cost which can be minimized using QP.

The existence of stabilizing finite terminal weights for RHHC is considered using Lyapunov theory. First, it is shown that for controllable plants in the absence of input limits, there exist terminal weights which achieve closed-loop stability while keeping the effect of disturbances within prescribed bounds for any stabilizing feedback gain F. Through a simple example, it is shown that closed-loop predictions can yield guaranteed H_∞-norm bounds smaller than those of Barsar [13] and Petersen [15], which are obtained from Algebraic Riccati Equations. In the case of input saturation, the feedback gain F must satisfy additional conditions determined by the existence of feasible inputs, which steer the current state into an feasible and invariant set in a finite number of steps. Unlike the earlier work [15], polyhedral types of feasible sets are considered and the resulting RHHC algorithm are given in the form of Quadratic Programming.

2. Problem Formulation

Consider a linear time invariant system described by

$$\boldsymbol{x}_{k+1} = A\,\boldsymbol{x}_k + B\,\boldsymbol{u}_k + D\,\boldsymbol{\omega}_k \tag{1}$$

and a constant state feedback gain F,

where $\boldsymbol{x}_k \in R^{n}$, $\boldsymbol{u}_k \in R^{m}$, $\boldsymbol{\omega}_k \in R^{q}$,

$$-(\boldsymbol{u}_{\lim})_i \le (\boldsymbol{u}_k)_i \le (\boldsymbol{u}_{\lim})_i \quad \text{for } i = 1, 2, \cdots, m, \tag{2}$$

$\boldsymbol{u}_k$ and $\boldsymbol{y}_k$ are the system input and output at time k, and $\boldsymbol{\omega}_k$ is a disturbance on the system and $(\boldsymbol{u}_k)_j$ is the j^{th} low element of u_k. We shall use the notation $\boldsymbol{x} = \{(x)_i\}$ for vectors and $A = \{(a)_{ij}\}$ for matrices. The gain F can be considered to be an LQ optimal feedback gain. Our aim here is to establish a strategy of finding perturbations, c_k, on the state feedback control $F\,\boldsymbol{x}_k$ *i.e.*

$$\boldsymbol{u}_k = F\,\boldsymbol{x}_k + \boldsymbol{c}_k \tag{3}$$

so that the H_∞-norm of the transfer function $T_{x\omega}$ from ω to x_k is bounded by a prescribed value γ; so that

$$\frac{\sum_{j=1}^{\infty} \|\boldsymbol{x}_{k+j}\|}{\sum_{j=1}^{\infty} \|\boldsymbol{\omega}_{k+j}\|} \le \|T_{x\omega}\|_\infty \le \gamma^2 \tag{4}$$

The system (1) with the control (3) can be rewritten as:

$$\boldsymbol{x}_{k+1} = A_c\,\boldsymbol{x}_k + B\,\boldsymbol{c}_k + D\,\boldsymbol{\omega}_k, \quad A_c = A + BF \tag{5}$$

with constraints

$$|(F\,\boldsymbol{x}_k + \boldsymbol{c}_k)_i| \le (\boldsymbol{u}_{\lim})_i \quad \text{for } i = 1, 2, \cdots, m. \tag{6}$$

By applying the well known link between the H_∞ optimal control and the linear quadratic games theory[10][13], we introduce the following cost index with weights $Q>0, R>0$ and $\Psi>0$ of conformal dimensions:

$$J_k(\bar{\boldsymbol{c}}_k, \bar{\boldsymbol{\omega}}_k) = J_k^{RHC}(\bar{\boldsymbol{c}}_k, \bar{\boldsymbol{\omega}}_k) - \gamma^2 \sum_{i=0}^{N-1} \|\omega_{k+i|k}\| \tag{7}$$

$$J_k^{RHC}(\bar{\boldsymbol{c}}_k, \bar{\boldsymbol{\omega}}_k) = \sum_{i=1}^{N-1} \|\boldsymbol{x}_{k+i|k}\|_Q + \sum_{i=0}^{N-1} \|\boldsymbol{c}_{k+i|k}\|_R + \|\boldsymbol{x}_{k+N|k}\|_\Psi. \tag{8}$$

$\|\boldsymbol{x}\|_Q = \boldsymbol{x}' Q \boldsymbol{x}$, $\bar{\boldsymbol{c}}_k = [\,\boldsymbol{c}_{k|k}'\ \boldsymbol{c}_{k+1|k}' \cdots \boldsymbol{c}_{k+N-1|k}'\,]'$,

$\bar{\boldsymbol{\omega}}_k = [\,\boldsymbol{\omega}_{k|k}'\ \boldsymbol{\omega}_{k+1|k}' \cdots \boldsymbol{\omega}_{k+N-1|k}'\,]'$, and $\boldsymbol{x}_{k+i|k}$ is the predicted value of $\boldsymbol{x}_{k+i}$ calculated on the basis of available data at time k, we define the discrete game as the following problem:

$$\min_{\bar{\boldsymbol{c}}_k} \max_{\bar{\boldsymbol{\omega}}_k} J_k(\bar{\boldsymbol{c}}_k, \bar{\omega}_k) \tag{9}$$

We will compute the optimal min-max pair $\bar{\boldsymbol{c}}_k^*$, $\bar{\boldsymbol{\omega}}_k^*$ of the optimization problem in the case of input saturation and assume that only the first element of $\bar{c}_k^*$ is applied at time k; at the next time $k+1$, $\bar{c}_{k+1}^*$ will be obtained for the receded future horizon and the same procedure will be repeated thereafter. Stability properties and disturbance H_∞ norm bounds will be considered. Furthermore, for the case of input saturation, the region of attraction of the proposed controller will be examined in conjunction with allowable disturbance bounds.

3. Receding Horizon H_∞ Predictive Control

At the present time k, the future states $\boldsymbol{x}_{k+j}$, $j = 1, 2, \cdots, N$ can be predicted as follows:

$$\bar{\boldsymbol{x}}_k = \bar{A}\,\boldsymbol{x}_k + \bar{B}\,\bar{\boldsymbol{c}}_k + \bar{D}\,\bar{\boldsymbol{\omega}}_k \tag{10}$$

$$\bar{\boldsymbol{x}}_k = [\,\boldsymbol{x}_{k+1|k}'\ \boldsymbol{x}_{k+2|k}' \cdots \boldsymbol{x}_{k+N|k}'\,]' \quad \bar{A} = [\,A_c'\ A_c^{2\prime} \cdots A_c^{N\prime}\,]'$$

$$\bar{B} = \begin{bmatrix} B & 0 & \cdots & 0 \\ A_cB & B & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ A_c^{N-1}B & A_c^{N-2}B & \cdots & B \end{bmatrix} \quad \bar{D} = \begin{bmatrix} D & 0 & \cdots & 0 \\ A_cD & D & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ A_c^{N-1}D & A_c^{N-2}D & \cdots & D \end{bmatrix}.$$

Using Equation (10), the cost (7) can be rewritten as:

$$J_k(\bar{\boldsymbol{u}}_k, \bar{\boldsymbol{\omega}}_k) = \|\bar{A}\,\boldsymbol{x}_k + \bar{B}\,\bar{\boldsymbol{c}}_k + \bar{D}\,\bar{\boldsymbol{\omega}}_k\|_{\bar{Q}} + \|\bar{\boldsymbol{c}}_k\|_{\bar{R}} - \gamma^2 \|\bar{\omega}_k\| \tag{11}$$

where

$\bar{Q} = diag(Q, \cdots, Q, \Psi)$, $\bar{R} = diag(R, \cdots, R)$ and $diag(R, \cdots, R)$ is a block diagonal matrix with $R, \cdots, R$ as diagonal block matrices. The maximizing disturbance $\bar{\omega}_k^*(\bar{\boldsymbol{c}}_k)$ is obtained by $\frac{\partial J_k}{\partial \bar{\boldsymbol{\omega}}_k} = 0$ as:

$$\bar{\boldsymbol{\omega}}_k^*(\bar{\boldsymbol{c}}_k) = (\gamma^2 I - \bar{D}'\bar{Q}\bar{D})^{-1}\bar{D}'\bar{Q}(\bar{A}\,\boldsymbol{x}_k + \bar{B}\,\bar{\boldsymbol{c}}_k) \tag{12}$$

$$\frac{\partial^2 J}{\partial \bar{\boldsymbol{\omega}}_k^2} = 2(\bar{D}'\bar{Q}\bar{D} - \gamma^2 I) < 0, \tag{13}$$

where $M<0$ denotes that M is negative definite. The cost of (11), for the $\bar{\boldsymbol{\omega}}_k^* = [\,\omega^*_{k|k}(\bar{\boldsymbol{c}}_k)'\,\omega^*_{k+1|k}(\bar{\boldsymbol{c}}_k)' \cdots$

$\omega^*_{k+N-1|k}(\bar{c}_k)']'$ of (12), can be written as:

$$J_k(\bar{c}_k, \bar{\omega}_k^*(\bar{c}_k)) = \|\bar{A}x_k + \bar{B}\bar{c}_k\|_{\bar{Q}_{off}} + \|\bar{c}_k\|_{\bar{R}} \tag{14}$$

where $\bar{Q}_{off} = (\bar{Q} + \overline{QD}\Omega^{-1}\bar{D}'\bar{Q})$, $\Omega = \gamma^2 I - \bar{D}'\overline{QD}$, and (9) is converted to:

$$\bar{c}_k^* = \arg\min_{\bar{c}_k} J_k(\bar{c}_k, \bar{\omega}_k^*(\bar{c}_k)) \tag{15}$$

$$-u_{\lim} \le F x_{k+i|k}(\bar{\omega}_k^*) + c_{k+i|k} \le u_{\lim}$$

for $i = 0, \cdots, N-1$, where $x_{k+i|k}(\bar{\omega}_k)$ is the state at $k+i$ when disturbances $\bar{\omega}_k^*(\bar{c}_k)$ are applied to the system for a given x_k and perturbations $c_k, c_{k+1}, \cdots, c_{k+i-1}$. Note that the cost index (14) is quadratic in $\bar{c}_k$ and the minimization problem of (15) can be solved via well-known QP methods.

Remark 1 : Given that the cost of (14) is quadratic in x_k and $\bar{c}_k$, it is obvious that for zero initial condition, $x_k = 0$, the optimal cost index $J_k(\bar{c}_k^*, \bar{\omega}_k^*(\bar{c}_k)) = 0$ with $\bar{c}_k^* = 0$, since by (13) Ω and Q_{off} are positive definite.

If there is no saturation in the inputs, the minimizing control sequence $\bar{c}_k^* = [c_{k|k}^{*\prime}\ c_{k+1|k}^{*\prime} \cdots c_{k+N-1|k}^{*\prime}]'$ can be obtained by setting to zero the derivative of $J_k(\bar{c}_k, \bar{\omega}_k^*(\bar{c}_k))$ with respect to $\bar{c}_k$:

$$\bar{c}_k^* = \{\bar{R} + \bar{B}'(\bar{Q} + \overline{QD}\Omega^{-1}\bar{D}'\bar{Q})\bar{B}\}^{-1}\bar{B}'(\bar{Q} + \overline{QD}\Omega^{-1}\bar{D}'\bar{Q})\bar{A}x_k \tag{16}$$

Substituting $\bar{c}_k^*$, obtained either by (16) or for the case of input saturation by solving the QP problem of (15), into (12), we have $\bar{\omega}_k^* = \omega_k^*(\bar{c}_k^*)$. For this min-max pair $\bar{c}_k^*$, $\bar{\omega}_k^*$ and $\bar{c}_k$ satisfying (15), the following inequality holds:

$$J_k(\bar{c}_k^*, \bar{\omega}_k) \le J_k(\bar{c}_k^*, \bar{\omega}_k^*) \le J_k(\bar{c}_k, \bar{\omega}_k^*(\bar{c}_k)),\ \forall\ \bar{\omega}_k \tag{17}$$

In the next section, consideration is given to the stability and disturbance H_∞ norm boundedness of RHHC, which applies $\bar{c}_k^*$ in a receding horizon manner.

4. Stability and H_∞ norm bound without Input Saturation

First, the stability of RHHC without input saturation is examined. The proof of stability is based on (17) which applies regardless of input saturation.

Theorem 1: Consider system (5) with $|(u_{\lim})_i| = \infty$ for $i = 1, 2, \cdots, m$. If the positive definite weights Ψ, Q, R and γ satisfy the following inequalities for some matrix $K(\in R^{m\times n})$ and scalar $\rho(>1)$:

$$\gamma^2 I - \bar{D}'\overline{QD} > 0 \tag{18}$$

$$\Psi \ge (1 + (\rho-1)^{-1})(A_c + BK)'\Psi(A_c + BK) + Q + K'RK \tag{19}$$

$$\gamma^2 I - \rho D'\Psi D \ge 0 \tag{20}$$

then the receding horizon control law implied by (16) guarantees closed loop stability, where $\Psi > 0 (\ge 0)$ denotes that Ψ is positive (semi) definite.

Proof : The stability of linear systems is not affected by the value of disturbances.
Therefore, for simplicity, the disturbances which are actually fed to the system will be taken to be zero. Stability can be established through the monotonic decreasing property of the cost index. Define

$$\hat{c}_{k+1} = [c_{k+1|k}^{*\prime}\ c_{k+2|k}^{*\prime} \cdots c_{k+N-1|k}^{*\prime}\ K x_{k+N|k}']'$$

$$\hat{\omega}_k(\hat{c}_{k+1}) = [0\ \omega^*_{k+1|k+1}(\hat{c}_{k+1})'\ \omega^*_{k+2|k+1}(\hat{c}_{k+1})' \cdots$$
$$\cdots \omega^*_{k+N-1|k+1}(\hat{c}_{k+1})']'.$$

Then, from (17) and following the similar procedure of [12], we have:

$$J_k(\bar{c}_k^*, \bar{\omega}_k^*(\bar{c}_k^*)) \ge \|x_{k+1}\|_Q + \|c_{k|k}^*\|_R + J_{k+1}(\bar{c}_{k+1}^*, \bar{\omega}_{k+1}^*(\bar{c}_{k+1}^*)) + M_k \tag{21}$$

where

$$M_k = \|x_{k+N|k}(\bar{c}_k^*, \hat{\omega}_k(\hat{c}_{k+1}))\|_\Psi - \|x_{k+N|k}(\bar{c}_k^*, \hat{\omega}_k(\hat{c}_{k+1}))\|_Q$$
$$-\|K x_{k+N|k}(\bar{c}_k^*, \hat{\omega}_k(\hat{c}_{k+1}))\|_R + \gamma^2\|\omega^*_{k+N|k+1}(\hat{c}_{k+1})\|$$
$$-\|x_{k+N+1|k+1}(\hat{c}_{k+1}, \bar{\omega}_{k+1}^*(\hat{c}_{k+1}))\|_\Psi$$

and $x_{k+i|k}(\bar{c}_k^*, \hat{\omega}_k(\hat{c}_{k+1}))$ is the prediction of x_{k+i} based on x_k and the vectors of control and disturbance sequence $\bar{c}_k^*$ and $\hat{\omega}_k(\hat{c}_{k+1})$ for $[k, k+i-1]$. Relation (21) establishes that the optimal cost index decreases provided $M_k \ge 0 (k > 0)$.

The fact that

$$x_{k+N+1|k+1}(\hat{c}_{k+1}, \bar{\omega}_{k+1}^*(\hat{c}_{k+1}))$$
$$= (A_c + BK)x_{k+N|k}(\bar{c}_k^*, \hat{\omega}_k(\hat{c}_{k+1})) + D\omega^*_{k+N|k+1}(\hat{c}_{k+1}), \tag{22}$$

and $\|a+b\|_\Psi = \|a\|_{(1+(\rho-1)^{-1})\Psi} + \|b\|_{\rho\Psi} - \|a-(\rho-1)b\|_{(\rho-1)^{-1}\Psi}$, imply:

$$M_k \ge \|x_{k+N|k}(\bar{c}_k^*, \hat{\omega}_k(\hat{c}_{k+1}))\|_{\Psi-(1+(\rho-1)^{-1})(A_c+BK)'\Psi(A_c+BK)-Q-K'RK}$$
$$+\|\omega^*_{k+N|k+1}\|_{\gamma^2 I - \rho D'\Psi D}.$$

Under conditions of (19) and (20), $M_{k+i} \ge 0$ for $i = 0, 1, 2, \cdots$, and thus:

$$J_k(\bar{c}_k^*, \bar{\omega}_k^*(\bar{c}_k^*)) \ge \sum_{i=1}^{L}\|x_{k+i}\|_Q + \|c_{k+i-1|k+i-1}^*\|_R + J_{k+L}(\bar{c}_{k+L}^*, \bar{\omega}_{k+L}^*(\bar{c}_{k+L}^*)). \tag{23}$$

Since $J_{k+L}(\bar{c}_{k+L}^*, \bar{\omega}_{k+L}^*(\bar{c}_{k+L}^*)) \ge 0$ for all $L \ge 0$, as indicated by (14), and $\|x\|_Q \ge 0$, $\|c\|_R \ge 0$, inequality (23) implies that $\|x\|_Q$ and $\|c\|_R$ converge to zero as time increase. This establishes asymptotic closed-loop stability.

In the next theorem the H_∞ norm boundedness of RHHC without input saturation is established using (16) in the receding horizon manner.

Theorem 2 : When the control (16) is applied to the system (5) in the receding horizon manner and the conditions of Theorem 1 are met, the induced disturbance H_∞ norm, for zero initial conditions, is bounded by γ, *i.e.*:

$$\frac{\sum_{i=1}^{\infty}\| x_{k+i}\|_Q}{\sum_{i=0}^{\infty}\| \omega_{k+i}\|} \le \gamma^2. \qquad (24)$$

Proof : Follow similar arguments to those used for Theorem1, use the vectors hat $\bar{c}_{k+1}$ of Theorem 1, but define $\hat{\omega}_k$ as:

$$\hat{\omega}_k = [\ \omega_k{}'\ \omega^*_{k+1|k+1}{}'\ \omega^*_{k+2|k+1}{}' \cdots \omega^*_{k+N-1|k+1}{}'\ ]'.$$

where ω_k represents the disturbance which is actually applied to the system at time k. Then, from (17), we have:

$$\begin{aligned} J_k(\bar{c}_k^*, \bar{\omega}_k^*(\bar{c}_k^*)) &\ge J_k(\bar{c}_k^*, \hat{\omega}_k(\hat{c}_{k+1})) \\ &\ge \| x_{k+1}\|_Q + \| c_{k|k}^*\|_R - \gamma^2\|\omega_k\| \qquad (25) \\ &\quad + J_{k+1}(\bar{c}_{k+1}^*, \bar{\omega}_{k+1}^*(\bar{c}_{k+1}^*)) + M_k \end{aligned}$$

where the definition of M_k is the same as that of Theorem 1. Under the conditions of Theorem 1, the terms $M_{k+i} \ge 0$ for $i \ge 0$ and hence, as per the inequality (23), which is reinforced when the terms ω_k are omitted, we have:

$$\begin{aligned} J_k(\bar{c}_k^*, \bar{\omega}_k^*(\bar{c}_k^*)) &\ge \sum_{i=1}^{L}\| x_{k+i}\|_Q + \| c_{k+i-1|k+i-1}^*\|_R - \gamma^2\|\omega_{k+i-1}\| \qquad (26) \\ &\quad + J_{k+L}(\bar{c}_{k+L}^*, \bar{\omega}_{k+L}^*(\bar{c}_{k+L}^*)). \end{aligned}$$

The interest here is only in state response to ω, hence x_k is taken to be zero to remove the effect of the initial state on x_{k+i}, $i \ge 1$. By (14), $J_k(\bar{c}_k^*, \bar{\omega}_k^*(\bar{c}_k^*)) = 0$ for $x_k = 0$ and $J_{k+L}(\bar{c}_{k+L}^*, \bar{\omega}_{k+L}^*(\bar{c}_{k+L}^*)) \ge 0$for all $L \ge 0$ implying:

$$\gamma^2 \sum_{i=0}^{L-1}\|\omega_{k+i}\| \ge \sum_{i=1}^{L}(\| x_{k+i}\|_Q + \| c^*_{k+i-1|k+i-1}\|_R). \qquad (27)$$

As L goes to infinity the above inequality yields condition (24).

The theorems above give sufficient conditions for closed loop stability and disturbance H_∞ norm boundedness. The determination of Ψ, γ and K which satisfy conditions (18-20) for a given set of $\rho, R,$ and Q can be accomplished using an LMI (Linear Matrix Inequality) approach, to be described later.

Remark 2 : Let $\mu = (1 + (\rho - 1)^{-1})^{1/2}, A_m = \mu A_c$ and $B_m = \mu B$, then (19) becomes:

$$\Psi \ge (A_m + B_m K)' \Psi (A_m + B_m K) + Q + K' R K. \qquad (28)$$

From classical Lyapunov stability theory therefore we have that there exist $\Psi > 0$which satisfy (28) for any stable $A_m + B_m K$. Thus there will always exist $\Psi > 0, K, \mu > 1$ satisfying relation (19) for any $\rho > 1$ provided that $\{A_c, B\}$ is completely controllable. Note that $\{A_c, B\}$ is completely controllable if $\{A, B\}$ is completely controllable.

Remark 2 suggests that, for any finite scalar $\rho > 1$, we can always find terminal weights Ψ that satisfy (19) provided that (A, B) is controllable. In the case of no input saturation, where feasibility is not an issue, the strategy should be to select Ψ which satisfy (18-20) for the smallest possible value of γ.

In the next section, consideration is given to the stability and disturbance H_∞ norm bound of RHHC with input saturation.

5. Stability and H_∞ norm bound with Input Saturation

The ideas used in Theorem 2 can also be applied to the case of input saturation, except that now the input constraint condition (15b) should be taken into account. Here we will follow the approach used in [16], *i.e.* first define a polyhedral set of states, $\mathcal{R}_F^W$, which is robustly invariant with respect to a state feedback law $u = Fx$ in the presence of bounded disturbances and then compute perturbations $\bar{c}_k$ which minimizes $J(\bar{c}_k, \bar{\omega}_k^*(\bar{c}_k))$ while guaranteeing that $F x_{k+i}(\bar{\omega}_k) + c_{k+i} (i = 0, 1, \cdots, N-1)$ are feasible and $x_{k+N|k}(\bar{c}_k, \bar{\omega}_k) \in \mathcal{R}_F^W$ dispite disturbances. Note that, because of the invariance property of $\mathcal{R}_F^W$ with respect to the feedback gain F, $\bar{c}_k^*$ satisfying the above mentioned feasibility and membership conditions guarantees the feasibility of $\hat{c}_{k+1}$ (of the proof of theorem 2) with $K = 0$ provided that ω_k is bounded properly.

We assume that the disturbance is bounded as:

$$|\,\omega_k| \le \omega_{\lim}, \tag{29}$$

where $|\omega| = \{|(\omega)_i|\}$ and the inequalities between two vectors apply on an element-by-element basis. In order to define a feasible and and invariant set, we consdier a state transformation matrix W , which is a design parameter to make the sets non-empty. Using the state transformation $z = Wx$, the closed-loop state equation (5) can be transformed into:

$$z_{k+1} = A_c^W z_k + WB\, c_k + WD\, \omega_k. \tag{30}$$

where $A_c^W = WA_cW^{-1}$. Based on the relation (30), feasible and invariant set, $\mathcal{R}_F^W$, can be defined as per the following lemma.

Lemma 1. Consider the system (1) with transformed state $z = Wx$ and a stabilizing state feedback gain F. A set of states defined as:

$$\mathcal{R}_F^W(\alpha) = \{x \mid |z| \le \alpha\} \tag{31}$$

is feasible and invariant with respect to F despite the bounded disturbance *i.e.* for any $x \in \mathcal{R}_F^W(\alpha)$ and ω satisfying (29), $u = Fx$ satisfies the input constraint (2) and makes the state remain in the set if and only if:

$$|A_c^W|\,\alpha + |WD|\,\omega_{\lim} \le \alpha \tag{32}$$

$$|FW^{-1}|\,\alpha \le \omega_{\lim} \tag{33}$$

are met.

Proof : Assume that $x_k \in \mathcal{R}_F^W(\alpha)$, from (30) and (32), use of $u_k = F x_k$ yields:

$$\begin{aligned} |z_{k+1}| &= |A_c^W z_k + WD\,\omega_k| \\ &\le |A_c^W z_k| + |WD\,\omega_k| \\ &\le |A_c^W|\,\alpha + |WD|\,\omega_{\lim} \\ &\le \alpha \end{aligned} \tag{34}$$

and from (33) we have:

$$|F x_k| = |FW^{-1} z_k| \le |FW^{-1}|\,\alpha \le \omega_{\lim}. \tag{35}$$

Relation (34) implies $x_{k+1} \in \mathcal{R}_F^W(\alpha)$ and (35) guarantees that $u = Fx$ satisfies the input constraint (2) for any x in the set. This proves the sufficiency part of the theorem. The observation that we can choose z_k and ω_k among the states of $\mathcal{R}_F^W(\alpha)$ and bounded disturbances, respectively, so that the relations (34-35) hold with equality (elementwise) for any given A_c^W, F and WD proves that conditions (33-34) are necessary for the invariance and feasibility of the set.

For states inside the set $\mathcal{R}_F^W(\alpha)$, it would be possible to find perturbations which minimize the worst case performance index (14) while making the state remain in the set. When a state is outside of the set, one possible strategy is to steer the state into the set in finite time steps. Predictions of future states can be made as follows based on (30):

$$z_{k+i|k} = (A_C^W)^i z_k + \sum_{j=1}^{i} (A_c^W)^{i-j} WB\, c_{k+j-1} + \sum_{j=1}^{i} (A_c^W)^{i-j} WD\, \omega_{k+j-1}. \tag{36}$$

It is possible to compute elementwise maximum/ minimum values of $z_{k+i|k}$ based on (29) and (36). Following lemma summarizes conditions under which x_k is steered into $\mathcal{R}_F^W(\alpha)$, *i.e.* $x_{k+M|k} \in \mathcal{R}_F^W(\alpha)$ using feasible perturbations $c_{k|k}, c_{k+1|k}, \cdots, c_{k+N-1|k}$.

Lemma 2. Consider the system (1) with transformed state $z = Wx$ and a set $\mathcal{R}_F^W(\alpha)$ defined as (31) with respect to a state feedback gain F. A state x_k is guaranteed to be steered into $\mathcal{R}_F^W(\alpha)$ in N control steps, *i.e.* $x_{k+M|k} \in \mathcal{R}_F^W(\alpha)$ dispite bounded disturbances (29) if:

$$|\,F^{W+} z^{\max}_{k+i-1|k} - F^{W-} z^{\min}_{k+i-1|k} + c_{k+i-1|k}| \le \omega_{\lim} \tag{37}$$

$$|\,F^{W+} z^{\min}_{k+i-1|k} - F^{W-} z^{\max}_{k+i-1|k} + c_{k+i-1|k}| \le \omega_{\lim}, \tag{38}$$

$$|z^{\max}_{k+M|k}| \le \alpha, \quad |z^{\min}_{k+M|k}| \le \alpha \tag{39}$$

are met for

$$z^{\max}_{k+i|k} = (A_C^W)^i z_k + \sum_{j=1}^{i} (A_c^W)^{i-j} WB\, c_{k+j-1|k} + \sum_{j=1}^{i} |(A_c^W)^{i-j} WD|\, \omega_{\lim} \tag{40}$$

$$z^{\min}_{k+i|k} = (A_C^W)^i z_k + \sum_{j=1}^{i} (A_c^W)^{i-j} WB\, c_{k+j-1|k} - \sum_{j=1}^{i} |(A_c^W)^{i-j} WD|\, \omega_{\lim} \tag{41}$$

and $i = 1, 2, \cdots, N$ with $z^{\max}_{k|k} = z^{\min}_{k|k} = z_k$, where $F^W = FW^{-1}$, $M^+ = \max(M, 0)$, and $M^- = \max(-M, 0)$.(the choice of maximum value is done elementwise)

Proof : The proof is based on the fact that the min/max value of Mz for $z_{\min} \le z \le z_{\max}$ can be represented as:

$$\max_{z_{\min} \le z \le z_{\max}} Mz = M^+ z_{\max} - M^- z_{\min}$$

$$\min_{z_{\min} \le z \le z_{\max}} Mz = M^+ z_{\min} - M^- z_{\max}.$$

Applying the above facts to (36), we obtain $z^{\max}_{\cdot|k}$, $z^{\min}_{\cdot|k}$ in (40)(41), which are maximum and minimum possible values of $z_{\cdot|k}$, respectively, subject to disturbances satisfying (29). For the given bounds $z^{\max}_{\cdot|k}$, $z^{\min}_{\cdot|k}$ on $z_{\cdot|k}$, bounds on $F^W z_{\cdot|k} + c_{\cdot|k}$ also can be obtained in the same way to yield the feasibility condition (37-38). It is easy to see that (39) ensures $x_{k+M|k} \in \mathcal{R}_F^W(\alpha)$, since

$z^{\max}_{k+N|k}$, $z^{\min}_{k+N|k}$ are maximum and minimum possible values of the transformed terminal state.

Note that relations (32-33) and (37-39) are linear inequalities with respect to α, $c_{\cdot|k}$, and $z^{\max}_{\cdot|k}$, $z^{\min}_{\cdot|k}$. Now, we are ready to summarize the constrained receding horizon algorithm as follows:

Algorithm RHHC

Step1. (off-line) Make choice of γ, Ψ, F, ρ and W so that conditions (18-20) are satisfied and there exist α satisfying the invariance condition (32).

Step2. At time instant k and for the measured state x_k, solve the QP problem of minimizing cost index (14) subject to constraints (32-33) and (37-39) with variables α, $c_{\cdot|k}$, and $z^{\max}_{\cdot|k}$, $z^{\min}_{\cdot|k}$.

Step3. Apply $u_k = F x_k + c_{k|k}$ to the system.

Step4. At the next time $k+1$, repeate Steps 2 and 3.

Allowing α searched on-line enables us to use the union of $\mathcal{R}^W_F(\alpha)$ as our target set *i.e.* Step2 of Algorithm RHHC ensures:

$$x_{k+N|k} \in \mathcal{R}^W_F = \bigcup_{\alpha \in S_\alpha} \mathcal{R}^W_F(\alpha), \tag{42}$$

where S_α denotes the set of α satisfying (32-39). The stability of Algorithm RHHC
can be summarized as per the following theorem.

Theorem 3: Algorithm RHHC is guaranteed to be feasible and keeps the state bounded while the truncated induced H_∞ norm is bounded by γ:

$$\frac{\sum_{i=1}^{L} \| x_{k+i} \|_Q}{\sum_{i=0}^{L-1} \| \omega_{k+i} \|} \le \gamma^2 \tag{43}$$

provided that

(i) ω_{k+i} is bounded as (29) for all $i \ge 0$.

(ii) F, γ, ρ and Ψ satisfy (18-20) with $K=0$.

(iii) an initial feasible solutions are obtained in Step2.

Proof : If feasible perturbations $\bar{c}_k$ are obtained at time k, then the existence of feasible perturbations $\bar{c}_{k+1}$ at time $k+1$ is guaranteed, since perturbations $\hat{c}_{k+1} = [\, c_{k+1|k}', c_{k+2|k}, \cdots, c_{k+N-1|k}, 0]$ will provide one feasible set of perturbations at time k+1. This argument can be applied recursively to yield the guaranteed feasibility based on the initial feasibility. The procedure of Theorem 2 with $K=0$ gives:

$$J_k(\bar{c}_k^*, \bar{\omega}_k^*(\bar{c}_k^*)) \ge \sum_{i=1}^{L}(\|x_{k+i}\|_Q + \| c^*_{k+i-1|k+i-1}\|_R) - \gamma^2 \sum_{i=0}^{L-1} \| \omega_{k+i}\|. \tag{44}$$

Since we are interested in the induced l_2 norm from disturbance to state, we take x_k to be zero to remove the effect of initial state on x_{k+i}, $i \ge 1$. By (14), $J_k(\bar{c}_k^*, \bar{\omega}_k^*(\bar{c}_k^*)) = 0$ for zero x_k. Thus, we obtain the condition (43).

Now for the boundedness of the state consider a set of states $\mathcal{R}^W_F(N)$ such that for any state $x \in \mathcal{R}^W_F(N)$, there exist a bounded input sequence $[\, u_0(x), u_1(x), \cdots, u_{N-1}(x)]$ which steers x into $\mathcal{R}^W_F$. It is easy to see that the set $\mathcal{R}^W_F(N)$ is bounded because any state of it can be steered into a bounded set of states $\mathcal{R}^W_F$ in a finite number of steps using bounded inputs. Due to the guaranteed feasibility of Algorithm RHHC, it is obvious that the actual state of the system must lie in $\mathcal{R}^W_F(N)$ and therefore will be bounded.

In the case when the disturbance is an l_2 signal, we can establish additional asymptotic stability result.

Corollary 3.1: If the disturbance ω_k is bounded as (29) for every $k \ge 0$ and defines a sequence in l_2, then the closed loop system is asymptotically stable with guaranteed disturbance H_∞ norm bound *i.e.*

$$\frac{\sum_{i=1}^{\infty} \| x_{k+i} \|_Q}{\sum_{i=0}^{\infty} \| \omega_{k+i} \|} \le \gamma^2. \tag{45}$$

under the conditions of Theorem 3.

Proof : The procedure of Theorem 2 and the guarantee of feasibility, gives:

$$\sum_{i=1}^{\infty}(\| x_{k+i}\|_Q + \| c^*_{k+i-1|k+i-1}\|_R)$$
$$\le \gamma^2 \sum_{i=1}^{\infty} \| \omega_{k+i-1}\| - J_k(\bar{c}_k^*, \bar{\omega}_k^*(\bar{c}_k^*)) < \infty.$$

This suggests that x_{k+i} goes to zero asymptotically for $Q>0$, since its sum is bounded from above. Similarly c_{k+i} goes to zero asymptotically for $R>0$ and this proves asymptotic stability. For $x_k = 0$, $J_k(\bar{c}_k^*, \bar{\omega}_k^*(\bar{c}_k^*)) = 0$ and this in turn implies the disturbance H_∞ norm bound of (45).

Unlike Theorems 1 and 2 which consider the saturation free case, the result above takes into account the input limits and therefore requires that $x_{k+N|k} \in \mathcal{R}^W_F$, $|\omega_{k+i}| \le \omega_{\lim}$ for $i \ge 0$ as well as that the matrices F, Ψ should satisfy the relations (19-20) with $K=0$ for some $\rho > 1$. The values of ρ can be regarded as a design factor. As ρ

approaches 1, condition (19) is satisfied by only those stabilizing controllers F which result in closed loop poles that are close to origin. Such controllers however, would result in dead-beat like predicted trajectories for which, in the presence of input saturation, the feasibility region would be rather limited. Conversely, for arbitrary large ρ , condition (19) can be satisfied for almost the entire class of stabilizing F, thereby yielding an enlarged feasibility region. By (20) however, large ρ imply large γ and this suggests that there exists a trade off between feasibility, which requires large ρ, and disturbance rejection, which makes small γ desirable.

6. Numerical Example

For clarity of illustration consider the trivial scalar dynamics [13]:

$$x_{k+1} = x_k + u_k + \omega_k. \qquad (46)$$

For F=-0.8, the values $N=3, \Psi=1.15, Q=R=1$ satisfy stability conditions (18-20) with $K=0, \alpha=1.6348$ and $\gamma^2=1.9$. The norm bound $\gamma^2=1.9$ is smaller than 2 which is the norm bound obtained in [13] by solving Algebraic Riccati Equation. The choice $F=-1$, gives the even smaller H_∞ norm bound $\gamma^2=1.21$ with $N=3$, $\Psi=1.1, Q=R=1, K=0, \alpha=1.1$. In the presence of input saturation however, use of $F=-1$ yields a smaller stabilizable set of states. It is noted that proper choice of F allows one to guarantee smaller H_∞ norm bound than is obtainable via [13] for this particular example.

7. Conclusions

A RHHC method was derived by solving the min-max problem in non-recursive forms for systems with input constraint. The min-max cost index was converted to the quadratic cost index (14) for which minimizing inputs can be calculated via QP for the case of input saturation. The cost index (14) resembles those of [7] except for the fact that the weighting matrix $\overline{Q}_{off}$ contains non-zero off diagonal elements. The two cost indices and resulting control law become equivalent only for the case when γ becomes infinite, but then one loses the disturbance boundedness property.

The stability conditions of RHHC, which use closed-loop predictions, were obtained in the presence of input constraints and it was shown that an induced l_2 norm from disturbance to state, *i.e.* disturbance H_∞ norm is bounded by γ. By selecting a design factor ρ, a trade off between norm boundedness and feasibility was accomplished. Through a simple example, it was shown that by selecting the gain F properly, an improved H_∞ -norm bound can be obtained.

Without input saturation, the existence of stabilizing terminal weights Ψ is guaranteed for any choice of F provided that the plant is controllable. When there are input constraints, local stability of RHHC is guaranteed for F which satisfy additional conditions. In this case, the stability is assured over a set of states for which there exist feasible perturbations $\overline{c}_k$ which bring the current state into $\mathcal{R}_F^W$ despite bounded disturbances.

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