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ON REGULARITY OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR THE POLYHARMONIC OPERATOR

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ABSTRACT. Polyharmonic operator with Dirichlet boundary condition is considered in a n–dimensional cone. The regularity properties of weak solutions are studied. In particular, it is proved the Hölder continuity of solutions near the vertex of the cone for dimensions $n = 2m+3, 2m+4$, where $2m$ is the order of the polyharmonic operator.

1. Introduction

Let G be a bounded polyhedral domain in $\mathbb{R}^n$. Consider the Dirichlet problem

$$(-\Delta)^m U = F \quad \text{in } G, \tag{1}$$

$$\partial_\nu^k U = 0 \quad \text{on } \partial G, \text{ for } k = 0, \dots, m-1, \tag{2}$$

where Δ is the Laplace operator in $\mathbb{R}^n$, m is an integer, $m \geq 2$, and ν is the outward normal.

This problem has a unique solution in the Sobolev space $\mathring{W}_2^m(G)$ if, for example, $F \in C^\infty(\overline{G})$. If $n < 2m$ then it is Hölder continuous by the Sobolev embedding theorem. In [1], Maz'ya and Plamenevskii proved that for the biharmonic operator ($m = 2$) the solution is Hölder continuous for all dimensions n. If $m > 2$ and $n = 2m, 2m+1, 2m+2$ then the Hölder continuity of U follows from Maz'ya and Donchev [2] (where it was proved for more general class of domains). One of the results of this

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paper is the Hölder continuity of U for dimensions $n = 2m+3, 2m+4$. The main difficulty here lies in investigation of the spectrum of operator pencils associated with singularities of the boundary. In order to obtain the pencil corresponding to a boundary singularity one should take the tangent cone at this point and apply the Mellin transformation to the operators of the boundary value problem. The essential part of the paper is devoted to the study of the spectrum of this pencil. Let us introduce the operator pencil more accurate.

Consider the cone $\mathcal{K} = \{x \in \mathbb{R} : 0 < r < \infty, \omega \in \Omega\}$, where Ω is a domain on S^{n-1}. Here and in what follows, we use the spherical coordinates (r, ω) in $\mathbb{R}^n$, where $r = |x|$ and $\omega = x/|x|$. We suppose that $S^{n-1} \setminus \Omega$ contains a nonempty open set.

We define the differential operator L on S^{n-1} polynomially depending on $\lambda \in \mathbb{C}$ by

$$L(\lambda)u(\omega) = (-1)^m r^{2m-\lambda} \Delta^m (r^\lambda u(\omega)) .$$

Direct calculations show that it can be written as

$$L(\lambda) = (-1)^m \prod_{j=0}^{m-1} \left(\delta + (\lambda - 2j)(\lambda - 2j + n - 2)\right) , \tag{3}$$

where δ is the Laplace-Beltrami operator on S^{n-1}.

We introduce the operator pencil

$$\mathcal{L}(\lambda) : \mathring{W}_2^m(\Omega) \to W_2^{-m}(\Omega) \tag{4}$$

by $\mathcal{L}(\lambda)u = L(\lambda)u$.

Clearly, the function

$$U(x) = r^{\lambda_0} \sum_{s=0}^{N} \frac{(\log r)^{N-s}}{(N-s)!} u_s(\omega) , \tag{5}$$

where $u_s \in \mathring{W}_2^m(\Omega)$, satisfies

$$(-\Delta)^m U(x) = 0 \quad \text{on } \mathcal{K}, \tag{6}$$

if and only if λ_0 is an eigenvalue of the pencil $\mathcal{L}$, u_0 is an eigenfunction and $u_1, \dots, u_N$ are generalized eigenfunctions of $\mathcal{L}$ corresponding to λ_0. Thus the description of solutions (5) to equation (6) is reduced to a spectral analysis of the operator pencil (4), which is the main our goal. We state the main result of this paper

THEOREM 1. (i) *The strip*

$$m - 2 - n/2 \leq \Re\lambda \leq m + 2 - n/2 \quad \text{if } 2m \leq n - 4, \tag{7}$$

$$-3 \leq \Re\lambda \leq 0 \quad \text{if } 2m = n - 3 \tag{8}$$

$$-2 \leq \Re\lambda \leq 0 \quad \text{if } 2m = n - 2 \tag{9}$$

and

$$m - (n+1)/2 \leq \Re\lambda \leq m - (n-1)/2 \quad \text{if } 2m \geq n - 1 \tag{10}$$

contains no eigenvalues of the pencil $\mathcal{L}$.

(ii) *Let* $2m = n - 3$. *Then the strip* $-3 - 1/2 \leq \Re\lambda \leq 1/2$ *contains only real eigenvalues of* $\mathcal{L}$, *the corresponding eigenfunctions have no generalized eigenfunctions. There is at most one eigenvalue on the interval* $(0, 1/2]$ *which increases when the domain* Ω *decreases.*

(iii) *Let* $2m = n - 2$. *Then the strip* $-3 \leq \Re\lambda \leq 1$ *contains only real eigenvalues of* $\mathcal{L}$, *the corresponding eigenfunctions have no generalized eigenfunctions. There is at most one eigenvalue on the interval* $(0, 1]$ *which increases when the domain* Ω *decreases.*

(vi) *Let* $2m \geq n-1$. *Then the strip* $m-1-n/2 \leq \Re\lambda \leq m+1-n/2$ *contains only real eigenvalues of* $\mathcal{L}$, *the corresponding eigenfunctions have no generalized eigenfunctions. The eigenvalues on the interval* $(m-(n-1)/2, m+1-n/2]$ *increase when the domain* Ω *decreases.*

The case $n = 2$ and m is arbitrary follows from [3] and [4] where operator pencils corresponding to general elliptic operators of order $2m$ with real coefficients are investigated. The case of biharmonic operator (m=2) in n-dimensional cone was considered in [5], where one can find, in particular, almost all results of the above theorem.

As an immediate consequence of Theorem 1 we obtain

COROLLARY 1. *The strip*

$$2m - n - \varepsilon \leq \Re\lambda \leq \varepsilon \tag{11}$$

is free of eigenvalues of $\mathcal{L}$ *if* $n = 2m$, $2m+1$, $2m+2$, $2m+3$, $2m+4$. *Here* ε *is a positive number depending on* Ω.

The above corollary and regularity results for solutions to elliptic problems in domains with piecewise smooth boundaries (see [1], [7]) imply the results on Hölder continuity formulated at the beginning of the introduction.

We note that the estimate (10) for the strip free of spectrum is sharp. For $2m = n-1$ it is shown in [8] and for $2m \geq n$ in [9]. Clearly, the estimates (10), (8) and (7) for $2m = n-4$ can not be improve for arbitrary cones. It remains an important question. Is it possible to improve the estimate (7) for $2m < n-4$? The conjecture here is that the strip $m - n/2 \leq \lambda \leq 0$ should be free of the eigenvalues of $\mathcal{L}$. It would lead to the Hölder continuity of solutions to (1), (2) for all m and n. We note that this conjecture is not true for arbitrary elliptic operator with constant real coefficients (see [10]).

2. Some properties of $\mathcal{L}$

We begin with a positivity property of $\mathcal{L}$.

LEMMA 1. *For every $\lambda = m - n/2 + i\tau$, $\tau \in \mathbb{R}$*

$$\big(\mathcal{L}(\lambda)u, u\big) \geq c\,(||u||^2_{W_2^m(\Omega)} + |\lambda|^{2m}) \tag{12}$$

for all $u \in \mathring{W}_2^m(\Omega)$, where c is a positive constant. Here and elsewhere we use the notation $(\cdot,\cdot)$ for the scalar product in $L_2(\Omega)$.

Proof. (i) Let $m = 2k$. Then for $\lambda = m - n/2 + i\tau$

$$L(\lambda) = L_1(\lambda)L_1(\overline{\lambda})\ ,$$

where

$$L_1(\lambda) = \prod_{j=0}^{k-1}\big(\delta + (\lambda - 2j)(\lambda - 2j + n - 2)\big).$$

Hence

$$\big(\mathcal{L}(\lambda)u, u\big) = ||L_1(\lambda)u||^2_{L_2(\Omega)}. \tag{13}$$

The right-hand side can be equal to zero if $L_1(\lambda)u = 0$. Since the order of L_1 is m and $u \in \mathring{W}_2^m(\Omega)$, we have that $u = 0$. Relation (13) implies (12) for large $|\tau|$. If $|\tau| \leq N$, where N is a constant then $||L_1(\lambda)u||_{L_2(\Omega)} \geq c\,||u||_{W_2^m(\Omega)}$ with some positive c depending on N because of $\ker L_1(\lambda) = 0$ on $\mathring{W}_2^m(\Omega)$.

(ii) Let $m = 2k+1$. Then

$$\begin{aligned} -L(\lambda) &= \big(\delta + (\lambda - 2k)(\lambda - 2k + n - 2)\big)L_1(\lambda)L_1(\overline{\lambda}) \\ &= \big(\delta - (1 - n/2)^2 - \tau^2\big)L_1(\lambda)L_1(\overline{\lambda}), \end{aligned}$$

which implies

$$(14)\qquad \big(\mathcal{L}(\lambda)u,u\big) = \big((-\delta+(1-n/2)^2+\tau^2)L_1(\lambda)u, L_1(\lambda)u\big).$$

The equality $L_1(\lambda)u = 0$ implies $u = 0$ because of the boundary conditions for u. Therefore (12) follows from (14).

The above lemma shows, in particular, that the operator (4) is isomorphic for $\lambda = m - n/2$. Clearly, the operator

$$\mathcal{L}(\lambda) - \mathcal{L}(m-n/2) : \mathring{W}_2^m(\Omega) \to W_2^{-m}(\Omega)$$

is compact for all $\lambda \in \mathbb{C}$. Therefore, the operator (4) is Fredholm for every λ and hence, the spectrum of $\mathcal{L}$ consists of eigenvalues of finite algebraic multiplicity and that the only accumulation point for the spectrum is infinity.

Direct verification gives

$$L(\lambda) = L(2m-n-\lambda)\ .$$

This implies, in particular, that the spectrum of $\mathcal{L}$ is symmetric with respect to the line $\Re\lambda = m - n/2$. Moreover, the numbers λ is an eigenvalue if and only if the number $2m-n-\lambda$ is an eigenvalue and they have the same eigenfunctions and generalized eigenfunctions. Thus, it suffices to study the spectral properties of $\mathcal{L}$ only in the half-plane $\Re\lambda > m - n/2$. □

Remark 2. In the case $m < n/2$ the assertion of Lemma 1 is true for arbitrary domain on $S^{(n-1)}$. This follows from the fact that $L_1(\lambda)u = 0$ for $u \in W_2^m(S^{(n-1)})$ implies $u = 0$, which can be verified by going back to x variable.

3. On the eigenvalues on the line $\Re\lambda = m + 1 - n/2$

Lemma 2. *The number $\lambda = m + 1 - n/2 + i\tau$, with nonzero real τ, is not an eigenvalue of $\mathcal{L}$.*

Proof. (i) Let $m = 2k+1$ and let $\lambda = m+1-n/2+i\tau$. Then

$$-L(\lambda) = \big(\delta + \lambda(\lambda+n-2)\big)L_2(\lambda)L_2(\overline{\lambda})\ ,$$

where

$$L_2(\lambda) = \prod_{j=1}^{k}\big(\delta + (\lambda-2j)(\lambda-2j+n-2)\big)\ .$$

Therefore,

$$\Im(L(\lambda)u,u) = 2m\tau||L_2(\lambda)u||_{L_2(\Omega)} \, . \tag{15}$$

If $\tau \neq 0$ and the left-hand side is zero then $L_2(\lambda)u = 0$. Using the Dirichlet boundary conditions for u, we obtain $u = 0$. The result follows for odd m.

(ii) Let $m = 2k+2$. Then

$$L(\lambda) = (\delta + \lambda(\lambda+n-2))(\delta + (\lambda-2k-2)(\lambda-2k+n-4))L_2(\lambda)L_2(\overline{\lambda}).$$

Since

$$\begin{aligned} &\big(\delta + \lambda(\lambda+n-2)\big)\big(\delta + (\lambda-2k-2)(\lambda-2k+n-4)\big) \\ &= \big(\delta + (i\tau+m)^2 - (1-n/2)^2\big)\big(\delta - \tau^2 - (1-n/2)^2\big), \end{aligned}$$

it follows that

$$\Im(L(\lambda)u,u) = 2m\tau\big((-\delta+\tau^2+(1-n/2)^2)L_2(\lambda)u, L_2(\lambda)u\big). \tag{16}$$

If u is an eigenfunction corresponding to λ with $\tau \neq 0$ then $L_2(\lambda)u = 0$ and hence $u = 0$. The proof is complete. □

LEMMA 3. *Let $\lambda_0 = m+1-n/2$ be the eigenvalue of $\mathcal{L}$ and u_0 be a corresponding eigenfunction. Then*

$$\frac{d}{d\lambda}\big(L(\lambda)u_0,u_0\big)\Big|_{\lambda=\lambda_0} > 0 \, . \tag{17}$$

Proof. The quadratic form $(L(\lambda)u_0,u_0)$ is a polynomial with respect to λ and we represent it as

$$(L(\lambda)u_0,u_0) = p(\sigma,\tau) + i\tau q(\sigma,\tau) \, ,$$

where $\sigma + i\tau = \lambda$, and p and q are real-valued polynomials with respect to σ and τ. Since

$$\frac{d}{d\lambda}(L(\lambda)u_0,u_0) = \frac{d}{id\tau}(L(\lambda)u_0,u_0) \, ,$$

we have

$$\frac{d}{d\lambda}(L(\lambda)u_0,u_0) = \frac{d}{id\tau}p(\sigma,\tau) + q(\sigma,\tau) + \tau\frac{d}{d\tau}q(\sigma,\tau).$$

Using that the left-hand side in the last relation is real, we obtain

$$\frac{d}{d\lambda}\big(L(\lambda)u_0,u_0\big)\Big|_{\lambda=\lambda_0} = q(\lambda_0,0) \, .$$

Thus,

$$\frac{d}{d\lambda}\big(L(\lambda)u_0,u_0\big)\Big|_{\lambda=\lambda_0}=\frac{1}{\tau}\Im\big(L(\lambda_0)u_0,u_0\big)$$

and inequality (17) follows from positivity (after division by τ) of the right hand sides in (15) and (16). □

COROLLARY 3. *If* $\lambda_0 = m+1-n/2$ *is the eigenvalue of* $\mathcal{L}$ *then it has no generalized eigenfunctions.*

Proof. Let λ_0 be an eigenvalue and u_0 be a corresponding eigenfunction. Then the equation for a generalized eigenfunction u_1 is

$$\mathcal{L}(\lambda_0)u_1=-\frac{d}{d\lambda}L(\lambda)u_0|_{\lambda=\lambda_0}\,.$$

In order to show that it is unsolvable we multiply (in $L_2(\Omega)$) both sides of the equation by u_0. Then the left-hand side is equal to zero because of u_0 is an eigenfunction and the operator $\mathcal{L}(\lambda_0)$ is selfadjoint. But the right-hand side differs from zero by Lemma 3. This contradiction proves Corollary. □

4. Eigenvalues of $\mathcal{L}$ in the strip $m-n/2\le\Re\lambda\le m+1-n/2$

Consider the operator pencil $\mathcal{L}$ for real λ. Clearly, the operator $\mathcal{L}(\lambda)$ is selfadjoint and semibounded from below. For every $\lambda\ge m-n/2$ we denote by $\{\mu_j(\lambda)\}_{j\le 1}$ a nondecreasing sequence of eigenvalues of $\mathcal{L}(\lambda)$, counted with their multiplicities. Furthermore, we denote by $v_j(\lambda)$ an eigenfunction of $\mathcal{L}(\lambda)$ corresponding to $\mu_j(\lambda)$. One can suppose that the system $\{v_j(\lambda)\}_{j\le 1}$ forms an orthonormal basis in $L_2(\Omega)$. It is known (see, for example, [Kato, Ch.VII, Th,1.8]) that the functions μ_j are continuous (moreover, piecewise analytic) on $[m-n/2,\infty)$. From the variational principle for $\mu_j(\lambda)$ it follows that the functions μ_j increase when the domain Ω becomes smaller.

We shall denote by M the index satisfying $\mu_M(m+1-n/2)\le 0$ and $\mu_{M+1}(m+1-n/2)>0$.

LEMMA 4. *The strip*

$$m-n/2\le\Re\lambda\le m+1-n/2 \tag{18}$$

contains exactly M eigenvalues of the operator pencil $\mathcal{L}$, counted with their geometric multiplicities. All these eigenvalues are real and have no *generalized eigenfunctions. Every function μ_j has at most one zero on the interval $[m-n/2, m+1-n/2]$.*

Proof. (i) Consider functions $\mu_1, \dots, \mu_M$. By Lemma 1 the operator $\mathcal{L}(m-n/2)$ is positive. Therefore, $\mu_j(m-n/2) > 0$ for all j. If $j \le M$ then $\mu_j(m+1-n/2) \le 0$ therefore the function μ_j has at least one zero, say λ_j, on the interval $(m-n/2, m+1-n/2]$. Clearly, λ_j is an eigenvalue of the pencil $\mathcal{L}$ and v_j is an eigenfunction of the pencil corresponding to λ_j. Thus, we have proved that the total geometric multiplicity of eigenvalues situated on the interval $(m-n/2, m+1-n/2]$ is greater or equal to M. Moreover, if one of the functions μ_j, $j \le M$, has more than one zero on $(m-n/2, m+1-n/2]$ then the total geometric multiplicity of eigenvalues of $\mathcal{L}$ in the strip (18) is greater than M.

(ii) Consider the family of operator pencils $\mathcal{L}_t(\lambda) = \mathcal{L}(\lambda) + tI$, where $t \ge 0$ and I is the identity operator. Clearly, the set $\{\mu_j(\lambda) + t\}_{j \le 1}$ represents a nondecreasing sequence of eigenvalues of $\mathcal{L}_t(\lambda)$, counted with their multiplicities. If t is sufficiently big then there are no eigenvalues of $\mathcal{L}_t$ in the strip (18).

Now we keep t decreasing. For $t > -\mu_1(m+1-n/2)$ this strip is still free of eigenvalues of $\mathcal{L}_t$. In fact, from Lemmas 1 and 2 it follows that there are no eigenvalues on the lines $\Re\lambda = m-n/2$ and $\Re\lambda = m+1-n/2$ and there are no eigenvalues in the strip (18) for large $|\Im\lambda|$ hence, by the Rouché operator theorem (see [12], Sect.XI.9) there are no eigenvalues in the strip (18) for all $t > -\mu_1(m+1-n/2)$.

Let $t = -\mu_1(m+1-n/2)$. Then the strip $m-n/2 < \Re\lambda < m+1-n/2$ is free of eigenvalues of $\mathcal{L}_t$, otherwise by the Rouché operator theorem the same strip should contain eigenvalues of $\mathcal{L}_t$ for some t bigger than $-\mu_1(m+1-n/2)$. Let κ be a such index that $\mu_2(m+1-n/2) = \dots = \mu_\kappa(m+1-n/2) = -t$ and $t > -\mu_{\kappa+1}(m+1-n/2)$. Then the strip (18) contains exactly one eigenvalue $\lambda = m+1-n/2$ of the pencil $\mathcal{L}_t$ of geometric multiplicity κ and by Lemma 2 this eigenvalue has no generalized eigenfunctions. Thus, the total algebraic multiplicity of the eigenvalues in the strip (18) is equal to their total geometric multiplicity and equals to κ. $\lambda \in [m-n/2, m+1-n/2)$. If we take t a little bit smaller than $-\mu_1(m+1-n/2)$ then both lines $\Re\lambda = m-n/2$ and $\Re\lambda = m+1-n/2$ are free again of eigenvalues of the pencil $\mathcal{L}_t$ and by continuous dependence of eigenvalues on a parameter the total algebraic multiplicity of eigenvalues of $\mathcal{L}_t$ in the strip (18) is not greater than κ.

The next time when this total algebraic multiplicity changes will be for $t = -\mu_{\kappa+1}(m+1-n/2)$. Let κ_1 be the index such that

$$\mu_{\kappa+1}(m+1-n/2) = \cdots = \mu_{\kappa+\kappa_1}(m+1-n/2) = -t$$

and $t > -\mu_{\kappa+\kappa_1}(m+1-n/2)$. Then reasoning as above we conclude that the total algebraic multiplicity of $\mathcal{L}_t$ in the strip (18) does not exceed $\kappa+\kappa_1$ for this t and for t which is a little bit smaller that $-\mu_{\kappa_1}$. Continuing this procedure we obtain that the total algebraic multiplicity of $\mathcal{L}_0 = \mathcal{L}$ in the strip (18) is not greater than M.

Finally, combining (i) and (ii), we arrive at the assertions of Lemma. □

5. Eigenvalues of the pencil $\mathcal{L}$ in the strip $m - n/2 \leq \Re\lambda \leq m+2-n/2$

LEMMA 5. *Let $m \leq n/2 - 1$. Then the set $\{\lambda = i\tau + m + 2 - n/2\}$, where τ is a nonzero real number, does not contain eigenvalues of the pencil $\mathcal{L}$.*

Proof. (i) Let $m = 2k+2$ and $\lambda = i\tau + m + 2 - n/2$, $\tau \neq 0$. Then

$$\mathcal{L}(\lambda) = (\delta + \lambda(\lambda+n-2))(\delta + (\lambda-2)(\lambda+n-4))L_3(\lambda)L_3(\overline{\lambda}),$$

where

$$L_3(\lambda) = \prod_{j=2}^{k+1}\big(\delta + (\lambda-2j)(\lambda-2j+n-2)\big).$$

Since

$$\begin{aligned}&\Im\big((\delta + \lambda(\lambda+n-2))(\delta+(\lambda-2)(\lambda+n-4))u,u\big)\\ &= \ 4m\tau\big((\delta - \tau^2 + m^2 - \frac{n^2}{4} + n - 2)u,u\big)\,,\end{aligned}$$

it follows that

$$\Im\big(\mathcal{L}(\lambda)u,u\big) = 4m\tau\big((\delta-\tau^2+m^2-\frac{n^2}{4}+n-2)L_3(\lambda)u, L_3(\lambda)u\big).$$

The operator $-\delta + \tau^2 - m^2 + \frac{n^2}{4} - n + 2$ is positive if $m \leq n/2 - 1$. Therefore, the left hand side can be zero only if $L_3(\lambda)u = 0$. But this implies $u = 0$ because of $u \in \mathring{W}_2^m(\Omega)$.

(ii) Consider the case $m = 2k+1$. Then

$$-L(\lambda) \;=\; \big(\delta + \lambda(\lambda+n-2)\big)\cdot\big(\delta + (\lambda-2)(\lambda+n-4)\big) \cdot\big(\delta + (\lambda - 2(k+1))(\lambda - 2k + n - 4)\big) L_4(\lambda) L_4(\overline{\lambda}),$$

where

$$L_4(\lambda) = \prod_{j=2}^{k} (\delta + (\lambda - 2j)((\lambda - 2j) + n - 2)).$$

After simple calculations, we get

(19)

$$\Im\Big(\big(\delta{+}\lambda(\lambda{+}n{-}2)\big)\big(\delta{+}(\lambda{-}2)(\lambda{+}n{-}4)\big)\big(\delta{+}(\lambda{-}2(k{+}1))(\lambda{-}2k{+}n{-}4)\big)v,\ v\Big)$$

$$= 4m\tau\Big(\big(\delta{-}\tau^2{-}(1{-}n/2)^2{+}m^2{-}1\big)v,\ \big(\delta{-}\tau^2{-}(1{-}n/2)^2\big)v\Big).$$

Let $w = (\delta - \tau^2 - (1-n/2)^2)v$. Then $||w||_{L_2(\Omega)} \le (1-n/2)^2||v||_{L_2(\Omega)}$. Using this estimate together with the Cauchy inequality, we obtain

$$\big|(w, w + (m^2-1)v)\big| \le \Big(1 - \frac{m^2-1}{1-n/2)^2}\Big)||w||^2_{L_2(\Omega)}\,.$$

Since $m \le n/2 - 1$, the left-hand side of the last equality is positive and (19) implies

$$-\frac{1}{\tau}\Im\big(\mathcal{L}(\lambda)u, u\big) > 0 \quad \text{for } u \in \mathring{W}_2^m(\Omega), \quad u \neq 0.$$

The proof is complete. □

In the same way as Lemma 3, 4 and Corollary 3, one can prove

LEMMA 6. *Let $m \le n/2-1$ and let $\lambda_0 = m+2-n/2$ be the eigenvalue of $\mathcal{L}$ and u_0 be a corresponding eigenfunction. Then*

$$\frac{d}{d\lambda}\big(L(\lambda)u_0, u_0\big)\Big|_{\lambda=\lambda_0} < 0. \tag{20}$$

COROLLARY 4. *If $\lambda_0 = m+2-n/2$ is the eigenvalue of $\mathcal{L}$ then it has no generalized eigenfunctions.*

Let M be the maximal index j such that $\mu_j(m+2-n/2) \le 0$.

LEMMA 7. *Let $m \le n/2 - 1$. The strip $m - n/2 \le \Re\lambda \le m + 2 - n/2$ contains exactly M eigenvalues of the operator pencil $\mathcal{L}$, counted with their algebraic multiplicities. All these eigenvalues are real and have no generalized eigenfunctions. Every function μ_j has at most one zero in the interval $[m - n/2, m + 1 - n/2]$.*

6. On real eigenvalues of the pencil $\mathcal{L}$

Denote by $\Lambda = \Lambda(\Omega)$ the real eigenvalue of $\mathcal{L}$ which is closest to and greater than $m - n/2$. If there are no such eigenvalues of $\mathcal{L}$ on $(m - n/2, \infty)$ we put $\Lambda = \infty$. Clearly, Λ is the smallest zero of the function μ_1 on $(m - n/2, \infty)$. In the following lemma we give some properties of Λ whose proof is quite straightforward.

LEMMA 8. *Let $\Lambda < \infty$.*

(i) *The number Λ is the smallest number λ on $(m - n/2, \infty)$ such that the operator $\mathcal{L}(\lambda)$ has non-trivial kernel.*

(ii) *For $u \in \mathring{W}_2^m(\Omega)$, $u \neq 0$, we denote by $R(u)$ the smallest root of the equation $(\mathcal{L}(\lambda)u, u) = 0$ on the interval $(m - n/2, \infty)$. Then*

$$\Lambda = \inf R(u),$$

where the infimum is taken over all nonzero $u \in \mathring{W}_2^m(\Omega)$.

COROLLARY 5. (i) *The number Λ increases when the domain Ω decreases. Moreover, $\Lambda(\Omega_2) > \Lambda(\Omega_1)$ if $\Omega_2 \subset \Omega_1$ and $\Omega_1 \setminus \overline{\Omega_2}$ is not empty.*

(ii) *If $m - n/2 < 0$ then $\Lambda > 0$.*

(iii) *If $m - n/2 \ge 0$ then $\Lambda > m - n/2 + 1/2$.*

Proof. (i) The monotonicity of Λ with respect to Ω follows from Lemma 8(ii). The strict monotonicity is proved in the same way as it was done in Theorem 7.2 [13].

(ii) By Remark 2 the definition of Λ without changes can be applied for all domains in $S^{(n-1)}$ when $m < n/2$. Moreover, Lemma 8 is valid in this case also. If $\Omega = S^{(n-1)}$ then the spectrum of $\mathcal{L}$ consists of the integers $0, 1, \dots$ and $2m - n, 2m - n - 1, \dots$. Therefore, $\Lambda(S^{(n-1)}) = 0$. Now, the assertion follows from Lemma 8(ii).

(iii) Let $\mathcal{K}_0$ be the complement of a small closed circular cone and let Ω_0 is the intersection of $\mathcal{K}_0$ and $S^{(n-1)}$. We can suppose that $\Omega \subset \Omega_0$. Clearly, the boundary of $\mathcal{K}_0$ is Lipschitz and by Theorem 1 [14] the strip $|\Re\lambda - m + n/2| \leq 1/2$ is free of eigenvalues of the pencil corresponding Ω_0. This implies $\Lambda(\Omega_0) > m - n/2 + 1/2$. The result for Ω follows from Lemma 8(ii). $\square$

7. Proof of Theorem 1

(i) Let $2m \leq n-4$. We introduce the operator pencil $\mathcal{L}_t(\lambda) = \mathcal{L}(\lambda) + tI$, $t \geq 0$. Using Lemma 1 we obtain

$$\big(\mathcal{L}(\lambda)u, u\big) \geq c\,(||u||^2_{W_2^m(\Omega)} + |\lambda|^{2m} + t)$$

for all λ on the line $\Im\lambda = m - n/2$. This inequality implies that there are no eigenvalues of $\mathcal{L}_t$ for $|\Im\lambda| \geq N$, $\Re\lambda \in [m - n/2, m + 2 - n/2]$ where N is a sufficiently large number. From Lemma 5 and Corollary 5(ii) the pencils $\mathcal{L}_t$ have no eigenvalues on the line $\Re\lambda = m + 2 - n/2$ neither. By the operator Rouché theorem all pencils $\mathcal{L}_t$ have the same number of eigenvalues in the strip (7). Since for large t the pencil $\mathcal{L}_t$ has no eigenvalues in this strip we conclude that the same is true for $\mathcal{L}$.

The cases $2m = n - 3$ and $2m = n - 2$ are considered analogously. Let $2m > n - 2$. Then reasoning as in the proof of inequalities (7) and using Corollary 5(iii) instead of Corollary 5(ii), we obtain the proof in this case.

(ii) By Lemma 7, the strip $-7/2 \leq \Re\lambda \leq 1/2$ contains only real eigenvalues of $\mathcal{L}$ which has no generalized eigenfunctions. By (8), there are no eigenvalues on the interval $[-3, 0]$. If $\Omega = S^{(n-1)}$ then the interval $[0, 1/2]$ contains exactly one eigenvalue of $\mathcal{L}$: $\lambda_0 = 0$ of multiplicity 1. This eigenvalue is the zero of the functions μ_1. Clearly, this function increases when the domain Ω decreases. Therefore, if $S^{(n-1)}\backslash\overline{\Omega}$ is nonempty then the interval $(0, 1/2]$ contains at most one simple eigenvalue which increases when the decreases.

The proofs of (iii) and (iv) are essentially the same as those of (ii). $\square$

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