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# CORRELATION DIMENSIONS OF QUASI-PERIODIC ORBITS WITH FREQUENCIES GIVEN BY QUASI ROTH NUMBERS

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Abstract. In this paper, we estimate correlation dimensions of discrete quasi periodic orbits with frequencies, irrational numbers, which are called quasi Roth numbers. We specify the lower estimate values of the dimensions by using the parameters which are derived from the rational approximable properties of the quasi Roth numbers.

## 1. Introduction

In our previous papers, we have estimated box dimensions ([1], [2], [3]) or correlation dimensions ([4]) for quasi periodic orbits by using Diophantine approximations. In the present paper we also consider a two-frequency q.p. function from $R$ to a Banach space $X$ given by

$$f(t) = g(wt, t)$$

where the frequency $w : w > 1$ is an irrational number and $g : R \times R \to X$ is 1-periodic with respect to each variable. Our purpose is to estimate correlation dimensions for the sequence, given by a Poincaré section of $f(t)$,

$$\Sigma = \{\varphi(n) : \varphi(n) = f(n\tau) = g(n, \tau n), \;\; n = 0, 1, 2, ...\} \subset X, \quad \tau = 1/w.$$

In [4], we have already shown the lower estimate of the correlation dimension under the two Hölder conditions, the usual type and the reverse inequality type, on the function $g$. In the present paper, we can

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show the lower estimate under only the reverse inequality condition and simplify its proof. In [4], we have also specified the lower estimate values of the correlation dimensions by using the parameters which are derived from the algebraic properties of the frequencies, exactly, the rational approximable properties of the irrational frequencies. In the present paper, for the irrational numbers called quasi Roth numbers, which have not so badly approximable properties by rational numbers, we can give further examples and characterize the approximable properties by using the Diophantine approximations.

We consider the following cases, which are classified by the rational approximable properties of the frequency.
(i) Constant type; there exists a constant $c_0 > 0$ such that

$$|\tau - \frac{r}{q}| \geq \frac{c_0}{q^2} \tag{1.1}$$

for every positive integers $r, q$.
(ii) Quasi Roth number type; there exists a constant $\alpha_0 > 0$ such that for every $\alpha \geq \alpha_0$ there exists a constant $c_\alpha > 0$ which satisfies

$$|\tau - \frac{r}{q}| \geq \frac{c_\alpha}{q^{2+\alpha}} \tag{1.2}$$

for every positive integers $r, q$.
(iii) Roth number type; for every $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ which satisfies

$$|\tau - \frac{r}{q}| \geq \frac{c_\varepsilon}{q^{2+\varepsilon}} \tag{1.3}$$

for every positive integers $r, q$.

REMARK. The irrational numbers of constant type are also called "badly approximable" numbers and Roth numbers are also called the numbers with good Diophantine properties.

DEFINITION OF CORRELATION DIMENSION. Let $S = \{x_1, x_2, \ldots, x_n, \ldots\ldots\}$ be an infinite sequence of elements in $X$ and, for a small number $\varepsilon > 0$, define

$$\underline{N}(\varepsilon) = \liminf_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} H(\varepsilon - \|x_i - x_j\|),$$

$$\overline{N}(\varepsilon) = \limsup_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} H(\varepsilon - \|x_i - x_j\|),$$

where $H(\cdot)$ is a Heaviside function:

$$H(a) = \begin{cases} 1 & a \geq 0 \\ 0 & a < 0 \end{cases}$$

and if the limit exits, $N(\varepsilon) := \underline{N}(\varepsilon) = \overline{N}(\varepsilon)$. The upper and lower correlation dimension of $S$, $\overline{C}(S)$ and $\underline{C}(S)$, are defined as follows:

$$\overline{C}(S) = \limsup_{\varepsilon \downarrow 0} \frac{\log \underline{N}(\varepsilon)}{\log \varepsilon},$$
$$\underline{C}(S) = \liminf_{\varepsilon \downarrow 0} \frac{\log \overline{N}(\varepsilon)}{\log \varepsilon}.$$

If $N(\varepsilon)$ exists and $\overline{C}(S) = \underline{C}(S)$, we say that $S$ has the correlation dimension $C(S) = \overline{C}(S) = \underline{C}(S)$.

As in our pervious papers, assuming Hölder's continuity on the function $g(\cdot,\cdot)$, we estimate the dimensions by using Hölder's exponents.

**(G1)** There exist constants $\delta, c_1 : 0 < \delta \leq 1, c_1 > 0$:

$$|g(t,s) - g(t',s)| \leq c_1|t-t'|^\delta,$$
$$|g(t,s) - g(t,s')| \leq c_1|s-s'|^\delta, \quad t,t',s,s' \in R$$

**(G2)** There exist constants $\delta, c_2 : 0 < \delta \leq 1, c_2 > 0$:

$$|g(t,s) - g(t',s)| \geq c_2|t-t'|^\delta,$$
$$|g(t,s) - g(t,s')| \geq c_2|s-s'|^\delta, \quad t,t',s,s' \in R : |t-t'|, |s-s'| < 1/2.$$

The plan of this paper is as follows; In section 2, we estimate the correlation dimensions in the case (ii) and then (i) and (iii). In section 3, we introduce some examples of Roth and quasi Roth numbers.

## 2. Correlation dimension of quasi Roth numbers case

Consider the following continued fraction of the number $\tau$:

$$\tau = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}} \qquad (a_i \in N) \tag{2.1}$$

and take the rational approximation as follows. Let $m_0 = 1, n_0 = 0, m_{-1} = 0, n_{-1} = 1$ and define the pair of sequences of natural numbers

$$m_i = a_i m_{i-1} + m_{i-2}, \tag{2.2}$$

$$n_i = a_i n_{i-1} + n_{i-2}, \quad i \geq 1, \tag{2.3}$$

then the elementary number theory gives the Diophantine approximation

$$\frac{1}{m_i(m_{i+1} + m_i)} < |\tau - \frac{n_i}{m_i}| < \frac{1}{m_i m_{i+1}} < \frac{1}{m_i^2} \tag{2.4}$$

where the sequence $\{n_i/m_i\}$ is the best approximation in the sense that

$$|\tau - \frac{n_i}{m_i}| \leq |\tau - \frac{r}{l}|$$

holds for every rational $r/l : \ l \leq m_i$.

First we consider the case (ii) quasi Roth number type with (1.2). Then we can obtain the following estimate:

$$\|\varphi(m) - \varphi(n)\| \geq c_2 (\frac{c_\alpha}{|m-n|^{1+\alpha}})^\delta, \quad \forall \alpha \geq \alpha_0 \tag{2.5}$$

for every $m, n \in N : m \neq n$. In fact, since we can find an integer $n'$:

$$|m\tau - n\tau - n'| < \frac{1}{2}$$

(in case $m > n$), Hypothesis **(G2)** and the periodicity of $g$ yield the following estimates.

$$\begin{aligned}\|\varphi(m) - \varphi(n)\| &= \|g(m, m\tau) - g(n, n\tau)\| \\ &= \|g(m, m\tau - n') - g(m, n\tau)\| \\ &\geq c_2|(m-n)\tau - n'|^{\delta}.\end{aligned}$$

Thus (1.2) yields (2.5).

For the case of the constant type (i), it is well known (cf. [6]) that the uniform boundedness of the sequence $\{a_j\}$ is equivalent to the property (1.1). On the other hand, by using the increasing sequence $\{m_j\}$, we can show an equivalent condition to the badly approximable property of quasi Roth numbers.

**(B)** There exist constants $\beta, K > 0$:

$$m_{j+1} \leq K m_j^{1+\beta}, \quad \forall j. \tag{2.6}$$

We can show the following lemmas.

LEMMA 1. *If the condition* (B) *is satisfied for an irrational number* $\tau$, *then* $\tau$ *is a quasi Roth number for the constant*

$$\alpha_0 = \beta(\beta + 3). \tag{2.7}$$

*Proof.* For every positive integer $l$, there exists a number $j$:

$$m_{j-1} \leq l < m_j \leq K m_{j-1}^{\beta+1} \leq K l^{\beta+1}. \tag{2.8}$$

Since $n_j/m_j$ is a best approximation of $\tau$, we have

$$\begin{aligned}|\tau - \frac{r}{l}| &\geq |\tau - \frac{n_j}{m_j}| \\ &\geq \frac{1}{(m_{j+1} + m_j)m_j} \\ &\geq \frac{1}{2m_{j+1}m_j} \geq \frac{c}{m_j^{\beta+2}} \\ &\geq \frac{c}{l^{(\beta+1)(\beta+2)}}\end{aligned}$$

where we denote by $c$ a suitable constant in each term. Thus for every rational number $r/l$ we have

$$|\tau - \frac{r}{l}| \geq \frac{c}{l^{2+\beta(\beta+3)}}. \tag{2.9}$$

□

LEMMA 2. *If $\tau$ is a quasi Roth number, then for every $\beta \geq \alpha_0$, there exists $K_\beta > 0$ which satisfies* (B)*:*

$$m_{j+1} \leq K_\beta m_j^{1+\beta}, \quad \forall j. \tag{2.10}$$

*Proof.* It follows from the definition of quasi Roth numbers that, for every $\beta \geq \alpha_0$, there exists $K_\beta > 0$:

$$\frac{K_\beta^{-1}}{m_j^{2+\beta}} \leq |\tau - \frac{n_j}{m_j}| \leq \frac{1}{m_{j+1}m_j}. \tag{2.11}$$

Thus we obtain the conclusion. □

For the quasi periodic sequence $\Sigma = \{\varphi(n) : n \in N\}$, we can estimate its correlation dimension from below.

THEOREM 1. *Assume Hypotheses* (ii) *and* (G2)*. Then we have*

$$\underline{C}(\Sigma) \geq \frac{1}{(1+\alpha_0)^2\delta}. \tag{2.12}$$

*Proof.* Let $k, i : k < i$ be sufficiently large numbers and consider a small positive constant $\varepsilon_k$, given by

$$\varepsilon_k = (\frac{1}{m_{k+1}})^\delta.$$

It follows from Lemma 2 that

$$\begin{aligned}\varepsilon_{k+1} = (\frac{1}{m_{k+2}})^\delta &\geq (\frac{1}{Km_{k+1}^{1+\alpha_0}})^\delta \\ &= K^{-\delta}\varepsilon_k^{1+\alpha_0}\end{aligned}$$

where it is sufficient to consider the case $0 < K < 1$. In fact, for every $\beta : \beta > \alpha_0$, from Lemma 2 we obtain

$$m_{j+1} < (K m_{j_0}^{\alpha_0 - \beta}) m_j^{1+\beta}, \quad \forall j > j_0 \tag{2.13}$$

for some $j_0$. Then we can substitute $K$ by a sufficiently small $K m_{j_0}^{\alpha_0-\beta}$. Following the argument below, we can obtain the conclusion for every $\beta : \beta > \alpha_0$.

Let $\alpha_1 > 0$; $\alpha_1 > \alpha_0$, be a constant, which satisfies

$$\alpha_1 + 1 > (1 + \alpha_0)^2, \tag{2.14}$$

and, take a small constant $\varepsilon : \varepsilon_{k+1}^{1+\alpha_0} \le \varepsilon \le \varepsilon_k^{1+\alpha_0}$. Then, since we have

$$\varepsilon_{k+1}^{1+\alpha_0} \ge (K^{-\delta})^{1+\alpha_0} \varepsilon_k^{(1+\alpha_0)^2} > (K^{-\delta})^{1+\alpha_0} \varepsilon_k^{1+\alpha_1}, \tag{2.15}$$

there exists a constant $\alpha : \alpha_0 \le \alpha \le \alpha_1$, which satisfies

$$\varepsilon = (K^{-\delta})^{1+\alpha_0} \varepsilon_k^{1+\alpha}. \tag{2.16}$$

Now, consider an $\varepsilon$-neighborhood of $\varphi(1)$, say $B_\varepsilon$, in $X$. Then, for a large integer $n \in N$, we estimate an upper bound of the number of the elements $\varphi(l), l \in I_n = \{1, ..., n\}$, which satisfy $\varphi(l) \in B_\varepsilon$:

$$M_n(\varepsilon) := \#\{\varphi(l) \in B_\varepsilon : \ l \in I_n\}.$$

Assume that $\varphi(n_1) \in B_\varepsilon$ for some $n_1 \in I_n$. Then, for any $m \in I_n$, $m \ne n_1$, we can estimate

$$\begin{aligned} \|\varphi(m) - \varphi(1)\| &\ge \|\varphi(m) - \varphi(n_1)\| - \|\varphi(n_1) - \varphi(1)\| \\ &\ge c_2 c_\alpha^\delta (\frac{1}{|m - n_1|})^{(1+\alpha)\delta} - \varepsilon, \quad \forall \alpha \ge \alpha_0. \end{aligned}$$

It follows that, if

$$\begin{aligned} c_2 c_\alpha^\delta (\frac{1}{|m - n_1|})^{(1+\alpha)\delta} &\ge 2\varepsilon \\ &= 2(K^{-\delta})^{1+\alpha_0} \varepsilon_k^{1+\alpha} \\ &= 2(K^{-\delta})^{1+\alpha_0} (\frac{1}{m_{k+1}})^{\delta(1+\alpha)}, \end{aligned}$$

that is, if

$$|m-n_1| \le c_\alpha^{\frac{1}{1+\alpha}} (\frac{c_2}{2})^{\frac{1}{(1+\alpha)\delta}} (K^{-\delta})^{-\frac{1+\alpha_0}{(1+\alpha)\delta}} m_{k+1},$$

then $\varphi(m) \notin B_\varepsilon$. Thus the numbers of the elements, which are in the neighborhood, satisfy

$$\begin{aligned} M_n(\varepsilon) &\le c_\alpha^{-\frac{1}{1+\alpha}} (\frac{c_2}{2})^{-\frac{1}{(1+\alpha)\delta}} (K^{-\delta})^{\frac{1+\alpha_0}{(1+\alpha)\delta}} m_{k+1}^{-1} n \\ &< M_0 m_{k+1}^{-1} n, \\ M_0 &= \sup_{\alpha_0<\alpha<\alpha_1} c_\alpha^{-\frac{1}{1+\alpha}} (\frac{c_2}{2})^{-\frac{1}{(1+\alpha)\delta}} (K^{-\delta})^{\frac{1+\alpha_0}{(1+\alpha)\delta}}. \end{aligned}$$

Following the argument above for each $\varepsilon$-neighborhood $\varphi(l), l \in I_n$, we have

$$\frac{1}{n^2} \sum_{l,m=1}^{n} H(\varepsilon - \|\varphi(l)-\varphi(m)\|) \le \frac{1}{n^2} n M_n(\varepsilon) = \frac{M_n(\varepsilon)}{n}.$$

Thus we have

$$\begin{aligned} \frac{1}{n^2} \sum_{l,m=1}^{n} H(\varepsilon - \|\varphi(l)-\varphi(m)\|) &\le M_0 (\frac{1}{m_{k+1}}) \\ &= M_0 \varepsilon_k^{\frac{1}{\delta}} \\ &= M_0 ((K^{-\delta})^{-(1+\alpha_0)} \varepsilon)^{\frac{1}{(1+\alpha)\delta}} \\ &\le M_0 K \varepsilon^{\frac{1}{(1+\alpha_1)\delta}}. \end{aligned}$$

It follows that

$$\overline{N}(\varepsilon) = \limsup_{n\to\infty} \frac{1}{n^2} \sum_{l,m=1}^{n} H(\varepsilon - \|\varphi(l)-\varphi(m)\|) \le c\varepsilon^{\frac{1}{\delta(1+\alpha_1)}}$$

for every $\varepsilon > 0$. From the definition, we obtain

$$\begin{aligned} \underline{C}(\Sigma) &= \liminf_{\varepsilon\downarrow 0} \frac{\log \overline{N}(\varepsilon)}{\log \varepsilon} \\ &\ge \liminf_{\varepsilon\downarrow 0} \frac{\log c\varepsilon^{\frac{1}{\delta(1+\alpha_1)}}}{\log \varepsilon} \\ &= \frac{1}{(1+\alpha_1)\delta}, \quad \forall \alpha_1 > (1+\alpha_0)^2 - 1, \end{aligned}$$

which completes the proof. $\square$

For the case of Roth numbers, it follows from Theorem 1 that we can estimate its dimension from below by $\alpha_0 \to 0$.

THEOREM 2. *Assume* (iii) *and Hypothesis* (G2). *Then we have*

$$\underline{C}(\Sigma) \geq \frac{1}{\delta}. \tag{2.18}$$

Next we consider the relation between the box counting dimension $D_B$ and the correlation dimension $C$. For a subset $S \subset X$, let $M_\varepsilon(S)$, $\varepsilon > 0$, denote the minimum number of balls of $X$ with its radius $\varepsilon$ which is necessary to cover the subset $S$. The box dimension of $S$ is the number

$$D_B(S) = \lim_{\varepsilon \to 0} \frac{\log M_\varepsilon(S)}{\log 1/\varepsilon}. \tag{2.18}$$

Furthermore, $\overline{D_B}$ and $\underline{D_B}$ can be also defined by $\limsup, \liminf$ as $\varepsilon \to 0$, respectively. By using the well known relation between the box dimension and the correlation dimension, we can obtain the exact values of these dimensions in (i) the constant type case and (iii) the Roth numbers case.

COROLLARY 1. *Assume Hypotheses* (G1), (G2) *and* (iii). *Then we have*

$$D_B(\Sigma) = C(\Sigma) = \frac{1}{\delta}. \tag{2.19}$$

*Proof.* In [1], we have obtained the upper bound of the box dimension;

$$\frac{1}{\delta} \geq \overline{D_B}(\Sigma). \tag{2.20}$$

Thus it is sufficient to show that

$$\overline{D_B}(S) \geq \overline{C}(S) \tag{2.21}$$

for an infinite sequence $S = \{x_1, x_2, ..., x_n, ...\}$, which is relatively compact. Define $S_n = \{x_1, x_2, ..., x_n\}$. For a small $\varepsilon > 0$, instead of $M_\varepsilon(S)$, to estimate the box dimension, we can use the largest number, say $L_\varepsilon(S)$, of disjoint balls of radius $\varepsilon$ with centers in $S$ (see [5]). On the other hand, we can take $L_n$ disjoint balls of radius $\varepsilon$ with its centers, say $\{x_{i_1}, x_{i_2}, ..., x_{i_{L_n}}\}$, in $S_n$, which admit the property that there exists a constant $c : 0 < c < 1$ which satisfies

$$c < \frac{\sum_{j=1}^{L_n} l_j}{n} < 1$$

for every large $n$, where, for each $\varepsilon$-ball, $l_j$ denotes the number of the elements of $B_\varepsilon(x_{i_j}) \cap S_n$. Then we have

$$\sum_{j=1}^{L_n} l_j \leq n, \quad N_{2\varepsilon}^n(S_n) \geq \sum_{j=1}^{L_n} l_j^2$$

where $N_{2\varepsilon}^n(S_n) := \sum_{i,j=1}^n H(2\varepsilon - \|x_i - x_j\|)$. It follows that

$$N_{2\varepsilon}^n(S_n)L_n \geq (\sum_{j=1}^{L_n} l_j^2)L_n \geq (\sum_{j=1}^{L_n} l_j)^2.$$

Thus we have

$$\frac{1}{n^2}N_{2\varepsilon}^n(S_n) \geq L_n^{-1}(\frac{\sum_{j=1}^{L_n} l_j}{n})^2 > L_\varepsilon(S)^{-1}c^2,$$

since $L_\varepsilon$ is the largest number. Taking the limit as $n \to \infty$, we have

$$\underline{N}_{2\varepsilon}(S) \geq L_\varepsilon(S)^{-1}c^2.$$

It follows that

$$\frac{\log L_\varepsilon(S)c^2}{-\log \varepsilon} \geq \frac{\log \underline{N}_{2\varepsilon}(S)}{\log \varepsilon} = \frac{\log \underline{N}_{2\varepsilon}(S)}{\log 2\varepsilon - \log 2}.$$

By taking the limsup as $\varepsilon \to 0$ of the both side terms above, we obtain (2.21). $\square$

## 3. Examples of quasi Roth numbers

In this section, we introduce some examples of quasi Roth numbers and Roth numbers.

LEMMA 3. *Let $\{a_j\}$ be the partial quotients in the continued fraction expansion of $\tau$. Assume that, for a given constant $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$, which satisfies*

$$a_{j+1}a_j^2 \leq C_\varepsilon(a_{j-1}a_{j-2}\cdots a_1)^\varepsilon, \quad \forall j. \tag{3.1}$$

*Then we have*

$$|\tau - \frac{r}{l}| \geq \frac{c_\varepsilon}{l^{2+\varepsilon}}, \quad \forall l, r \in N \tag{3.2}$$

*where $c_\varepsilon = 1/(16C_\varepsilon)$.*

*Proof.* Let $l$ be a positive integer, then there exists a number $j$ : $m_{j-1} \leq l \leq m_j$ and we have

$$m_{j-1} \leq l \leq m_j \leq (a_j + 1)m_{j-1} \leq (a_j + 1)l. \tag{3.3}$$

Since $n_j/m_j$ is the best rational approximation, it follows from (3.3) that we have

$$\begin{aligned} |\tau - \frac{r}{l}| &\geq |\tau - \frac{n_j}{m_j}| \geq \frac{1}{(m_{j+1} + m_j)m_j} \\ &\geq \frac{1}{2(a_{j+1}+1)m_j^2} \geq \frac{1}{2(a_{j+1}+1)(a_j+1)^2 l^2} \end{aligned}$$

for every $r \in N$. Since

$$(a_{j+1} + 1)(a_j + 1)^2 \leq 8a_{j+1}a_j^2,$$

it follows from (3.1) that

$$(a_{j+1} + 1)(a_j + 1)^2 \leq 8C_\varepsilon(a_{j-1}a_{j-2}\cdots a_1)^\varepsilon.$$

On the other hand, we can estimate

$$\begin{aligned} l \geq m_{j-1} &\geq a_{j-1}m_{j-2} \geq \cdots \\ &\geq a_{j-1}a_{j-2}\cdots a_1 m_0 \\ &= a_{j-1}a_{j-2}\cdots a_1. \end{aligned}$$

Thus we obtain the conclusion. □

In the following examples, we can show that the properties of the rational approximation heavily depend on the growth rates of the continued fraction expansions $\{a_j\}$. For two sequences $\{a_j\}$ and $\{b_j\}$, we write $a_j \sim b_j$ if there exist constants $c_1, c_2 > 0$ such that

$$c_1 b_j < a_j < c_2 b_j.$$

EXAMPLE 1. If $a_j \sim j^\alpha,\ \alpha > 0$, then $\tau$ is a Roth number.
In fact, for every $\varepsilon > 0$ there exists $d_\varepsilon$:

$$(3.4) \qquad (j+1)^{\frac{3}{\varepsilon}} c_2^{\frac{3}{\alpha\varepsilon}} c_1^{-\frac{j-1}{\alpha}} \leq d_\varepsilon (j-1)!, \qquad \forall j.$$

It follows that

$$c_2^3 (j+1)^{3\alpha} \leq d'_\varepsilon \{c_1^{\frac{j-1}{\alpha}} (j-1)!\}^{\alpha\varepsilon}$$

and we have

$$a_{j+1}^3 < d'_\varepsilon (a_{j-1}a_{j-2}\cdots a_1)^\varepsilon.$$

Thus we can apply Lemma 3 for every $\varepsilon > 0$.

EXAMPLE 2. If $a_j \sim K^j,\ K > 1$, then $\tau$ is also a Roth number.

In fact, for every $\varepsilon > 0$ there exists $j_\varepsilon$:

$$c_2^3 K^{(3+\frac{\log c_1^{-\varepsilon}}{\log K})j_\varepsilon + 1} < c_1^{-\varepsilon} K^{\frac{(j_\varepsilon - 1)j_\varepsilon}{2}\varepsilon}.$$

Put

$$d_\varepsilon = c_2^3 K^{(3+\frac{\log c_1^{-\varepsilon}}{\log K})j_\varepsilon + 1}.$$

Then we have

$$c_2^3 K^{3j+1} < d_\varepsilon (c_1^{j-1} K^{j-1} \cdots K^2 K^1)^\varepsilon, \quad \forall j,$$

which yields Hypothesis of Lemma 3.

EXAMPLE 3. If $a_{j+1} \sim m_j^\beta, \ \ \beta > 0$, then Hypothesis **(B)** is satisfied and it follows from Lemma 1 that $\tau$ is a quasi Roth number: $\alpha_0 = \beta(\beta+3)$.

EXAMPLE 4. Here we consider the case that the growth rate of $a_j$ has the order $M^{\kappa^j}, \ M, \kappa > 1$.

THEOREM 3. *For constants $c_1, c_2, M, \kappa, \alpha : \ \ M, \kappa > 1, \ \ \alpha \geq 1$, assume that the partial quotients in the continued fraction expansion of $\tau$ satisfies*

$$c_1 M^{\kappa^j} < a_j < c_2 (M^\alpha)^{\kappa^j}. \tag{3.5}$$

*Then $\tau$ is a quasi Roth number: $\alpha_0 = (\kappa-1)(\kappa+2)\alpha$.*

*Proof.* First we consider the case $c_1 > 1$. Let $\varepsilon \geq (\kappa-1)(\kappa+2)\alpha$, then we have

$$\begin{aligned}
&\frac{\kappa}{\kappa-1}(\kappa^{j-1}-1)\varepsilon + \frac{\kappa}{\kappa-1}\varepsilon \\
&\quad = \frac{\kappa}{\kappa-1}\kappa^{j-1}\varepsilon \\
&\quad \geq \kappa\kappa^{j-1}(\kappa+2)\alpha.
\end{aligned}$$

It follows that

$$(M^\alpha)^{\kappa^{j+1}} (M^\alpha)^{2\kappa^j} \leq M^{\frac{\kappa}{\kappa-1}\varepsilon} M^{(\kappa^1+\kappa^2+\cdots+\kappa^{j-1})\varepsilon}.$$

Thus we can apply Lemma 3, since we have

$$\begin{aligned}
a_j^2 a_{j+1} &\leq c_2^3 (M^\alpha)^{\kappa^{j+1}} (M^\alpha)^{2\kappa^j} \\
&\leq c_2^3 M^{\frac{\kappa}{\kappa-1}\varepsilon} M^{(\kappa^1+\kappa^2+\cdots+\kappa^{j-1})\varepsilon} \\
&\leq C_\varepsilon (a_1 a_2 \cdots a_{j-1})^\varepsilon.
\end{aligned}$$

Next we consider the case $0 < c_1 < 1$. Take a constant $r : 0 < r < 1$, $Mr > 1$ and put $m = Mr$. Then, for a large $j_0$, we have

$$c_1 (r^{-1})^{\kappa^{j_0}} > 1$$

and

$$c_1 (r^{-1})^{\kappa^{j_0}} m^{\kappa^j} < a_j < c_2 M^{\alpha \kappa^j}$$

for every $j \geq j_0$. Since

$$M^\alpha = m^{\alpha \log M / (\log M + \log r)},$$

it follows from the above argument that there exists a constant $C'_\varepsilon$:

$$a_{j+1} a_j^2 \leq C'_\varepsilon (a_1 a_2 \cdots a_{j-1})^\varepsilon$$

for every $j \geq j_0$ and for every $\varepsilon$, which satisfies

$$\varepsilon \geq (\kappa + 2)(\kappa - 1)\alpha \cdot \frac{\log M}{\log M + \log r}. \tag{3.6}$$

Put

$$C_\varepsilon = \max_{j=1,\dots,j_0} \{C'_\varepsilon,\ a_{j+1} a_j^2 / (a_1 a_2 \cdots a_{j-1})^\varepsilon\}.$$

Then we can apply Lemma 3. Since (3.6) holds for every $r : 0 < r < 1$ and $Mr > 1$, we can conclude that

$$\alpha_0 = (\kappa + 2)(\kappa - 1)\alpha.$$

□

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