

COMPUTATION AND ANALYSIS OF MATHEMATICAL MODEL FOR MOVING FREE BOUNDARY FLOWS

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ABSTRACT. The nonlinear stage of the evolution of free boundary between a light fluid and a heavy fluid driven by an external force is studied by a potential flow model with a source singularity. The potential flow model is applied to a bubble and spike evolution for constantly accelerated interface (Rayleigh-Taylor instability) and impulsively accelerated interface (Richtmyer-Meshkov instability). The numerical results of the model show that, in constantly accelerated interface, bubble grows with constant velocity and the spike falls with gravitational acceleration at later times, while the velocity of the bubble in impulsively accelerated interface decay to zero asymptotically. We also derive the asymptotic solutions of the potential flow model for the bubble and spike for constantly accelerated interface and impulsively accelerated interface.

1. Introduction

A free boundary interface accelerated by a gravitational force pointing from a heavy fluid to a light fluid, or by a shock wave, is unstable. The former is known as Rayleigh-Taylor instability [1] and the latter is known as Richtmyer-Meshkov instability [2]. These interfacial instabilities play an important role in the study of inertial confinement fusion and supernova. Progress on the study of the gravity-induced and shock-induced instabilities can be found in and traced from the references [3-12].

Small perturbations at the gravity-induced and shock-induced unstable interfaces grow into nonlinear structures having the form of *bubbles*

Received December 25, 1999. Revised May 10, 2000.

2000 Mathematics Subject Classification: 76B07, 76E17, 76L05.

Key words and phrases: potential flow, source singularity, bubble, spike, hydrodynamic instability.

The author was partially supported by KOSEF.

and *spikes*. A bubble (spike) is a portion of the light (heavy) fluid penetrating into the heavy (light) fluid. The evolution of the gravity-induced and shock-induced instabilities proceeds along the three stages: linear growth, early nonlinear growth, and late nonlinear growth. At the first linear stage, the amplitude of the interface is small and the shape of the spike and bubble are approximately equal. At the early nonlinear stage, the bubble and spike are distinctively formed and move with different velocity. At the late nonlinear stage, the bubble has a steady-state growth rate and the spike has a mushroom-shaped vortex structure. The linear stage is well understood through the literature [1, 2, 9]. However, more important one is the dynamics of nonlinear stages. The theory presented in this paper is for the development of the gravity-induced and shock-induced instabilities from early nonlinear to late nonlinear stages.

A number of numerical simulations on the nonlinear growth of the gravity-induced and shock-induced instabilities have been performed [6, 7]. However, most of these simulations are for the time range of the early nonlinear stage. In fact, it is difficult to investigate the late nonlinear stage by numerical simulations, since the problems involves highly distorted interfaces. The late nonlinear stage requires a several factor of longer time simulations than the linear or early nonlinear stage and, in the shock-induced case, needs a long computational domain to include the effect of the transmitted shock and the reflected wave to the interface. The current computational resources and numerical methods do not permit this computation in a finite time. Therefore, the theoretical study of the late nonlinear stage for these interfacial instabilities is very important and is the main purpose of this paper.

Our theory is based on the potential flow model by Zufria[5]. The potential flow model studies the motion at the tip of the bubble in a system with an infinite density ratio. The bubble is modeled by the complex potential with a source singularity inside the bubble. The model approximates the shape of the material interface near the tip of the bubble as a parabola and derives a set of ordinary differential equations which determines the position and local radius of the bubble tip, and the source strength.

The Zufria's potential flow model with a source singularity has been applied to the gravity-induced bubble only and the numerical solutions for the gravity-induced bubble are studied [5]. In this paper, we show

that the potential flow model with a source singularity model is applicable to the gravity-induced spike and the shock-induced bubble as well. Furthermore, we derive, for the first time, the analytic solutions of the potential flow model with source singularity for both bubble and spike at asymptotic large times. So far, the theory for the spike has been rarely developed and the analytic solutions of the potential flow model are not known due to the complication of the point source in the velocity potential.

The earliest analytic model for the bubble is the another potential flow model of Layzer[3]. The main difference between the two models is that, in the Layzer's model, the velocity potential is an analytical function, while in the Zufria's model, the velocity potential has a point source (singularity). Layzer pointed out in his paper that the bubble is better modeled by the potential with source [3]. We compare the results of two models in Section 3 and 4.

In Section 2, we present the potential flow model with a source singularity for the bubble and spike in incompressible fluids with infinite density ratio. In Section 3, we apply the potential flow model to the gravity-induced spike and the shock-induced bubble. The analytic expressions of asymptotic solutions for the gravity-induced bubble and spike, and the shock-induced bubble are derived in Section 4. Section 5 gives conclusions.

2. Potential flow model

In this section, we present an analytical model for the bubble and spike evolution, based on the potential flow with a source singularity in incompressible fluids of an infinite density ratio. We derive the equations for a bubble evolution. The equation for a bubble is directly applicable to the case of spike by changing signs of variables.

The potential with a source singularity gives a nice description for the front of the bubble and spike, but not correct for the flow in the far field behind the bubble and spike. In reality, the spikes (bubbles) are formed in the behind of the bubbles (spikes). Therefore, the potential flow model assumes that the spike (bubble) gives a little influence to the dynamic of the bubble (spike).

Consider the bubble rising in a vertical channel filled with an incompressible irrotational inviscid flow. The density of the light fluid is zero.

From the assumption of the flow, there exists a complex potential for the bubble $W(z) = \phi + i\psi$, where ϕ and ψ are the velocity potential and the stream function, respectively. The position of the bubble tip are $Z(t) = X(t) + iY(t)$. The bubble is characterized by the velocity of the bubble tip U , the local radius of curvature R , and the location of the bubble tip with respect to a frame of reference attached to the channel. We describe the evolution in a comoving frame (\hat{x}, \hat{y}) rising with the bubble. Then the origin of coordinate (\hat{x}, \hat{y}) is attached to the bubble tip $x = X(t), y = L/2$ moving with the bubble velocity U in the x direction, where L is a channel width. The shape of the interface near the bubble tip, under the parabola approximation [3], is

$$(1) \quad \eta(\hat{x}, \hat{y}, t) = \hat{y}^2 + 2R(t)\hat{x} = 0.$$

The bubble evolution can be determined by the kinematic boundary condition

$$(2) \quad \frac{D\eta(\hat{x}, \hat{y}, t)}{Dt} = 2\frac{dR}{dt}\hat{x} + 2Ru + 2\hat{y}v = 0$$

and the Bernoulli equation

$$(3) \quad \frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + g\hat{x} + \frac{dU}{dt}\hat{x} = 0$$

where u and v are two components of fluid velocity in the comoving frame with the vertex of the bubble, respectively, and g is an external acceleration.

The potential for the bubble can be modeled by the complex potential with a source singularity inside the bubble. The potential for a source is described by $W = Q \log \hat{z}$ where Q denotes the source strength. Then, the potential of a source at the origin in a channel of width L , $W = Q \log \sinh(\pi\hat{z}/L)$, can be derived by Schwart-Christoffel transformation [13]. Combining a source located at a distance A below the bubble tip with a uniform incoming flow, we obtain the resulting source potential

$$(4) \quad W(\hat{z}) = Q \log \left[\frac{\sinh \frac{k}{2}(\hat{z} + A)}{\exp \frac{k}{2}(\hat{z} + A)} \right] - U\hat{z}.$$

Here $k = 2\pi/\lambda$ where $\lambda = L$ denotes a wavelength of the bubble. The velocity potential and the stream function are given by

$$(5) \quad \phi = \frac{Q}{2} \log[\cosh k(\hat{x} + A) - \cos k\hat{y}] - \left(\frac{1}{2}kQ + U\right)\hat{x},$$

$$(6) \quad \psi = Q \arctan[\coth \frac{k}{2}(\hat{x} + A) \tan \frac{k}{2}\hat{y}] - \left(\frac{1}{2}kQ + U\right)\hat{y}.$$

Notice that $\phi \rightarrow -U\hat{x}$ as $\hat{x} \rightarrow +\infty$, and $\phi \rightarrow -e^{kA}U\hat{x}$ as $\hat{x} \rightarrow -\infty$, from Eq. (5) and $\log[\cosh k(\hat{x} + A)] \sim -k\hat{x}$ as $\hat{x} \rightarrow -\infty$.

Expanding (4) in powers of \hat{z} , we have

$$(7) \quad W = Q \sum_{i=0}^{\infty} \frac{c_i}{i!} \hat{z}^i - U\hat{z}.$$

The expressions for coefficients c_i are given shortly.

From the relation $\frac{dW}{d\hat{z}} = u - iv$ and Eq. (1) and (7), the velocity u and v are

$$(8) \quad u = Q[c_1 + (c_2 + c_3R)\hat{x}] - U + O(\hat{x}^2),$$

$$(9) \quad v = -c_2Q\hat{y} + O(\hat{x}^{3/2}).$$

Substituting (8) and (9) into Eq. (2), and the zeroth and first order equations in \hat{x} gives

$$(10) \quad \frac{dX}{dt} = U = c_1Q,$$

$$(11) \quad \frac{dR}{dt} = -Q(3c_2 + c_3R)R.$$

The expressions for c_1, c_2 and c_3 are given shortly.

Substituting expansions in powers of \hat{x} for (5), or real part of (7), the first and second order equations in \hat{x} gives

$$(12) \quad (c_1 + c_2R)\frac{dQ}{dt} + Q(c_2 + c_3R)\frac{dA}{dt} = Q^2c_2^2R - g,$$

$$(13) \quad \left(\frac{c_2}{2} + c_3R + c_4\frac{R^2}{6}\right)\frac{dQ}{dt} + Q\left(\frac{c_3}{2} + c_4R + c_5\frac{R^2}{6}\right)\frac{dA}{dt} \\ = -\frac{Q^2}{2} \left(c_2^2 - 2c_2c_3R + (3c_3^2 - 4c_2c_4)\frac{R^2}{3} \right),$$

using the fact $\frac{d}{dt}c_i = c_{i+1}\frac{dA}{dt}$, $i \geq 1$. Here, the expression for c_i are

$$\begin{aligned}
 c_1 &= \frac{k}{e^{kA} - 1}, & c_2 &= -\frac{k^2 e^{kA}}{(e^{kA} - 1)^2}, \\
 c_3 &= \frac{k^3 e^{kA} (e^{kA} + 1)}{(e^{kA} - 1)^3}, & c_4 &= -\frac{k^4 e^{kA} (e^{2kA} + 4e^{kA} + 1)}{(e^{kA} - 1)^4}, \\
 (14) \quad c_5 &= \frac{k^5 e^{kA} (e^{3kA} + 11e^{2kA} + 11e^{kA} + 1)}{(e^{kA} - 1)^5}.
 \end{aligned}$$

The evolution of a bubble is determined by the system of four first order ordinary differential equations (10), (11) and (12), (13) with given initial conditions.

The streamline through an arbitrary reference point (\hat{x}_0, \hat{y}_0) is defined by $\psi(\hat{x}, \hat{y}, t) = \psi(\hat{x}_0, \hat{y}_0, t)$. The curve obtained by the streamline $\psi = 0$ through the stagnation point at $\hat{x} = \hat{y} = 0$ determines the bubble profile. The analytic expression of the bubble profile is found to be

$$(15) \quad y = \frac{1}{k} \log \left[\frac{\sin(\epsilon kx)}{\sin(\alpha \epsilon kx)} \right] - A,$$

where $\alpha = e^{-kA}$ and $\epsilon = \frac{1}{1-\alpha}$.

3. Numerical results

We show the results of the gravity-induced spike and the shock-induced bubble, based on the potential flow model presented in the previous section. We apply a standard numerical method for ordinary differential equations, i.e. the fourth order Runge-Kutta method, to solve the system of equations (10)-(13). The results for the bubble in gravity-induced case are studied in Ref. [5] and it has been found that gravity-induced bubble attains a constant asymptotic velocity.

For the shock-induced bubble, we consider the equations (10)-(13) with $g = 0$. This is an impulsive approximation for an incompressible flow. We showed that the shock-induced unstable system changes from the initial approximately linear and compressibility dominant regime to nonlinearity dominant and approximately incompressible regime [11]. The results of the shock-induced bubble evolution are given in Figure 1 for the initial conditions $R = 0.2, Q = 0.1, A = 0.1$ and the channel width $L = \pi$. The Figure shows that the bubble velocity U and the local radius of the bubble R the values 1.30. We have checked that and the source strength Q decay to zero asymptotically and the distance of

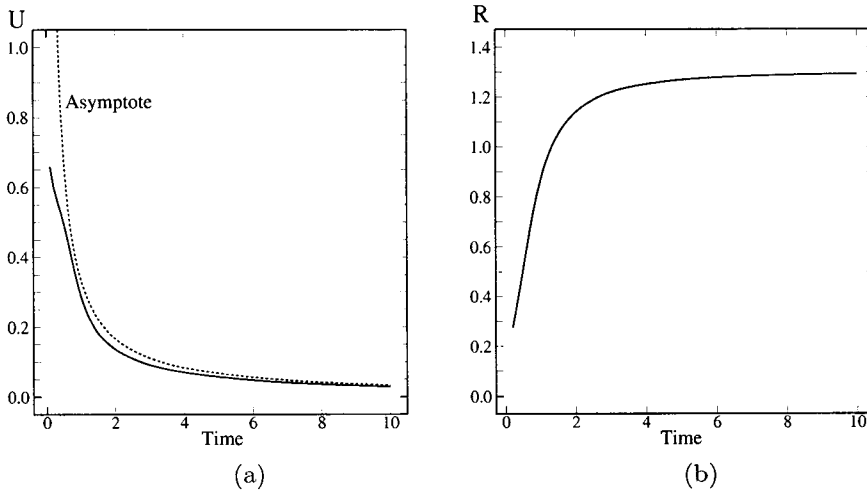


FIGURE 1. Results of the bubble evolution of shock-induced instability. (a) bubble velocity U and asymptote of bubble growth rate $2/3kt$, (b) Radius of the bubble tip R .

the source from the bubble tip A reach 1.33. In fact, for any initial conditions, R and A have a steady state solution $R^\infty = 1.30$ and $A^\infty = 1.33$, and U and Q decay to zero asymptotically. Recently, Hecht *et al.* [10] determined an asymptotic bubble growth rate of $2/3kt$ from the Layzer's analytic potential flow model. In Figure 1(a), we compare this asymptote with our result of the bubble velocity. It shows two solutions are in good agreement at late times. It is interesting to note that the shock-induced asymptotic bubble radius, $R_{\text{shock}}^\infty = 1.30$, is larger than the gravity-induced asymptotic bubble radius, $R_{\text{gravity}}^\infty = 0.866$ [5].

For the gravity-induced spike, we apply the same equations as the bubble case, i.e., (10)-(13), but need to change the sign of the state variables. Since the direction of the spike evolution is opposite to the one of the bubble evolution, the signs of state variables should be $R < 0$, $Q > 0$, $A < 0$ and $U < 0$. The results of the gravity-induced spike evolution is given in Figure 2 for the initial conditions $R = -0.2$, $Q = 0.1$, $A = -0.1$. g is set to 1 and the channel width is $L = \pi$. The Figure shows that the spike falls with a gravitational acceleration, which agrees with the

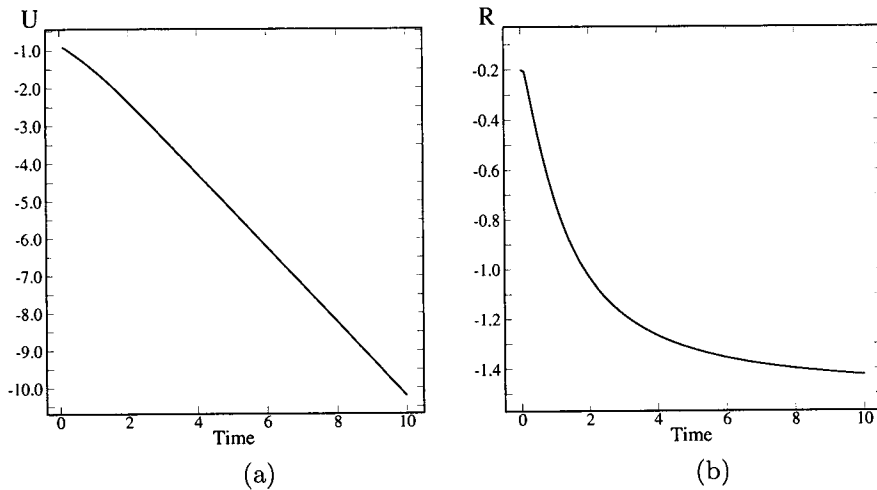


FIGURE 2. Results of the spike evolution of the gravity-induced instability. (a) spike velocity U , (b) Radius of the spike tip R .

previously published results of full numerical simulations [6], and the local radius of the spike R converges to the value -1.50 asymptotically. We have checked that the source strength Q grows linearly in time and the slope of the curve of source strength is 0.50 , and the distance of the source from the spike tip A have a logarithmic growth asymptotically.

4. Asymptotic analysis

We derive the analytical solutions for the large time of a gravity-induced bubble and spike, and a shock-induced bubble for the potential flow model presented in Section 2. We shall use the superscript ∞ to indicate a quantity at asymptotic large time.

For a bubble of the gravity-induced instability, $\frac{dR^\infty}{dt} = \frac{dQ^\infty}{dt} = \frac{dA^\infty}{dt} = 0$ and the asymptotic solutions are the critical points of the equations (10)-(13) :

$$(16) \quad U^\infty = c_1^\infty Q^\infty,$$

$$(17) \quad 3c_2^\infty + c_3^\infty R^\infty = 0,$$

$$(18) \quad (Q^\infty c_2^\infty)^2 R^\infty - g = 0,$$

$$(19) \quad (c_2^\infty)^2 - 2c_2^\infty c_3^\infty R^\infty + [(c_3^\infty)^2 - \frac{4}{3}c_2^\infty c_4^\infty](R^\infty)^2 = 0.$$

From (17), we have

$$(20) \quad R^\infty = -3 \frac{c_2^\infty}{c_3^\infty}.$$

After substituting this expression into (19), we have

$$(21) \quad \xi^2 - 4\xi + 1 = 0.$$

Here $\xi = e^{kA^\infty}$. Therefore there two possible solutions for A^∞ : $\frac{1}{k} \ln(2 + \sqrt{3})$ and $\frac{1}{k} \ln(2 - \sqrt{3})$. Since A^∞ is positive, only the former can be a solution.

From (16), (18), (20), and the expression for A , the solution for R, Q, U can be derived.

THEOREM 1. *The asymptotic solutions for large times of a gravity-induced bubble governed by the system of ordinary differential equations (10) – (13) with $g > 0$ for initial conditions $R > 0, A > 0, Q > 0, U > 0$ are*

$$(22) \quad \begin{aligned} R^\infty &= \frac{\sqrt{3}}{k}, & A^\infty &= \frac{1}{k} \ln(2 + \sqrt{3}), \\ Q^\infty &= \frac{2}{3^{1/4}} \sqrt{\frac{g}{k^3}}, & U^\infty &= \sqrt{1 - \frac{1}{(2 + \sqrt{3})^2}} \sqrt{\frac{g}{3k}}. \end{aligned}$$

In the Layzer’s analytic potential flow model, the asymptotic bubble velocity in gravity-induced instability is $U_{layzer}^\infty = \sqrt{\frac{g}{3k}}$ [3]. Therefore both the analytic potential flow model and the potential model with source singularity give same qualitative behavior of constant asymptotic bubble velocity. The quantitative difference between the asymptotic velocity of the two models is about 3 percent ($U_{layzer}^\infty = 0.577\sqrt{\frac{g}{k}}$ and $U_{zufiria}^\infty = 0.556\sqrt{\frac{g}{k}}$).

For a bubble of the shock-induced instability, one may express the late time bubble velocity in the form $U^\infty \sim \frac{\xi}{t^\alpha}$. From (10), Q^∞ is also proportional to $t^{-\alpha}$ at late time, namely $Q^\infty \sim \frac{\delta}{t^\alpha}$. From (11), (12) and (13), one concludes that α is equal to 1, and $\frac{dR^\infty}{dt}$ and $\frac{dA^\infty}{dt}$ have to tend to zero faster than t^{-1} . Otherwise it will contradict to the fact that A^∞

and Q^∞ have asymptotic limits. Thus at late time, (10)-(13) can be expressed as:

$$(23) \quad \epsilon = c_1^\infty \delta,$$

$$(24) \quad 3c_2^\infty + c_3^\infty R^\infty = 0,$$

$$(25) \quad -(c_1^\infty + c_2^\infty R^\infty) = \delta(c_2^\infty)^2 R^\infty,$$

$$(26) \quad [c_2^\infty + 2c_3^\infty R^\infty + \frac{1}{3}c_4^\infty (R^\infty)^2] = \delta\{(c_2^\infty)^2 - 2c_2^\infty c_3^\infty R^\infty + [(c_3^\infty)^2 - \frac{4}{3}c_2^\infty c_4^\infty](R^\infty)^2\}.$$

From (14) and the second and the third equations in (25), we have

$$(27) \quad \delta = -\frac{1}{c_2^\infty} \left[1 - \frac{c_1^\infty c_3^\infty}{3(c_2^\infty)^2} \right].$$

After substituting this expression and (14) into (26), we obtain a cubic equation:

$$(28) \quad \xi^3 - 15\xi^2 + 9\xi - 5 = 0$$

where $\xi = e^{kA^\infty}$. This equation has only one real root given by

$$(29) \quad \xi = 5 + (105 + \sqrt{377})^{1/3} + (105 - \sqrt{377})^{1/3}.$$

Thus, from (23)-(26), the asymptotic solution for a bubble at shock-induced unstable interface, from the point source model is obtained.

THEOREM 2. *The asymptotic solutions for large times of a shock-induced bubble governed by the system of ordinary differential equations (10)-(13) with $g = 0$ for initial conditions $R > 0$, $A > 0$, $Q > 0$, $U > 0$ behave as*

$$(30) \quad R^\infty = \frac{3(\xi - 1)}{k(\xi + 1)}, \quad A^\infty = \frac{1}{k} \ln \xi, \\ Q^\infty \sim \frac{(2\xi - 1)(1 - \xi)^2}{3\xi^2 k^2 t}, \quad U^\infty \sim \left(\frac{2}{3} - \frac{1}{\xi} + \frac{1}{3\xi^2} \right) \frac{1}{kt},$$

where ξ is given by (29).

In the Layzer's analytic potential flow model, the asymptotic bubble velocity in shock-induced instability is $U_{layzer}^\infty = 2/(3kt)$ [10]. Therefore both the analytic potential flow model and the potential model with source singularity give same qualitative late time behavior of $1/t$ decay.

For a spike of the gravity-induced instability, one may assume that $\frac{dU^\infty}{dt}$ has a limit at large time, since in this case the spike is a free fall down to the vacuum under the gravitational force. From (10), $\frac{dQ^\infty}{dt}$ also has a limit at large time. (we show this shortly.) Therefore we express $\frac{dU^\infty}{dt}$ and $\frac{Q^\infty}{dt}$ as $\frac{U^\infty}{dt} = \gamma$ and $\frac{Q^\infty}{dt} = \tau$. From (12) and (13), it is easy to see that $c_i, 2 \leq i \leq 5$, should tend to zero as $1/t^2$ and $\frac{dA^\infty}{dt} \sim -2/(kt)$. Thus, A behaves as

$$(31) \quad A \sim -\frac{2}{k} \ln t.$$

At asymptotic large time, (11) and (12) can be expressed as:

$$(32) \quad 3c_2^\infty + c_3^\infty R^\infty = 0,$$

$$(33) \quad (c_1^\infty + c_2^\infty R^\infty)\tau = -g.$$

Substituting (31) into (32) and (33), we have

$$(34) \quad R^\infty = -\frac{3}{k}, \quad \frac{dQ^\infty}{dt} = \frac{g}{k}.$$

Differentiating (10) gives

$$(35) \quad \frac{dU^\infty}{dt} = c_2 Q \frac{dA^\infty}{dt} + c_1 \frac{dQ^\infty}{dt}.$$

Since the first term on the right hand side decays to zero, this shows that the behavior of $\frac{dU^\infty}{dt}$ is same as $\frac{dQ^\infty}{dt}$. Therefore, we obtain the solution for $\frac{dU^\infty}{dt}$.

THEOREM 3. *The asymptotic solutions for large times of a gravity-induced spike governed by the system of ordinary differential equations (10)-(13) with $g > 0$ for initial conditions $R < 0, A > 0, Q < 0, U < 0$ are*

$$(36) \quad \begin{aligned} R^\infty &= -\frac{3}{k}, & A &\sim -\frac{2}{k} \ln t, \\ \frac{dQ^\infty}{dt} &= \frac{g}{k}, & \frac{dU^\infty}{dt} &= -g. \end{aligned}$$

We note that all asymptotic solutions derived in this section agree with the numerical results in the previous section.

5. Conclusions

The potential flow model with a source singularity has been applied to the bubble and spike evolution for the gravity-induced and the shock-induced instability in incompressible fluids of an infinite density ratio. The model predicts that the gravity-induced bubble grows with constant velocity and the gravity-induced spike falls with gravitation acceleration at later times, while the velocity of the shock-induced bubble decay to zero asymptotically. The results of the potential flow model with a source singularity agree with the analytic potential flow model and the full numerical simulations.

We derived, for the first time, the analytic expressions of asymptotic solutions of the potential flow model for the bubble and spike in the gravity-induced instability and the bubble in the shock-induced instability. The evolution of the spike in the shock-induced instability is more complex than that in the gravity-induced instability and the asymptotic solutions for this is left as a conjecture.

The potential flow model with a source singularity is applicable to the multiple bubble interactions [5, 8, 12]. At the late nonlinear stage of initial multi-mode perturbations, bubbles of different radii propagates with different velocities, and the leading bubbles grow in size at the expense of their neighboring bubbles. This phenomenon is known as bubble interaction, or bubble competition. The detailed study of multiple bubble interactions of the potential flow model and the comparison with full numerical simulations will be the next step of our research.

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