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BIFURCATION OF BOUNDED SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

JAMES ROBERT WARD, JR.

ABSTRACT. Conley index is used to study bifurcation from equilibria of full bounded solutions to parameter dependent families of ordinary differential equations of the form $\frac{dx}{dt} = \varepsilon F(x,t,\mu)$. It is assumed that $F(x,t,\mu)$ is uniformly almost periodic in t.

1. Introduction

The concepts and results of the Leray-Schauder degree and index theories have been very effectively applied to prove global continuation and bifurcation of solutions to nonlinear equations in a Banach space. The homotopy invariance of degree under certain conditions is a fundamental property generally used to establish continuation or bifurcation of solutions. The Conley index also has invariance properties that make it a useful tool to study some bifurcation phenomena in dynamical systems. In this paper we study non-autonomous ordinary differential systems, and use the Conley index to prove the existence of continua of full bounded solutions bifurcating from the trivial solution of these systems. More precisely, we study parameter dependent families of ordinary differential equations of the form

$$\frac{dx}{dt} = \varepsilon F(x,t,\mu) \tag{1}$$

where F is a continuous function of $(x,t,\mu) \in \Lambda := \Omega \times \mathbb{R} \times \mathbb{R}$, $\Omega \subset \mathbb{R}^m$ is an open set and $\varepsilon, \mu \in \mathbb{R}$ are parameters. By a (full) bounded solution

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of (1) we mean a solution $x = x(t)$ satisfying (1) for all $t \in \mathbb{R}$ and such that

$$\|x\| := \sup_{t\in\mathbb{R}} |x(t)| < \infty.$$

Throughout this paper, we assume $F(0,t,\mu) = 0$ for all $(t,\mu) \in \mathbb{R} \times R$, so that $x = 0$ is an equilibrium value for all μ and ε. The parameter μ will generally be a bifurcation parameter, while ε will generally be a real number of small magnitude. We will also assume that $F(x,t,\mu)$ is almost periodic in $t \in \mathbb{R}$, uniformly for (x,μ) in compact sets (see [3] or [10]). Let

$$F_0(x,\mu) := \lim_{T\to\infty} \frac{1}{T}\int_0^T F(x,t,\mu)\,dt.$$

We will study the bifurcation of full bounded solutions of (1) making use of the averaged equations

$$\frac{dx}{dt} = F_0(x,\mu). \tag{2}$$

Proofs are based upon applying Conley index theory to a family of skew-product flows associated with (1), and obtaining indices via homotopy to the associated averaged equations. Notice that the Conley index cannot be applied directly to nonautonomous differential equations, since the solutions of such equations do not define a flow (dynamical system) on the space of initial values. For this reason we use the associated skew product flows. For the properties of the homotopy index we refer the reader to [2] or [9], and for skew product flows to [10] or [5]; the use of these in the present context is discussed in [11], [12], and [13].

We will make use of a weak topology on our families of differential equations. This topology was studied by Artstein [1] in much greater depth and generality than will be needed here. Let $\Omega \subset \mathbb{R}^m$ be an open set and $f : \Omega\times\mathbb{R} \to \mathbb{R}^m$. Suppose f satisfies the Carathéodory conditions: for each $x \in \Omega$ the function $t \to f(x,t)$ is Lebesgue measurable, and for almost all $t \in \mathbb{R}$ (in the sense of Lebesgue measure) the function $x \to f(x,t)$ is continuous on Ω. Suppose

(C1) For every compact set $A \subset \Omega$ there exist two locally L^1 functions $m_A(t)$ and $k_A(t)$ such that if $x, y \in A$ and $t \in \mathbb{R}$ then:

(1) $|f(x,t)| \le m_A(t)$,

(2) $|f(x,t) - f(y,t)| \le k_A(t)\,|x-y|$,

(3) for every $\epsilon > 0$ there exists a $\delta = \delta_A(\epsilon) > 0$ such that if $E \subset \mathbb{R}$

is measurable, contained in an interval $[t, t+1]$, and with measure less than δ then $\int_E m_A(t)\, dt \leq \epsilon$, and

(4) there exists a number N_A such that $\int_t^{t+1} k_A(s)\, ds \leq N_A$ for all $t \in \mathbb{R}$.

Given a function f satisfying *(C1)* one can define an associated set of functions $\mathcal{G}$ on $\Omega \times \mathbb{R}$ that contains the time translates of f, defined for $\tau \in \mathbb{R}$ by $f_\tau(x,t) = f(x, \tau + t)$ for all $(x,t) \in \Omega \times \mathbb{R}$. With an appropriate topology on $\mathcal{G}$, one can then define skew-product flows on $\Omega \times \mathcal{G}$.

DEFINITION 1([1]). Let f satisfy *(C1)* and for every compact set $A \subset \Omega$ and $\epsilon > 0$ let N_A and $\delta_A(\epsilon)$ be given by *(C1)*. The family $\mathcal{G} = G(f)$ consists of all Carathéodory functions $g : \Omega \times \mathbb{R} \to \mathbb{R}^m$ satisfying : For every compact $A \subset \Omega$ there exist two locally L^1 functions $M_{A,g}$ and $K_{A,g}$ such that if $x, y \in A$ and $t \in \mathbb{R}$ then

(1) $|g(x,t)| \leq M_{A,g}(t)$,

(2) $|g(x,t) - g(y,t)| \leq K_{A,g}(t)\, |x-y|$, and the functions $M_{A,g}$ and $K_{A,g}$ *satisfy:*

(3) if $E \subset [t, t+1]$ and the Lebesgue measure of E is less than $\delta_A(\epsilon)$ then $\int_E M_{A,g}(s) ds \leq \epsilon$ and

(4) $\int_t^{t+1} K_{A,g}(s) ds \leq N_A$ for all $t \in \mathbb{R}$.

If $g \in \mathcal{G}(f)$ then so is g_τ for any $\tau \in \mathbb{R}$. Moreover, for each $x_0 \in \Omega$ the initial value problem

$$\frac{dx}{dt} = g(x,t),\ \ x(0) = x_0$$

has a unique solution $x(t; x_0, g)$ defined on a maximal interval of existence $I(x_0, g) = (\alpha(x_0, g), \beta(x_0, g))$. Artstein in [1] gives the space $\mathcal{G}$ a weak metrizable topology which we will impose. This is given by

DEFINITION 2([1]). Let $\{g_k\}$ be a sequence in $\mathcal{G}$. We say $\{g_k\}$ converges (weakly) to $g \in \mathcal{G}$ provided for every $x \in \Omega$ and $t \in \mathbb{R}$ the sequence $\{\int_0^t g_k(x,s)\, ds\}$ converges in $\mathbb{R}^m$ to $\int_0^t g(x,s)\, ds$.

Convergence in $\mathcal{G}$ is induced by a metric, which is explicitly given in [1]. The topological space $\mathcal{G}$ is compact and is closed under time translations. A local flow π can be defined on $\Omega \times \mathcal{G}$ by $\pi(x_0, g, t) = (x(t; x_0, g), g_t)$ for $t \in I(x_0, g)$. The flow (dynamical system) π is a *skew product flow* (see [10] or [5]).

We are not interested in flows on all of $\Omega \times \mathcal{G}$ however. Instead we take the closure in $\mathcal{G}$ of the time translates of f, the (weak) hull of f,which we denote by $H_w(f)$. Since $\mathcal{G}$ is compact and $H_w(f)$ is closed, it follows that $H_w(f)$ is compact. The flow π is locally invariant on $\Omega \times H_w(f)$ and we restrict our attention entirely to this local flow, which we also denote by π. If $f(x,t)$ is a continuous and uniformly almost periodic in $t \in \mathbb{R}$ then the usual hull of f is the closure of the time translates of f in the topology of uniform convergence on compact sets. We will call this the strong hull of f, and denote it by $H_s(f)$. Now if $\{f_{t_n}\}$ is a sequence of translates of the uniformly almost periodic function $f(x,t)$ converging in the weak topology to g, every subsequence of that sequence has in turn a subsequence converging uniformly on compact sets to some h in the strong hull; thus this subsequence converges to h in the weak topology also. Hence for each $x \in \Omega$, $g(x,t) = h(x,t)$ a.e., and in fact $\{f_{t_n}\}$ converges to h in the strong topology. Thus in the uniform almost periodic case the topologies are equivalent if we identify the equivalence classes of functions in $H_w(f)$ with their continuous representatives. Moreover, uniform convergence on compact sets of a sequence of uniformly almost periodic functions is equivalent to its uniform convergence on sets of the form $K \times \mathbb{R}$, $K \subset \Omega$ compact [3]. Thus our weak topology is equivalent in this case to this latter topology. Nevertheless, the weak topology will be useful to us in the study of parameter dependence. Since the hull of f is essentially independent of these topologies, we will simply denote it by $H(f)$. Recall that if $g \in H(f)$ then g is also uniformly almost periodic and $f \in H(g)$. The space $\Omega \times H(f)$ is a locally compact metric space in any of these topologies.

We will apply the following abstract continuation result regarding the Conley index; see [2] or [9]. Recall that if π is a flow on a metric space X, $N \subset X$ is said to be a compact isolating neighborhood for π if N is compact and the maximal π invariant set contained in N is contained in the interior of N. A compact set I is called a compact isolated invariant set if I has a compact neighborhood in which I is the maximal invariant set.

THEOREM 3. *Let X be a locally compact metric space and suppose for each $\lambda \in \mathbb{R}$ that π_λ is a local flow on X. Suppose:*(a) *The map $\lambda \to \pi_\lambda$ is continuous in the sense that if $\{\lambda_n\} \subset \mathbb{R}$, $\{x_n\} \subset X$, and $\{t_n\} \subset \mathbb{R}$ are sequences with $\lambda_n \to \lambda$, $x_n \to x$, $t_n \to t$, and $\pi_\lambda(x,t)$ is defined, then $\pi_{\lambda_n}(x_n,t_n)$ is defined for all large n and $\pi_{\lambda_n}(x_n,t_n) \to \pi_\lambda(x,t)$ as $n \to \infty$.*

(b) *There is a compact set N in X such that, for each $\lambda \in \mathbb{R}$, N is an isolating neighborhood for π_λ. Let $I(\lambda) = \{x \in N : \pi_\lambda(x,t) \in N$ for all $t \in \mathbb{R}\}$. Then the Conley (homotopy) index $h(\pi_\lambda, I(\lambda))$ is defined and its value is independent of $\lambda \in \mathbb{R}$.*

Recall that the Conley homotopy index is a homotopy class of compact pointed spaces, and that if the index of a compact isolated invariant set I_0 is not the homotopy class of $\overline{0}$, the one-point pointed space, then $I_0 \neq \emptyset$. We will write $h(\pi, I)$ for the Conley index of a compact isolated invariant set I in a flow π.

We also must explain our use of the term bifurcation. Let X be a locally compact metric space with distance function d. Let $J = [a,b]$ be a compact interval of real numbers containing the number λ_0 in its interior. Let π_λ, $\lambda \in J$, be a continuous family of (local) flows on X. A solution through x for a flow π_λ is a function σ mapping a real interval $I \ni 0$ into X such that $\sigma(t) = \pi_\lambda(x,t)$ for all $t \in I$. A full solution is one with $I = \mathbb{R}$. A full bounded solution is a full solution $\sigma = \sigma(t)$ such that

$$\text{closure}\{\sigma(t) : t \in \mathbb{R}\} \text{ is compact.}$$

We assume there is a compact set $A \subset X$ which is invariant for all π_λ, that is, for all $\lambda \in J$, $x \in A$, and $t \in \mathbb{R}$, $\pi_\lambda(x,t) \in A$. Since each $\pi_\lambda(\cdot,t)$ is a homeomorphism, $\pi_\lambda(A,t) = A$ for all $\lambda \in J$ and $t \in \mathbb{R}$. We define bifurcation from A.

DEFINITION 4. We will say that $\lambda_0 \in J$ is a bifurcation value from A if for every $\varepsilon > 0$ there is a pair $(\lambda, x) \in J \times X$ with $x \notin A$ and such that there is a full π_λ-solution $\sigma = \sigma(t)$ through x satisfying

$$d(\sigma(t), A) + |\lambda - \lambda_0| < \varepsilon \text{ for all } t \in \mathbb{R}.$$

Since we assume X is locally compact, it follows that σ is a full bounded solution. Notice that if $\sigma(t)$ is a full solution through x and $\sigma(t_0) = y$, then $\rho(t) = \sigma(t + t_0)$ is a full solution through y. If the π_λ form a family of semiflows, the definition may be modified to fit that situation. In this paper we study ordinary differential equations only and have no need for semiflows.

Let X be a locally compact metric space and let π_λ, $\lambda \in J$, be a continuous family of (local) flows on X. Let A be a compact set in X

invariant for all π_λ. Define the set $\mathcal{S}$ as follows:

$$\mathcal{S}^* = \{(\lambda, x) : \text{there is a full bounded solution for } \pi_\lambda \text{ through } x\}$$

$$\mathcal{S}_0 = \{(\lambda, x) \in \mathcal{S}^* : x \notin A\ \}$$

and let

$$\mathcal{S} = \overline{\mathcal{S}}_0.$$

THEOREM 5. *Assume:* (1) $\{\pi_\lambda : \lambda \in J\}$ *is a continuous family of local flows on* X. (2) *There is a compact connected set* A *which is invariant for all* $\lambda \in J$. (3) *There is a* λ_0 *in the interior of* J *such that for all* $\lambda \in J\backslash\{\lambda_0\}$, A *is an isolated invariant set for* π_λ. (4) *The Conley index* $h(\pi_\lambda, A)$ *is constant on the intervals* $[a, \lambda_0)$ *and* $(\lambda_0, b]$. (5) $h(\pi_a, A) \neq h(\pi_b, A)$.
Then (c1)*: For all* $\varepsilon > 0$, *there exists* $(\lambda, x) \in J \times X$, $x \notin A$, *and a full* $\pi_\lambda-$*solution* σ *through* x *satisfying* $d(\sigma(t), A) + |\lambda - \lambda_0| < \varepsilon$ *for all* $t \in \mathbb{R}$; *that is,* λ_0 *is a bifurcation point from* A. (c2)*: Let* $\mathcal{C}$ *denote the component of* $\mathcal{S}$ *containing* $\{\lambda_0\} \times A$; *then either* $\mathcal{C}$ *is unbounded or else* $\mathcal{C}$ *meets* $\{a, b\} \times X$.

This theorem extends one given in [14]; the proof will be given elsewhere.

2. Bifurcation in Almost Periodic Systems

Let $F : \Lambda = \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m$ be continuous. We assume there are continuous functions g, h, and κ such that

$$F(x, t, \mu) = g(x, \mu) + \kappa(\mu)h(x, t), \text{ and } F(0, t, \mu) = 0$$

for all $(t, \mu) \in \mathbb{R}^2$, with $g(x, \mu)$ and $h(x, t)$ locally Lipshitz continuous in $x \in \mathbb{R}^m$, uniformly with respect to t and μ in compact sets. Consider the parameterized family of differential equations

$$\frac{dx}{dt} = \varepsilon g(x, \mu) + \varepsilon\kappa(\mu)h(x, t). \tag{3}$$

By our assumptions, initial value problems for (3) have unique solutions, which vary continuously with the parameters ε and μ. We also assume that $h(x, t)$ is almost periodic in $t \in \mathbb{R}$, uniformly with respect to x in compact sets. We also assume that for each compact subset $K \subset \mathbb{R}^m$, $F(x, t, \mu)$ is continuous in $\mu \in \mathbb{R}$, uniformly with respect to $(x, t) \in K \times$

$\mathbb{R}$. Thus, given $\eta > 0$ and compact $K \subset \mathbb{R}^m$ there exists $\delta > 0$ such that $|F(x,t,\mu_1) - F(x,t,\mu_2)| < \eta$ whenever $(x,t) \in K \times \mathbb{R}$ and $|\mu_1 - \mu_2| < \delta$. For $\tau \in \mathbb{R}$, define the translated function h_τ by $h_\tau(x,t) = h(x,t+\tau)$ for all (x,t). We let

$$H = Hull(h) := \text{closure}\{h_\tau : \tau \in \mathbb{R}\},$$

where the closure is in the topology of uniform convergence on compact sets. It is known that since h is uniformly almost periodic in t, then H is compact. Moreover, if $h^* \in H$ then $Hull(h^*) = Hull(h)$. We may define a flow γ on H by $\gamma(h^*,t) = h^*_t$. For any $h^* \in H$, the positive limit set $\omega(h^*)$ and the negative limit set $\alpha(h^*)$ are each equal to H itself. Let $h_0(x)$ denote the average of $h(x,t)$, so that

$$h_0(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T h(x,t)\, dt.$$

We relate (3) to the averaged equation

$$\frac{dx}{dt} = g(x,\mu) + \kappa(\mu) h_0(x) := F_0(x,\mu). \tag{4}$$

It follows from our assumptions that initial value problems for (4) have unique solutions. We will denote by β_μ the flow on $\mathbb{R}^m$ generated by initial value problems for (4). Thus

$$\beta_\mu(x_0,t) = u\big(t; x_0, F_0(\cdot,\mu)\big),$$

where $u(t)$ satisfies

$$\frac{du}{dt} = F_0(u,\mu),\ \ u\big(0; x_0, F_0(.,\mu)\big) = x_0.$$

We will represent a generic member of H as $h^*(x,t)$. Consider the family of differential equations

$$\frac{dx}{dt} = \varepsilon g(x,\mu) + \varepsilon\kappa(\mu) h^*(x,t) \tag{5}$$

for $h^* \in H$.

Recall γ denotes the translation flow on H, $\gamma(h^*,t) = h^*_t$. Let $\mathbb{H}$ denote the homotopy type of the topological space H; $\mathbb{H}$ is compact and connected. Let H^* denote the Conley homotopy index of H under the flow γ. Then $H^* = [\mathbb{H}, p]$ where the distinguished point $p \notin \mathbb{H}$, since the flow on H has an empty exit set.

Fix μ. We wish to define a family of skew product flows associated with (5). Let $u(t; x_0, \varepsilon h^*)$ be the solution of (5) with $u(0; x_0, \varepsilon h^*) = x_0$. We define $\pi_{\mu,\varepsilon}$ as a flow on $\mathbb{R}^m \times H$ by

$$\pi_{\mu,\varepsilon}(x_0, h^*, t) = (u(t; x_0, \varepsilon h^*), h^*_t) \text{ for } (x_0, h^*) \in \mathbb{R}^m \times H.$$

A solution σ through $(x_0, h^*) \in \mathbb{R}^m \times H$ is a function $\sigma = \sigma(t)$ on an interval I into $\mathbb{R}^m \times H$ such that $\sigma(0) = (x_0, h^*)$ and $\sigma(t) = \pi_{\mu,\varepsilon}(x_0, h^*, t)$ for $t \in I$. A full solution is one with $I = \mathbb{R}$. A full bounded solution is a full solution with first component bounded: $\sup_{t \in \mathbb{R}} |u(t; x_0, \varepsilon h^*)| < \infty$. Since H is compact, the second component must be bounded in any case.

Define the set $\mathcal{S}_\varepsilon$ as follows:

$$\mathcal{S}^*_\varepsilon = \{(\mu, x_0, h^*) \in [a, b] \times \mathbb{R}^m \times H : \text{there is a full bounded solution for } \pi_{\mu,\varepsilon} \text{ through } (x_0, h^*)\}$$

$$\mathcal{S}_{\varepsilon,0} = \{(\mu, x) \in \mathcal{S}^*_\varepsilon : x \notin A\}$$

and let

$$\mathcal{S}_\varepsilon = \overline{\mathcal{S}}_{\varepsilon,0}.$$

THEOREM 6. *Let $J = [a, b]$ be a compact interval. Suppose:* (i) *There is a $\mu_0 \in J$, $a < \mu_0 < b$, such that $\{0\}$ is an isolated invariant set for β_μ, $\mu \in J \backslash \{\mu_0\}$. Moreover, for $a \le \mu < \mu_0$, $\{0\}$ is asymptotically stable for β_μ, and for $\mu_0 < \mu \le b$, $\{0\}$ is not asymptotically stable; the Conley index $h(\beta_\mu, \{0\})$ is non-zero and constant on the subintervals $[a, \mu_0)$ and $(\mu_0, b]$.* (ii) *$h(\beta_a, \{0\}) \neq h(\beta_b, \{0\})$.*

Then: There is an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$: (c1) μ_0 is a bifurcation point from $\{0\} \times H$ for the skew product flows $\pi_{\mu,\varepsilon}$ (for each fixed $0 < \varepsilon < \varepsilon_0$). (c2) Let $\mathcal{C}_\varepsilon$ denote the component of $\mathcal{S}_\varepsilon$ containing $(\mu_0, \{0\} \times H)$; then either $\mathcal{C}_\varepsilon$ is unbounded or else $\mathcal{C}_\varepsilon$ meets $\{a, b\} \times \mathbb{R}^m \times H$.

If, in addition to conditions (i) *and* (ii), *$\{0\}$ is a repeller for $\mu_0 \le \mu \le b$ then there is a continuum of bounded solutions of* (3) *bifurcating from $(\mu_0, 0)$ in $\mathbb{R} \times \mathbb{R}^m$.*

Proof. There are two parts to the proof. In the first part, we show that there is an $\varepsilon_0 > 0$ such that for each $0 < \varepsilon < \varepsilon_0$ and $\mu \in J \backslash \{\mu_C\}$ the invariant set $\{0\} \times H$ in the skew product flow $\pi_{\mu,\varepsilon}$ associated with

(5) has non-trivial Conley index, and this index changes as μ crosses μ_0. This will prove bifurcation in the skew product flows. We then consider bifurcation in the original family of differential equations, and almost periodicity.

Part 1: Let $\mu \in J\backslash\{\mu_0\}$. First notice that if $x = x(t)$ is a solution of (5) for some $\varepsilon > 0$ then $y(t) = x(t/\varepsilon)$ is a solution of

$$\frac{dy}{dt} = g(y,\mu) + \kappa(\mu)h^*(y, t/\varepsilon). \tag{6}$$

We will homotopy (6) for fixed small $\varepsilon > 0$ to the equation averaged at $\varepsilon = 0$. Let $0 < \varepsilon < 1$ and consider the homotopy

$$\frac{dy}{dt} = (1-\lambda)F_0(y) + \lambda F^*(y, t\varepsilon^{-1}, \mu) \tag{7}$$

for $F^*(y,t,\mu) = g(y,\mu)+\kappa(\mu)h^*(y,t/\varepsilon)$, and $F_0(y) = g(y,\mu)+\kappa(\mu)h_0(y)$. Let $B(r)$ be an isolating neighborhood of $\{0\}$ in the flow β_μ generated by (4). The family of equations (7) has $\overline{B}(r) = \{x \in \mathbb{R}^m : |x| \leq r\}$ as an isolating neighborhood. If this is not the case, then for each positive integer $n \in \mathbf{N}$ there is a bounded function $y_n = y_n(t)$, for all $t \in \mathbb{R}$, and numbers $\lambda_n \in [0,1]$, $\varepsilon_n \in (0, n^{-1}]$, $s_n \in \mathbb{R}$, and an $h_n \in H$, such that for all t, $y_n(t)$ satisfies the differential equation

$$\frac{dy_n}{dt} = (1-\lambda_n)F_0(y_n) + \lambda_n[g(y_n,\mu) + \kappa(\mu)h_n(y_n, t/\varepsilon_n)] \tag{8}$$

and $y_n(s_n) \in \partial\overline{B}(r)$. Let $z_n(t) = y_n(t+s_n)$; then z_n satisfies (8) translated by s_n. Now both z_n and dz_n/dt are uniformly bounded on $\mathbb{R}$ independently of $n \in \mathbf{N}$, so there is a subsequence of $\{z_n\}$ uniformly convergent on compact subsets of $\mathbb{R}$ to some $z = z(t)$ with $z(t) \in \overline{B}(r)$ and $z(0) \in \partial B(r)$ We can assume $\lambda_n \to \lambda_0 \in [0,1]$. Now the key here is that the sequence of functions $\{h_n(\cdot, \cdot\varepsilon_n^{-1})\}$ converges in our weak topology to $h_0(\cdot)$, ([13], Lemma 4.1). From all of this it follows that z satisfies

$$\frac{dz}{dt} = (1-\lambda_0)F_0(z(t)) + \lambda_0 F_0(z(t)) = F_0(z(t))$$

for all $t \in \mathbb{R}$. Since $z(0) \in \partial B(r)$ and $z(t) \in \overline{B}(r)$ for all t, this contradicts the hypothesis that $\overline{B}(r)$ is an isolating neighborhood for (4), and proves the existence of $0 < \varepsilon_0 \leq 1$ satisfying the claim.

Now fix $\varepsilon \in (0, \varepsilon_0)$, as well as $\mu \in J \backslash \{\mu_0\}$. We return to the equations (5) or (7) with $x = x(t) = y(\varepsilon t)$, for our fixed $0 < \varepsilon \leq \varepsilon_0$ and consider the homotopy

$$\frac{dx}{dt} = (1 - \lambda)\varepsilon F_0(x) + \lambda \varepsilon F^*(x, t, \mu) \tag{9}$$

for $\lambda \in [0, 1]$, where, as before, $F^*(x, t, \mu) = g(x, \mu) + \kappa(\mu) h^*(x, t)$. We define a family of skew product flows π_λ^μ on $\mathbb{R}^m \times H$ associated with (9) by

$$\pi_\lambda^\mu(x_0, F^*, t) = \big(x(t; x_0, \lambda\varepsilon\kappa(\mu)h^*), h_t^*\big)$$

where $x(t; x_0, \lambda\varepsilon\kappa(\mu)h^*)$ denotes the solution to (9) with $x(0) = x_0$. (Here we are holding μ fixed, but later will let it vary).

It follows, essentially from results of Artstein [1], that the family of flows $\{\pi_\lambda^\mu\}$ is continuous in the sense of Theorem 1. Moreover we have just shown that $\overline{B}(r) \times H$ is a compact isolating neighborhood for each π_λ^μ, $\lambda \in [0, 1]$. Thus the Conley index of the maximal invariant set in $\overline{B}(r) \times H$ for π_λ^μ is independent of $\lambda \in [0, 1]$. Let $I(0)$ denote this set for π_0^μ. Now in this case, $\lambda = 0$ so (9) becomes the equation

$$\frac{dx}{dt} = \varepsilon F_0(x, \mu). \tag{10}$$

The flow determined by (10) is the same as that of (4) except for a change of independent variable $t \to \varepsilon^{-1} t$. We denote this flow by β_μ. By hypothesis, $h(\beta_\mu, \{0\}) \neq \overline{0}$. Let γ denote the flow on H given by $\gamma(h^*, t) = h_t^*$. Then π_0^μ is a product flow on $\mathbb{R}^m \times H$ given by

$$\pi_0 = \beta_\mu \times \gamma$$

and

$$I(0) = \{0\} \times H.$$

It follows (see [1] or [9]) that

$$h(\pi_0^\mu, I(0)) = h(\beta_\mu \times \gamma, \{0\} \times H) = h(\beta_\mu, \{0\}) \wedge h(\gamma, H),$$

where $\wedge$ denotes the smash product of pointed topological spaces (homotopy types). By hypothesis, $h(\beta_\mu, \{0\}) \neq \overline{0}$, and $h(\gamma, H) = [\mathbb{H}^*, p]$ is of the form of a compact connected topological space with separated distinguished point (see above; recall H has an empty exit set under the flow γ). It follows from a result in [12] that $h(\pi_0^\mu, I(0)) \neq \bar{0}$. Hence

$$h(\pi_1^\mu, I(1)) = h(\pi_0^\mu, I(0)) \neq \bar{0},$$

where π_1 is the skew product flow on $\mathbb{R}^m \times H$ generated by the initial value problems

$$\frac{dx}{dt} = \varepsilon F^*(x, t; \mu),\ x(0) = x_0$$

where $h^* \in H$. Now $I(1) = \{0\} \times H$ also.

The idea now is that the index $h(\pi_1^\mu, I(1)) = h(\pi_1^\mu, \{0\} \times H)$ actually depends upon μ. For $a \leq \mu < \mu_0$, $h(\pi_1^\mu, \{0\} \times H) = h(\beta_\mu, \{0\}) \wedge [\mathbb{H}^*, p] = \sum^0 \wedge [\mathbb{H}^*, p] = [\mathbb{H}^*, p]$, since $\{0\}$ is asymptotically stable for (10). Moreover, $[\mathbb{H}^*, p]$ is not a connected topological space. On the other hand, for $\mu_0 < \mu \leq b$, $h(\pi_1^\mu, \{0\} \times H) = h(\beta_\mu, \{0\}) \wedge [\mathbb{H}^*, p]$, but in this case our hypothesis is that $h(\beta_\mu, \{0\})$ is connected. It is now easy to show that $h(\beta_\mu, \{0\}) \wedge [\mathbb{H}^*, p] = (h(\beta_\mu, \{0\}) \times [\mathbb{H}^*, p])/(h(\beta_\mu, \{0\}) \vee [\mathbb{H}^*, p])$ is also connected. (By $h(\beta_\mu, \{0\}) \vee [\mathbb{H}^*, p]$, we mean, as usual, the (homotopy type of) space obtained by joining the two spaces at their respective distinguished points). Thus the index changes as μ crosses μ_0. It follows from THEOREM 2 that there is a continuum of bounded solutions in the skew product flows bifurcating in $\mathbb{R}^m \times H$ from $\{0\} \times H$. This proves the first part of the theorem.

Part 2: We now attend to the question of bifurcating solutions to the original differential equation, and the question of almost periodic solutions. These are separate from the question of bifurcation in the skew product flows. This is because if $\sigma(t) = \pi_1^\mu(x_0, h^*, t)$ is a bifurcating solution in the skew product flow then $u(t) = u(t; x_0, \varepsilon\kappa(\mu)h^*)$, with $h^* \in H$, is a full bounded solution to

$$\frac{dx}{dt} = \varepsilon g(x, \mu) + \varepsilon\kappa(\mu)h^*(x, t), \tag{11}$$

but this does not insure the existence of a full bounded solution to our original differential equation

$$\frac{dx}{dt} = \varepsilon g(x, \mu) + \varepsilon\kappa(\mu)h(x, t). \tag{12}$$

This is because, while $h \in Hull(h^*)$, so there exists a sequence $\{\alpha_n\}_{n=1}^\infty$ with $h_{\alpha_n}^*(x, t) = h^*(x, t + \alpha_n) \to h(x, t)$ as $n \to \infty$, the corresponding sequence $\{u_{\alpha_n}\}$ may converge to $u = 0$, instead of converging to a nonzero solution of (12). The additional hypotheses in the second part of the theorem are designed to guarantee that this does not happen.

Thus we now assume in addition to conditions (i) and (ii), $\{0\}$ is a repeller for $\mu_0 \leq \mu \leq b$. We claim there is a continuum of bounded solutions of (3) bifurcating from $(\mu_0, 0)$ in $\mathbb{R}\times \mathbb{R}^m$, and these solutions are almost periodic near the bifurcation point. Let $(\mu, x_0, h^*) \in \mathcal{C}_\varepsilon$; then either $\mu < \mu_0$ or $\mu \geq \mu_0$. The cases are similar, and we treat only the first; so we suppose $\mu < \mu_0$. Now let $U(t) = u(t; x_0, \varepsilon\kappa(\mu)h^*)$, so $U(t)$ is a full bounded solution of (11), so

$$U'(t) = \varepsilon g(U(t), \mu) + \varepsilon\kappa(\mu)h^*(U(t), t)$$

for all $t \in \mathbb{R}$. Since $a \leq \mu < \mu_0$, the zero solution of (11) is asymptotically stable. Let $A = \alpha(x_0, h^*)$, i.e., A is the negative limit set of $(U(t), h^*_t)$ in $\mathbb{R}^m \times H$. Since $U(t)$ is a full bounded solution and $h(x,t)$ is uniformly almost periodic in t, A is compact. Moreover, $(\{0\}\times H)\cap A = \emptyset$ since the zero solution is asymptotically stable. Now there exists a sequence $\{\alpha_n\}$ with $\alpha_n \to -\infty$ as $n \to \infty$ and $h^*(x, t+\alpha_n) \to h(x,t)$ uniformly on compact sets. There is a subsequence of $\{\alpha_n\}$, which we relabel as $\{\alpha_n\}$, such that $U(\alpha_n) \to p_0 \in A$. We have that $(U(\alpha_n), h^*_{\alpha_n}) \to (p_0, H) \in A$. The solution of (12) passing through p_0 at $t = 0$ is the required full bounded solution.

Since the case of $\mu_0 \leq \mu \leq b$ is almost identical to the one considered, this completes the proof of the theorem. □

3. An application

Let $p, q \in C(\mathbb{R}, \mathbb{R})$ be continuous almost periodic functions. We consider the following example:

$$\frac{d^2x}{dt^2} + \varepsilon^2(\mu p(t) - x^2)\frac{dx}{dt} + \varepsilon^2 q(t)x = 0 \tag{13}$$

with $\varepsilon > 0$ and $\mu \in \mathbb{R}$. This is equivalent to the system

$$\begin{aligned} \tfrac{dx}{dt} &= \varepsilon y \\ \tfrac{dy}{dt} &= -\varepsilon(\mu p(t) - x^2)y - \varepsilon q(t)x. \end{aligned} \tag{14}$$

The averaged system is

$$\begin{aligned} \tfrac{dx}{dt} &= y \\ \tfrac{dy}{dt} &= -(\mu\overline{p} - x^2)y - \overline{q}x, \end{aligned} \tag{15}$$

where we assume the mean values of p and q are positive:

$$\overline{p} = \lim_{T\to\infty} \frac{1}{T}\int_0^T p(s)\,ds > 0, \text{ and } \overline{q} = \lim_{T\to\infty} \frac{1}{T}\int_0^T q(s)\,ds > 0.$$

When $\mu \le 0$ the origin is a repeller for the averaged equation (15), and hence has Conley index $h(\beta_\mu, \{0\}) = \sum^2$. However, when $\mu > 0$, the origin is asymptotically stable, and $h(\beta_\mu, \{0\}) = \sum^0$. It follows from THEOREM 3 that for all sufficiently small $\varepsilon > 0$, the system (14) has a bifurcating continuum of full bounded solutions in $(\mu, (x, y))$ space.

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Department of Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294
E-mail: ward@math.uab.edu