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ONE-PARAMETER GROUPS AND COSINE FAMILIES OF OPERATORS ON WHITE NOISE FUNCTIONS

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ABSTRACT. The main purpose of this paper is to study differentiable one-parameter groups and cosine families of operators acting on white noise functions and their associated infinitesimal generators. In particular, we prove the heredity of differentiable one-parameter group and cosine family of operators under the second quantization. As an application, we discuss the existence of the unique solution of the Cauchy problems for the first and second order differential equations.

1. Introduction

Let $\mathcal{W} \subset (L^2) \subset \mathcal{W}^*$ be the framework of white noise distribution theory recently introduced by Cochran, Kuo and Sengupta in [7] as a new class of white noise distributions which contains interesting noises such as the Poisson noise.

Gross[10] and Piech[27] initiated the study of Gross Laplacian and number operator, as natural infinite dimensional analogues of a finite dimensional Laplacian, in connection with the Cauchy problems in infinite dimensional abstract Wiener space. Recently, white noise approach to Cauchy problems in infinite dimension has been studied extensively by many authors ([1]–[4], [8], [15], [16], [20]–[23], [26]) and becomes an interesting area.

In [12], Hida, Obata and Saitô proved that the heredity of regular one-parameter group of operators on $(\mathcal{E})$ under the second quantization.

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Being motivated by this result, in this paper, we shall prove the heredity of differentiable one-parameter group and cosine family of operators under the second quantization. Our result shows that Theorem 4.1 in [12] is true without regularity condition.

The paper is organized as follows. In Section 2, we recall some basic notions and results in the white noise distribution theory. In Section 3, motivated by the results in [1]–[4] and [25], we introduce a class of transformations acting on $\mathcal{W}$ and then prove that the operator $\Xi_{0,m}(\kappa) + d\Gamma(K)$ is similar to $d\Gamma(K)$, where $\Xi_{0,m}(\kappa)$ and $d\Gamma(K)$ are the integral kernel operator and the second quantized differential operator of K, respectively, and $\kappa \in (\mathcal{E}_{\mathbb{C}}^{\otimes m})^*_{\mathrm{sym}}$, $K \in \mathcal{L}(\mathcal{E}_{\mathbb{C}})$ satisfy certain conditions. In Section 4, we discuss differentiable one-parameter groups of operators on $\mathcal{W}$ and their infinitesimal generators. In particular, we prove that for any differentiable one-parameter group $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ of operators on $\mathcal{E}_{\mathbb{C}}$, $\{\Gamma(\Omega_\theta)\}_{\theta\in\mathbb{R}}$ becomes a differentiable one-parameter group of operators on $\mathcal{W}$, where $\Gamma(\Omega_\theta)$ is the second quantization of Ω_θ. In Section 5, we first discuss differentiable cosine families of operators on $\mathcal{W}$ and their infinitesimal generators and then prove that $\{\Gamma(\Omega_\theta)\}_{\theta\in\mathbb{R}}$ becomes a differentiable cosine family of operators on $\mathcal{W}$ for any differentiable cosine family $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ of operators on $\mathcal{E}_{\mathbb{C}}$. In Section 6, as an application, we shall discuss the existence of the unique solution of the Cauchy problems for the first and second order differential equations.

2. Preliminaries on white noise distribution theory

In this section, we shall briefly recall the Cochran–Kuo–Sengupta space and operator theory on this space.

2.1. The Cochran–Kuo–Sengupta space

Let $\mathcal{H} = L^2(\mathbb{R}, dt)$ be the real Hilbert space of all square integrable functions on $\mathbb{R}$ with norm $|\cdot|_0$ and let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of $\mathbb{R}$-valued rapidly decreasing C^∞-functions and $\mathcal{S}'(\mathbb{R})$ be the strong dual space of $\mathcal{S}(\mathbb{R})$, i.e., the space of tempered distributions. Then we have a Gel'fand triple

$$\mathcal{E} \equiv \mathcal{S}(\mathbb{R}) \subset \mathcal{H} \equiv L^2(\mathbb{R}, dt) \subset \mathcal{S}'(\mathbb{R}) \equiv \mathcal{E}^*. \tag{2.1}$$

Note that the Gel'fand triple (2.1) can be reconstructed in the standard manner [13], [22], [25], using the positive self-adjoint operator $A = 1 +$

$t^2 - d^2/dt^2$ with Hilbert-Schmidt inverse. Since $\mathcal{E}$ is a nuclear Frechet space equipped with the Hilbertian norms $|\xi|_p = |A^p\xi|_0$, $p \in \mathbb{R}$, there exists a Gaussian measure μ on $\mathcal{E}^*$ whose characteristic function is given by

$$\int_{\mathcal{E}^*} \exp\{i\langle x, \xi\rangle\}\mu(dx) = \exp\left\{-\frac{1}{2}|\xi|_0^2\right\}, \qquad \xi \in \mathcal{E},$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $\mathcal{E}^* \times \mathcal{E}$. Then $(\mathcal{E}^*, \mu)$ is called the *white noise space or Gaussian space.*

We denote by (L^2) the complex Hilbert space of μ-square integrable functions on $\mathcal{E}^*$ with norm $\|\cdot\|_0$. By the Wiener-Itô decomposition theorem, each $\phi \in (L^2)$ admits an expression

$$\phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n\rangle, \qquad x \in \mathcal{E}^*, \quad f_n \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}, \tag{2.2}$$

where $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$is the n-fold symmetric tensor product of the complexification of $\mathcal{H}$ and $:x^{\otimes n}:$ denotes the Wick ordering of $x^{\otimes n}$. Moreover, the (L^2)-norm $\|\phi\|_0$ of ϕ is given by

$$\|\phi\|_0 = \left(\sum_{n=0}^{\infty} n!|f_n|_0^2\right)^{1/2},$$

where $|\cdot|_0$ denotes the $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$-norm for any n.

Now, let $\{\alpha(n)\}_{n=0}^{\infty}$ be a sequence satisfying the following conditions:

(A1) $\alpha(0) = 1$ and $\inf \alpha(n) > 0$;

(A2) The function $G_\alpha(t) = \sum_{n=0}^{\infty} (\alpha(n)/n!)\, t^n$ has an entire analytic extension;

(A3) The power series

$$\widetilde{G}_\alpha(t) = \sum_{n=0}^{\infty} t^n \frac{n^{2n}}{n!\alpha(n)} \left(\inf_{s>0} \frac{G_\alpha(s)}{s^n}\right)$$

has a positive radius of convergence.

There is a non-trivial example of $\{\alpha(n)\}$ satisfying (A1)–(A3) which is given by the k-th order Bell numbers $\{B_k(n)\}$ defined by

$$G_{\mathrm{Bell}(k)}(t) = \frac{\overbrace{\exp(\exp(\cdots(\exp t)\cdots))}^{k-\text{times}}}{\exp(\exp(\cdots(\exp 0)\cdots))} = \sum_{n=0}^{\infty} \frac{B_k(n)}{n!} t^n, \tag{2.3}$$

for more details see [7].

For each $p \geq 0$, define

$$\|\phi\|_{p,+} = \left(\sum_{n=0}^{\infty} n!\alpha(n)|f_n|_p^2 \right)^{1/2}, \qquad \phi \in (L^2),$$

where ϕ is given as in (2.2). Let $\mathcal{W}_p = \{\phi \in (L^2) : \|\phi\|_{p,+} < \infty\}$ and let $\mathcal{W}$ be the projective limit of $\{\mathcal{W}_p : p \geq 0\}$ and $\mathcal{W}^*$ be the topological dual space of $\mathcal{W}$. Then we have a Gel'fand triple

$$\mathcal{W} \subset (L^2) \subset \mathcal{W}^*, \tag{2.4}$$

which is called the *Cochran-Kuo-Sengupta space.* In particular, (2.4) is called the *Hida–Kubo–Takenaka space* or the *Kondratiev–Streit space* [17] according as $\alpha(n) = 1$ or $\alpha(n) = n!^{\beta}$, $0 \leq \beta < 1$ and denoted by

$$(\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^*, \qquad (\mathcal{E})_\beta \subset (L^2) \subset (\mathcal{E})_\beta^*,$$

respectively. The canonical bilinear form on $\mathcal{W}^* \times \mathcal{W}$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$.

2.2. S–transform

For each $\xi \in \mathcal{E}_{\mathbb{C}}$, the *exponential vector* $\phi_\xi \in \mathcal{W}$ is defined by

$$\phi_\xi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \frac{\xi^{\otimes n}}{n!} \right\rangle = \exp\left(\langle x, \xi \rangle - \frac{1}{2}\langle \xi, \xi \rangle \right).$$

It is well-known that $\{\phi_\xi; \xi \in \mathcal{E}_{\mathbb{C}}\}$ spans a dense subspace of $\mathcal{W}$.

For $\Phi \in \mathcal{W}^*$, the *S–transform* $S\Phi$ of Φ is a $\mathbb{C}$–valued function on $\mathcal{E}_{\mathbb{C}}$ defined by

$$S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle, \qquad \xi \in \mathcal{E}_{\mathbb{C}}.$$

Then $\Phi \in \mathcal{W}^*$ is uniquely determined by the S–transform $S\Phi$ of Φ. The following analytic characterization theorem for generalized white noise functional in terms of S–transform is proved in [7].

THEOREM 2.1. *Let F be a $\mathbb{C}$-valued function on $\mathcal{E}_{\mathbb{C}}$. Then F is the S–transform of some $\Phi \in \mathcal{W}^*$ if and only if F satisfies the following conditions:*

(F1) *for each $\xi, \eta \in \mathcal{E}_{\mathbb{C}}$, the function $z \mapsto F(z\xi + \eta)$ is entire on $\mathbb{C}$;*
(F2) *there exist $C \geq 0$ and $p \geq 0$ such that*

$$|F(\xi)|^2 \leq CG_\alpha(|\xi|_p^2), \qquad \xi \in \mathcal{E}_{\mathbb{C}}.$$

In this case, for each $q > 1/2$ *with* $\widetilde{G}_\alpha(\|A^{-1}\|_{\mathrm{HS}}^{2q}) < \infty$,

$$\|\Phi\|_{-(p+q),-}^2 \le C\widetilde{G}_\alpha(\|A^{-1}\|_{\mathrm{HS}}^{2q}).$$

For the analytic characterization theorem for S-transform of test white noise functional, we assume that G_α satisfies the following condition:

$$\limsup_{n\to\infty}\left\{\left(\frac{\alpha(n)}{n!}\right)^{1/n} n^2 \inf_{r>0}\left(\frac{1}{r}\left(G_{1/\alpha}(r)\right)^{1/n}\right)\right\} < \infty. \tag{2.5}$$

It is a sufficient condition for (2.5) that the sequence $1/(\alpha(n)n!)$ is log–concave (see, Theorem 4.3 in [7]). For example, if $\{\alpha(n)\}$ is a sequence of the second order Bell numbers, i.e., $\alpha(n) = B_2(n)$ for all $n \ge 1$, then $\alpha(n+1)^2 \le \alpha(n)\alpha(n+2)$ for all $n \ge 1$ (see, Proposition in [9]) and hence $1/(\alpha(n)n!)$ is log–concave. The condition (2.5) implies that the power series

$$\widetilde{G}_{1/\alpha}(t) = \sum_{n=0}^{\infty} t^n \frac{n^{2n}\alpha(n)}{n!}\left(\inf_{s>0}\frac{G_{1/\alpha}(s)}{s^n}\right)$$

has a positive radius of convergence.

The following characterization theorem for test white noise functional is proved by a simple modification of the proof of Theorem 2.1.

THEOREM 2.2. *Let* F *be a* $\mathbb{C}$*-valued function on* $\mathcal{E}_{\mathbb{C}}$. *Then* F *is the* S*–transform of some* $\phi \in \mathcal{W}$ *if and only if* F *satisfies the condition* (F1) *and*

(F2$'$) *for any* $p \ge 0$, *there exists* $K \ge 0$ *such that*

$$|F(\xi)|^2 \le KG_{1/\alpha}(|\xi|_{-p}^2), \qquad \xi \in \mathcal{E}_{\mathbb{C}}.$$

In this case, for each $p_0 > 1/2$ *with* $\widetilde{G}_{1/\alpha}(\|A^{-1}\|_{\mathrm{HS}}^{2p_0}) < \infty$,

$$\|\phi\|_{(p-p_0),+}^2 \le K\widetilde{G}_{1/\alpha}(\|A^{-1}\|_{\mathrm{HS}}^{2p_0}).$$

2.3. Operator symbols and integral kernel operators

For locally convex spaces $\mathfrak{X}, \mathfrak{Y}$, let $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ denote the space of continuous linear operators from $\mathfrak{X}$ into $\mathfrak{Y}$. We always assume that $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ is equipped with the topology of uniform convergence on every bounded subset. For the notational convenience, we write $\mathcal{L}(\mathfrak{X}) = \mathcal{L}(\mathfrak{X}, \mathfrak{X})$.

For $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, the *symbol* $\widehat{\Xi}$ of Ξ is a $\mathbb{C}$–valued function on $\mathcal{E}_{\mathbb{C}} \times \mathcal{E}_{\mathbb{C}}$ defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in \mathcal{E}_{\mathbb{C}}.$$

An operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is uniquely determined by its symbol $\widehat{\Xi}$.

The following is an analytic characterization theorem for operator symbols, the first assertion has been proved in [5] and the second assertion can be proved by using the similar arguments as in the proof of the first assertion.

THEOREM 2.3. *Let Θ be a $\mathbb{C}$-valued function defined on $\mathcal{E}_{\mathbb{C}} \times \mathcal{E}_{\mathbb{C}}$. Then Θ is the symbol of an operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ if and only if Θ satisfies the following conditions:*

(S1) *for each $\xi, \xi_1, \eta, \eta_1 \in \mathcal{E}_{\mathbb{C}}$, the function*

$$(z, w) \mapsto \Theta(z\xi + \xi_1, w\eta + \eta_1)$$

is an entire function on $\mathbb{C} \times \mathbb{C}$;

(S2) *there exist constant numbers $K \geq 0$ and $p \geq 0$ such that*

$$|\Theta(\xi, \eta)|^2 \leq K G_\alpha(|\xi|_p^2) G_\alpha(|\eta|_p^2), \qquad \xi, \eta \in \mathcal{E}_{\mathbb{C}}.$$

Moreover, Θ is the symbol of an operator $\Xi \in \mathcal{L}(\mathcal{W})$ if and only if Θ satisfies the condition (S1) and

(S2$'$) *for any $p \geq 0$, there exist constant numbers $K \geq 0$ and $q \geq 0$ such that*

$$|\Theta(\xi, \eta)|^2 \leq K G_\alpha(|\xi|_{p+q}^2) G_{1/\alpha}(|\eta|_{-p}^2), \qquad \xi, \eta \in \mathcal{E}_{\mathbb{C}}.$$

In this case, for each $r, p_0 > 1/2$ with $\widetilde{G}_\alpha(\|A^{-1}\|_{\mathrm{HS}}^{2r}) < \infty$ and $\widetilde{G}_{1/\alpha}(\|A^{-1}\|_{\mathrm{HS}}^{2p_0}) < \infty$,

$$\|\Xi_\theta \phi\|_{p-p_0,+}^2 \leq K \widetilde{G}_\alpha(\|A^{-1}\|_{\mathrm{HS}}^{2r}) \widetilde{G}_{1/\alpha}(\|A^{-1}\|_{\mathrm{HS}}^{2p_0}) \|\phi\|_{p+q+r,+}^2, \qquad \phi \in \mathcal{W}.$$

Let $l, m \geq 0$ be integers. For each $\kappa \in (\mathcal{E}_{\mathbb{C}}^{\otimes(l+m)})^*$, there exists a unique $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ such that

$$\widehat{\Xi}_{l,m}(\kappa)(\xi, \eta) = \langle \kappa, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle e^{\langle \xi, \eta \rangle}, \qquad \xi, \eta \in \mathcal{E}_{\mathbb{C}}. \tag{2.6}$$

Then $\Xi_{l,m}(\kappa)$ is called the *integral kernel operator* with kernel distribution κ (see; [5], [12], and [25]). To discuss about integral kernel operators acting on $\mathcal{W}$, we need more assumptions for the weighted sequence $\{\alpha(n)\}$:

(A4) there exists $C_1 > 0$ such that $\alpha(m)\alpha(n) \le C_1^{m+n}\alpha(m+n)$;
(A5) there exists $C_2 > 0$ such that $\alpha(m+n) \le C_2^{m+n}\alpha(m)\alpha(n)$.

If the weighted sequence $\{\alpha(n)\}$ satisfies the assumptions (A1)–(A5), then $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{W})$ if and only if $\kappa \in \mathcal{E}_{\mathbb{C}}^{\otimes l} \otimes (\mathcal{E}_{\mathbb{C}}^{\otimes m})^*$ (see; [5]).

Let $K \in \mathcal{L}(\mathcal{E}_{\mathbb{C}})$. Then by the kernel theorem, there exists a unique $\lambda_K \in \mathcal{E}_{\mathbb{C}} \otimes \mathcal{E}_{\mathbb{C}}^*$ such that

$$\langle \lambda_K, \xi \otimes \eta \rangle = \langle K\eta, \xi \rangle, \qquad \xi, \eta \in \mathcal{E}_{\mathbb{C}}. \tag{2.7}$$

In fact, for any $\eta \in \mathcal{E}_{\mathbb{C}}$, $K\eta$ equals to the right contraction $\lambda_K \otimes_1 \eta$. The integral kernel operator $\Xi_{1,1}(\lambda_K)$ with kernel distribution λ_K is called the *second quantized differential operator* of K and denoted by $d\Gamma(K)$, i.e., $d\Gamma(K)$ is defined by

$$\langle\langle d\Gamma(K)\phi_\xi, \phi_\eta \rangle\rangle = \langle K\xi, \eta \rangle e^{\langle \xi, \eta \rangle}, \qquad \xi, \eta \in \mathcal{E}_{\mathbb{C}}. \tag{2.8}$$

3. Transformations on white noise functions

From now on, the weighted sequence $\{\alpha(n)\}$ satisfies the conditions (A1)–(A5) and the condition

(A6) there exists a constant $K > 0$ such that $\exp\{e^t\} \le KG_\alpha(t)$, $t \ge 0$.

Let $\kappa \in (\mathcal{E}_{\mathbb{C}}^{\otimes m})^*$, $m \in \mathbb{N}$ and let $B \in \mathcal{L}(\mathcal{E}_{\mathbb{C}})$. Then by Theorem 2.3, there exists a unique $\mathcal{G}_{\kappa,B} \in \mathcal{L}(\mathcal{W})$ such that

$$\mathcal{G}_{\kappa,B}\phi_\xi = \exp\{\langle \kappa, \xi^{\otimes m} \rangle\}\, \phi_{B\xi}, \qquad \xi \in \mathcal{E}_{\mathbb{C}}. \tag{3.1}$$

In fact, by using the assumption (A6), we can easily show that the function $\Theta(\xi, \eta) = \langle\langle \mathcal{G}_{\kappa,B}\phi_\xi, \phi_\eta \rangle\rangle$ satisfies the condition (S2′) in Theorem 2.3.

By using (3.1), we can easily show that $\mathcal{G}_{0,B}$ coincides with the second quantization operator $\Gamma(B)$ of B, where $\Gamma(B)$ is defined by

$$\Gamma(B)\phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, B^{\otimes n} f_n \rangle$$

for any $\phi \in \mathcal{W}$ given as in (2.2). Moreover, $\mathcal{G}_{\kappa,I} = e^{\Xi_{0,m}(\kappa)}$. Hence we have the following expression:

$$\mathcal{G}_{\kappa,B} = \Gamma(B) \circ e^{\Xi_{0,m}(\kappa)}. \tag{3.2}$$

For a locally convex topological space $\mathfrak{X}$, let $GL(\mathfrak{X})$ denote the set of all linear homeomorphisms on $\mathfrak{X}$.

THEOREM 3.1. *Let* $G = \{\mathcal{G}_{\kappa,B} | \kappa \in (\mathcal{E}_{\mathbb{C}}^{\otimes m})^*, B \in GL(\mathcal{E}_{\mathbb{C}})\}$ *is a subgroup of* $GL(\mathcal{W})$. *Moreover, for each* $(\kappa, B) \in (\mathcal{E}_{\mathbb{C}}^{\otimes m})^* \times GL(\mathcal{E}_{\mathbb{C}})$, $\mathcal{G}_{-((B^{-1})^*)^{\otimes m}\kappa, B^{-1}} \in GL(\mathcal{W})$ *is the inverse of* $\mathcal{G}_{\kappa,B}$.

Proof. The proof follows from the following composition formula: for any $\kappa, \kappa' \in (\mathcal{E}_{\mathbb{C}}^{\otimes m})^*_{\mathrm{sym}}$ and $B, B' \in \mathcal{L}(\mathcal{E}_{\mathbb{C}})$

$$\mathcal{G}_{\kappa',B'}\mathcal{G}_{\kappa,B} = \mathcal{G}_{\kappa+(B^*)^{\otimes m}\kappa', B'B}. \tag{3.3}$$ □

THEOREM 3.2. *Let* $\kappa \in (\mathcal{E}_{\mathbb{C}}^{\otimes m})^*_{\mathrm{sym}}$, $\kappa' \in (\mathcal{E}_{\mathbb{C}}^{\otimes m'})^*_{\mathrm{sym}}$ *and* $B \in GL(\mathcal{E}_{\mathbb{C}})$, $K \in \mathcal{L}(\mathcal{E}_{\mathbb{C}})$. *Then we have the following properties:*

(i) $\mathcal{G}_{\kappa,B}\Xi_{0,m'}(\kappa') = \Xi_{0,m'}(((B^{-1})^*)^{\otimes m'}\kappa')\mathcal{G}_{\kappa,B}$;

(ii) *If* $(I^{\otimes(m-1)} \otimes K^*)\kappa = \alpha\kappa$, $\alpha \in \mathbb{C}$ *and* $[B, K] = 0$, *then we have*

$$\mathcal{G}_{\kappa,B}\Xi_{1,1}(\lambda_K) = \left(\Xi_{1,1}(\lambda_K) + m\alpha\Xi_{0,m}(((B^{-1})^*)^{\otimes m}\kappa)\right)\mathcal{G}_{\kappa,B};$$

(iii) *If* $(I^{\otimes(m-1)} \otimes K^*)\kappa = \alpha\kappa$, $\alpha \in \mathbb{C}$ *with* $\alpha \neq 0$ *and* $[B, K] = 0$, *then we have*

$$\mathcal{G}_{-1/(m\alpha)\kappa,B}(\Xi_{0,m}(\kappa) + \Xi_{1,1}(\lambda_K)) = \Xi_{1,1}(\lambda_K)\mathcal{G}_{-1/(m\alpha)\kappa,B}.$$

Proof. (i) For any $\xi \in \mathcal{E}_{\mathbb{C}}$, we obtain that

$$\begin{aligned}\mathcal{G}_{\kappa,B}\Xi_{0,m'}(\kappa')\phi_\xi &= \langle \kappa', \xi^{\otimes m'}\rangle \exp\{\langle \kappa, \xi^{\otimes m}\rangle\}\phi_{B\xi} \\ &= \Xi_{0,m'}(((B^{-1})^*)^{\otimes m'}\kappa')\mathcal{G}_{\kappa,B}\phi_\xi.\end{aligned}$$

Since $\mathcal{G}_{\kappa,B} \in \mathcal{L}(\mathcal{W})$ and all exponential vectors span a dense subspace of $\mathcal{W}$, the equality (i) is proved.

(ii) It follows from [5] that for any $\kappa \in (\mathcal{E}_{\mathbb{C}}^{\otimes m})^*$,

$$\begin{aligned}&\Xi_{1,1}(\lambda_K)\Xi_{0,m}(\kappa) \\ &= \Xi_{1,1}(\lambda_K) \diamond \Xi_{0,m}(\kappa)\end{aligned}$$

and

$$\Xi_{0,m}(\kappa)\Xi_{1,1}(\lambda_K) = \Xi_{1,1}(\lambda_K) \diamond \Xi_{0,m}(\kappa) + m\Xi_{0,m}(\kappa \otimes^1 \lambda_K),$$

where $\diamond$ is the Wick product (see [5]) and $\kappa \otimes^1 \lambda_K$ is the left contraction (see [25]). By the assumption $(I^{\otimes(m-1)} \otimes K^*)\kappa = \alpha\kappa$, $\Xi_{0,m}(\kappa \otimes^1 \lambda_K) = \alpha\Xi_{0,m}(\kappa)$. Hence we have

$$[\Xi_{1,1}(\lambda_K), \Xi_{0,m}(\kappa)] = -m\alpha\Xi_{0,m}(\kappa).$$

Therefore, we obtain that

$$e^{\Xi_{0,m}(\kappa)}\Xi_{1,1}(\lambda_K) = (\Xi_{1,1}(\lambda_K) + m\alpha\Xi_{0,m}(\kappa))e^{\Xi_{0,m}(\kappa)}.$$

On the other hand, we have from the assumption $[B, K] = 0$ that

$$\Gamma(B)\Xi_{1,1}(\lambda_K) = \Xi_{1,1}(\lambda_K)\Gamma(B).$$

Also, we can easily verify that

$$\Gamma(B)\Xi_{0,m}(\kappa) = \Xi_{0,m}(((B^{-1})^*)^{\otimes m}\kappa)\Gamma(B).$$

Hence by (3.2), the equation (ii) is satisfied.

(iii) From (i) and (ii), we obtain that for any $\kappa, \kappa' \in (\mathcal{E}_{\mathbb{C}}^{\otimes m})^*$

$$\begin{aligned}\mathcal{G}_{\kappa,B}&\left(\Xi_{0,m}(\kappa') + \Xi_{1,1}(\lambda_K)\right)\\ &= \left(\Xi_{0,m}(((B^{-1})^*)^{\otimes m}(\kappa' + m\alpha\kappa)) + \Xi_{1,1}(\lambda_K)\right)\mathcal{G}_{\kappa,B}.\end{aligned}$$

Thus we complete the proof. □

In the proof of (ii) in Theorem 3.2, we have shown that if $(I^{\otimes(m-1)} \otimes K^*)\kappa = 0$, then

$$[\Xi_{1,1}(\lambda_K), \Xi_{0,m}(\kappa)] = 0.$$

In Theorem 3.2, the equation (iii) is satisfied in the case $B = I$. Hence for the simple notation, we write $\mathcal{G}_\kappa = \mathcal{G}_{-1/(m\alpha)\kappa,I}$.

COROLLARY 3.3. *Let $\kappa \in (\mathcal{E}_{\mathbb{C}}^{\otimes m})^*_{\mathrm{sym}}$ and $B \in GL(\mathcal{E}_{\mathbb{C}})$. Then we have the following properties:*

(i) $\mathcal{G}_{\kappa,B}N = (N + m\Xi_{0,m}(((B^{-1})^*)^{\otimes m}\kappa))\,\mathcal{G}_{\kappa,B}$;

(ii) $\mathcal{G}_{-1/(m\alpha)\kappa,B}(\Xi_{0,m}(\kappa) + \alpha N) = \alpha N\mathcal{G}_{-1/(m\alpha)\kappa,B}$, *for any $\alpha \in \mathbb{C}$ with $\alpha \neq 0$.*

Proof. The proof is straightforward by applying Theorem 3.2 with $K = \alpha I$. □

4. Differentiable one–parameter transformation groups

From now on, let $\mathfrak{X}$ denote a barreled locally convex space over $\mathbb{C}$ and let $\{\|\cdot\|_p\}_{p\in\Lambda}$ be a set of seminorms which determines the topology of $\mathfrak{X}$.

DEFINITION 4.1. A family $\{\Omega_\theta\}_{\theta\in\mathbb{R}} \subset \mathcal{L}(\mathfrak{X})$ is called *differentiable* in $\theta \in \mathbb{R}$ if for each $\theta \in \mathbb{R}$, there exists $\Xi_\theta \in \mathcal{L}(\mathfrak{X})$ such that

$$\lim_{h\to 0}\left\|\frac{\Omega_{\theta+h}\phi - \Omega_\theta\phi}{h} - \Xi_\theta\phi\right\|_p = 0 \qquad \text{for all } \phi \in \mathfrak{X} \text{ and } p \in \Lambda.$$

If a family $\{\Omega_\theta\}_{\theta\in\mathbb{R}} \subset \mathcal{L}(\mathfrak{X})$ is an one-parameter subgroup of $GL(\mathfrak{X})$, i.e.,

$$\Omega_{\theta_1+\theta_2} = \Omega_{\theta_1}\Omega_{\theta_2}, \quad \theta_1, \theta_2 \in \mathbb{R}, \qquad \Omega_0 = I \text{ (identity)},$$

then $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ is differentiable in $\theta \in \mathbb{R}$ if and only if $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ is differentiable at 0, i.e., there exists $\Xi \in \mathcal{L}(\mathfrak{X})$ such that

$$\lim_{h\to 0}\left\|\frac{\Omega_h\phi - \phi}{h} - \Xi\phi\right\|_p = 0 \qquad \text{for all } \phi \in \mathfrak{X} \text{ and } p \in \Lambda.$$

In that case, Ξ is unique and is called the *infinitesimal generator* of a differentiable one-parameter subgroup $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$. Note that if $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ is a differentiable one-parameter subgroup of $GL(\mathfrak{X})$ with infinitesimal generator $\Xi \in \mathcal{L}(\mathfrak{X})$, then it is easily shown that $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ is infinitely many differentiable and

$$\frac{d^n\Omega_\theta}{d\theta^n}\phi = \Omega_\theta\Xi^n\phi = \Xi^n\Omega_\theta\phi, \qquad \phi \in \mathfrak{X}, \quad \theta \in \mathbb{R}.$$

THEOREM 4.2. *Let $K \in \mathcal{L}(\mathcal{E}_\mathbb{C})$. Then K is the infinitesimal generator of a differentiable one-parameter subgroup $\{\Omega_\theta\}_{\theta\in\mathbb{R}} \subset GL(\mathcal{E}_\mathbb{C})$ if and only if $d\Gamma(K)$ is the infinitesimal generator of a differentiable one-parameter subgroup $\{\Gamma(\Omega_\theta)\}_{\theta\in\mathbb{R}} \subset GL(\mathcal{W})$.*

Proof. It is obvious that $\{\Gamma(\Omega_\theta)\}_{\theta\in\mathbb{R}}$ is a one-parameter subgroup of $GL(\mathcal{W})$. We now prove that $\Gamma(\Omega_\theta)\phi$ is differentiable in $\theta \in \mathbb{R}$ and $d\Gamma(K)$ is the infinitesimal generator of $\{\Gamma(\Omega_\theta)\}_{\theta\in\mathbb{R}}$. We put

$$f(\theta) = \langle\langle \Gamma(\Omega_\theta)\phi_\xi, \phi_\eta\rangle\rangle = e^{\langle\Omega_\theta\xi,\eta\rangle}, \qquad \xi, \eta \in \mathcal{E}_\mathbb{C}.$$

Then we have

$$f'(\theta) = \langle K\Omega_\theta\xi, \eta\rangle e^{\langle\Omega_\theta\xi,\eta\rangle}$$

and

$$f''(\theta) = \left(\langle K\Omega_\theta\xi, \eta\rangle^2 + \langle K^2\Omega_\theta\xi, \eta\rangle\right) e^{\langle\Omega_\theta\xi,\eta\rangle}.$$

Let $\theta_0 > 0$ be fixed. Then by applying Baire category theorem, for any $r \geq 0$ there exist $r' \geq 0$ and $C_1 \geq 0$ such that

$$\max_{|\theta| \leq \theta_0} |K\Omega_\theta \xi|_r \leq C_1 |\xi|_{r+r'}, \qquad \max_{|\theta| \leq \theta_0} |K^2\Omega_\theta \xi|_r \leq C_1 |\xi|_{r+r'}$$

since Ω_θ is continuous in $\theta \in \mathbb{R}$. Therefore, for any $p \geq 0$, there exist $C = C(K, \theta_0) \geq 0$ and $q \geq 0$ such that

$$\max_{|\theta| \leq \theta_0} |f''(\theta)| \leq C G_\alpha(|\xi|_{p+q}^2) G_{1/\alpha}(|\eta|_{-p}^2).$$

Now we put

$$g_\theta(\xi, \eta) = f(\theta) - f(0) - f'(0)\theta.$$

Then by the Taylor theorem, whenever $|\theta| \leq \theta_0$ it follows that

$$\begin{aligned} |g_\theta(\xi, \eta)| &\leq \frac{|\theta|^2}{2} \max_{|\theta| \leq \theta_0} |f''(\theta)| \\ &\leq \frac{|\theta|^2}{2} C G_\alpha(|\xi|_{p+q}^2) G_{1/\alpha}(|\eta|_{-p}^2), \qquad \xi, \eta \in \mathcal{E}_{\mathbb{C}}. \end{aligned}$$

It then follows from Theorem 2.3 that there exists $\Xi_\theta \in \mathcal{L}(\mathcal{W})$ such that $\widehat{\Xi}_\theta = g_\theta$ and

(4.1)

$$\|\Xi_\theta \phi\|_{p-p_0,+}^2 \leq \frac{|\theta|^2}{2} C \widetilde{G}_\alpha(\|A^{-1}\|_{\mathrm{HS}}^{2r}) \widetilde{G}_{1/\alpha}(\|A^{-1}\|_{\mathrm{HS}}^{2p_0}) \|\phi\|_{p+q+r,+}^2, \quad |\theta| \leq \theta_0,$$

where $r, p_0 > 1/2$ with $\widetilde{G}_\alpha(\|A^{-1}\|_{\mathrm{HS}}^{2r}) < \infty$ and $\widetilde{G}_{1/\alpha}(\|A^{-1}\|_{\mathrm{HS}}^{2p_0}) < \infty$. On the other hand,

$$f'(0)(\xi, \eta) = \langle K\xi, \eta \rangle e^{\langle \xi, \eta \rangle}, \qquad \xi, \eta \in \mathcal{E}_{\mathbb{C}}.$$

Hence we have

$$\Xi_\theta = \Gamma(\Omega_\theta) - I - \theta d\Gamma(K).$$

It follows from (4.1) that

$$\begin{aligned} &\sup_{\|\phi\|_{p-q-r,-} \leq 1} \left\| \frac{\Gamma(\Omega_\theta)\phi - \phi}{\theta} - d\Gamma(K)\phi \right\|_{p-p_0,+} \\ &\leq \frac{|\theta|}{2} C \widetilde{G}_\alpha(\|A^{-1}\|_{\mathrm{HS}}^{2r}) \widetilde{G}_{1/\alpha}(\|A^{-1}\|_{\mathrm{HS}}^{2p_0}) \to 0 \end{aligned}$$

as $\theta \to 0$.

The proof of the converse is obvious since

$$\Gamma(\Omega_\theta)\langle x, \xi \rangle = \langle x, \Omega_\theta \xi \rangle, \qquad d\Gamma(K)\langle x, \xi \rangle = \langle x, K\xi \rangle, \qquad x \in \mathcal{E}^*, \quad \xi \in \mathcal{E}_{\mathbb{C}}$$

and $\|\langle x, \xi \rangle\|_{p,+} = \alpha(1)|\xi|_p$ for all $\xi \in \mathcal{E}_{\mathbb{C}}$ and $p \geq 0$. □

REMARK. Theorem 4.2 is true for the Hida–Kubo–Takenaka space, i.e. $K \in \mathcal{L}(\mathcal{E}_{\mathbb{C}})$ is the infinitesimal generator of a differentiable one-parameter subgroup $\{\Omega_\theta\}_{\theta\in\mathbb{R}} \subset GL(\mathcal{E}_{\mathbb{C}})$ if and only if $d\Gamma(K)$ is the infinitesimal generator of a differentiable one-parameter subgroup $\{\Gamma(\Omega_\theta)\}_{\theta\in\mathbb{R}} \subset GL((\mathcal{E}))$.

THEOREM 4.3. *Let $\{\Omega_\theta\}_{\theta\in\mathbb{R}} \subset GL(\mathcal{E}_{\mathbb{C}})$ be a differentiable one-parameter subgroup with the infinitesimal generator $K \in \mathcal{L}(\mathcal{E}_{\mathbb{C}})$ and let $\kappa \in (\mathcal{E}_{\mathbb{C}}^{\otimes m})_{\mathrm{sym}}^*$ satisfy $(I^{\otimes(m-1)} \otimes K^*)\kappa = 0$. Then $\{\mathcal{G}_{\theta\kappa,\Omega_\theta}\}_{\theta\in\mathbb{R}}$ is a differentiable one-parameter subgroup of $GL(\mathcal{W})$ with the infinitesimal generator $\Xi_{0,m}(\kappa) + d\Gamma(K)$.*

Proof. The assumption $(I^{\otimes(m-1)} \otimes K^*)\kappa = 0$ implies that

$$\frac{d}{d\theta}\langle(\Omega_\theta^*)^{\otimes m}\kappa, \xi^{\otimes m}\rangle = m\langle(I^{\otimes(m-1)} \otimes K^*)\kappa, \Omega_\theta^{\otimes m}\xi^{\otimes m}\rangle = 0, \qquad \xi \in \mathcal{E}_{\mathbb{C}}.$$

It follows that

$$\langle(\Omega_\theta^*)^{\otimes m}\kappa - \kappa, \xi^{\otimes m}\rangle = 0, \qquad \xi \in \mathcal{E}_{\mathbb{C}}.$$

Hence $(\Omega_\theta^*)^{\otimes m}\kappa = \kappa$ and by (3.3), we have

$$\mathcal{G}_{\theta_1\kappa,\Omega_{\theta_1}}\mathcal{G}_{\theta_2\kappa,\Omega_{\theta_2}} = \mathcal{G}_{(\theta_1+\theta_2)\kappa,\Omega_{\theta_1+\theta_2}}, \qquad \theta_1, \theta_2 \in \mathbb{R}.$$

Since $\mathcal{G}_{0,I} = I$, $\{\mathcal{G}_{\theta\kappa,\Omega_\theta}\}_{\theta\in\mathbb{R}}$ is a one-parameter subgroup of $GL(\mathcal{W})$. Also, by the similar arguments as in the proof of Theorem 4.2, we can prove that $\{\mathcal{G}_{\theta\kappa,\Omega_\theta}\}_{\theta\in\mathbb{R}}$ is a differentiable one-parameter subgroup of $GL(\mathcal{W})$ with the infinitesimal generator $\Xi_{0,m}(\kappa) + d\Gamma(K)$. □

The following lemma is obvious.

LEMMA 4.4. *If $\Xi \in \mathcal{L}(\mathcal{W})$ is the infinitesimal generator of a differentiable one-parameter subgroup $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ of $GL(\mathcal{W})$, then for any $\mathcal{G} \in GL(\mathcal{W})$, $\mathcal{G}^{-1}\Xi\mathcal{G}$ is the infinitesimal generator of a differentiable one-parameter subgroup $\{\mathcal{G}^{-1}\Omega_\theta\mathcal{G}\}_{\theta\in\mathbb{R}}$ of $GL(\mathcal{W})$.*

THEOREM 4.5. *Let $\{\Omega_\theta\}_{\theta\in\mathbb{R}} \subset GL(\mathcal{E}_{\mathbb{C}})$ be a differentiable one-parameter subgroup with the infinitesimal generator $K \in \mathcal{L}(\mathcal{E}_{\mathbb{C}})$ and let $\kappa \in (\mathcal{E}_{\mathbb{C}}^{\otimes m})_{\mathrm{sym}}^*$ satisfy $(I^{\otimes(m-1)} \otimes K^*)\kappa = \alpha\kappa$, $\alpha \in \mathbb{C}$ with $\alpha \neq 0$. Then $\{\Omega'_\theta\}_{\theta\in\mathbb{R}}$ is a differentiable one-parameter subgroup of $GL(\mathcal{W})$ with the infinitesimal generator $\Xi_{0,m}(\kappa) + d\Gamma(K)$, where*

$$\Omega'_\theta = \mathcal{G}_{1/(m\alpha)((\Omega_\theta^*)^{\otimes m}-1)\kappa,\Omega_\theta}, \qquad \theta \in \mathbb{R}.$$

Proof. By Theorem 4.2, $\{\Gamma(\Omega_\theta)\}_{z\in\mathbb{C}}$ is a differentiable one-parameter subgroup of $GL(\mathcal{W})$ with the infinitesimal generator $d\Gamma(K)$. On the other hand, $\Gamma(\Omega_\theta) = \mathcal{G}_{0,\Omega_\theta}$, $\theta \in \mathbb{R}$. Hence, by Lemma 4.4, $\{\mathcal{G}_\kappa \mathcal{G}_{0,\Omega_\theta} \mathcal{G}_\kappa^{-1}\}_{\theta\in\mathbb{R}}$ is a differentiable one-parameter subgroup of $GL(\mathcal{W})$ with the infinitesimal generator $\Xi_{0,m}(\kappa) + d\Gamma(K)$, where $\mathcal{G}_\kappa^{-1} = \mathcal{G}_{1/(m\alpha)\kappa,I}$. Therefore, by (3.3), we have

$$\mathcal{G}_\kappa^{-1}\mathcal{G}_{0,\Omega_\theta}\mathcal{G}_\kappa = \mathcal{G}_{1/(m\alpha)\left((\Omega_\theta^*)^{\otimes m}-1\right)\kappa,\Omega_\theta}.$$

This completes the proof. □

REMARK. In general, Theorem 4.3 and 4.5 is true for the Kondratiev–Streit space with $0 \le \beta < 1$ depending on $m \in \mathbb{N}$.

5. Differentiable cosine families of operators

We start with the definition of cosine family of operators.

DEFINITION 5.1. A one parameter family $\{\mathcal{C}_\theta\}_{\theta\in\mathbb{R}} \subset \mathcal{L}(\mathfrak{X})$ is called a *differentiable cosine family* if

(i) $\mathcal{C}_{\theta_1+\theta_2} + \mathcal{C}_{\theta_1-\theta_2} = 2\mathcal{C}_{\theta_1}\mathcal{C}_{\theta_2}$ for all $\theta_1, \theta_2 \in \mathbb{R}$
(ii) $\mathcal{C}_0 = I$
(iii) for each $\phi \in \mathfrak{X}$, $\mathcal{C}_\theta\phi$ is twice differentiable in $\theta \in \mathbb{R}$.

The operator $\Xi \in \mathcal{L}(\mathfrak{X})$ defined by

$$\Xi\phi = \frac{d^2}{d\theta^2}\mathcal{C}_\theta\phi\bigg|_{\theta=0}, \qquad \phi \in \mathfrak{X}$$

is called the *infinitesimal generator* of $\{\mathcal{C}_\theta\}_{\theta\in\mathbb{R}}$. Then the associated *sine family* $\{\mathcal{S}_\theta\}_{\theta\in\mathbb{R}} \subset \mathcal{L}(\mathfrak{X})$ is defined by

$$\mathcal{S}_\theta\phi = \int_0^\theta \mathcal{C}_u\phi du, \qquad \phi \in \mathfrak{X}, \quad \theta \in \mathbb{R},$$

where the integral is in the Pettis sense.

Note that if $\{\mathcal{C}_\theta\}_{\theta\in\mathbb{R}} \subset \mathcal{L}(\mathfrak{X})$ is a *differentiable cosine family* with the infinitesimal generator $\Xi \in \mathcal{L}(\mathfrak{X})$, then it is easily shown that $\{\mathcal{C}_\theta\}_{\theta\in\mathbb{R}}$ is infinitely many differentiable and

(5.1)

$$\frac{d^{2n}\mathcal{C}_\theta}{d\theta^{2n}}\phi = \Xi^n\mathcal{C}_\theta\phi, \qquad \frac{d^{2n+1}\mathcal{C}_\theta}{d\theta^{2n+1}}\phi = \Xi^n\mathcal{S}_\theta\phi, \qquad \phi \in \mathfrak{X}, \quad \theta \in \mathbb{R}, \quad n \ge 1.$$

Moreover, $[\mathcal{C}_\theta, \Xi] = 0$ and $[\mathcal{S}_\theta, \Xi] = 0$ for all $\theta \in \mathbb{R}$.

THEOREM 5.2. *Let $K \in \mathcal{L}(\mathcal{E}_{\mathbb{C}})$. Then K is the infinitesimal generator of a differentiable cosine family $\{\mathcal{C}_\theta(K)\}_{\theta\in\mathbb{R}} \subset \mathcal{L}(\mathcal{E}_{\mathbb{C}})$ if and only if $d\Gamma(K)$ is the infinitesimal generator of a differentiable cosine family $\{\Gamma(\mathcal{C}_\theta(K))\}_{\theta\in\mathbb{R}} \subset \mathcal{L}(\mathcal{W})$.*

Proof. Obviously, $\{\Gamma(\mathcal{C}_\theta(K))\}_{\theta\in\mathbb{R}}$ satisfies the conditions (i) and (ii) in Definition 5.1. To prove that for each $\phi \in \mathcal{W}$, $\Gamma(\mathcal{C}_\theta(K))\phi$ is twice differentiable in $\theta \in \mathbb{R}$ and $\Gamma(\mathcal{C}_0(K))''\phi = d\Gamma(K)\phi$, we use the same arguments as in the proof of Theorem 4.2. For each $\theta \in \mathbb{R}$, we put

$$f(h) = \langle\langle \Gamma(\mathcal{C}_{\theta+h}(K))\phi_\xi, \phi_\eta\rangle\rangle = e^{\langle \mathcal{C}_{\theta-h}(K)\xi,\eta\rangle}, \qquad \xi,\eta \in \mathcal{E}_{\mathbb{C}}, \quad h \in \mathbb{R}.$$

Then we have

$$f'(h) = \langle K\mathcal{S}_{\theta+h}(K)\xi,\eta\rangle e^{\langle \mathcal{C}_{\theta-h}(K)\xi,\eta\rangle}$$

and

$$f''(h) = \left(\langle K\mathcal{S}_{\theta+h}(K)\xi,\eta\rangle^2 + \langle K\mathcal{C}_{\theta+h}(K)\xi,\eta\rangle\right) e^{\langle \mathcal{C}_{\theta-h}(K)\xi,\eta\rangle}.$$

Let $h_0 > 0$ be fixed. Then for any $p \geq 0$, there exist $C = C(K,h_0) \geq 0$ and $q \geq 0$ such that

$$\max_{|h|\leq h_0} |f''(h)| \leq CG_\alpha(|\xi|_{p+q}^2)G_{1/\alpha}(|\eta|_{-p}^2).$$

Now we put

$$g_h(\xi,\eta) = f(h) - f(0) - f'(0)h.$$

Then we obtain that

$$|g_h(\xi,\eta)| \leq \frac{|h|^2}{2}CG_\alpha(|\xi|_{p+q}^2)G_{1/\alpha}(|\eta|_{-p}^2), \qquad \xi,\eta \in \mathcal{E}_{\mathbb{C}}, \quad |h| \leq h_0.$$

It then follows from Theorem 2.3 that there exists $\Xi_h \in \mathcal{L}(\mathcal{W})$ such that $\widehat{\Xi}_h = g_h$ and

(5.2)

$$\|\Xi_h\phi\|_{p-p_0,+}^2 \leq \frac{|h|^2}{2}C\widetilde{G}_\alpha(\|A^{-1}\|_{\mathrm{HS}}^{2r})\widetilde{G}_{1/\alpha}(\|A^{-1}\|_{\mathrm{HS}}^{2p_0})\|\phi\|_{p+q+r,+}^2, \quad |h| \leq h_0,$$

where $r, p_0 > 1/2$ with $\widetilde{G}_\alpha(\|A^{-1}\|_{\mathrm{HS}}^{2r}) < \infty$ and $\widetilde{G}_{1/\alpha}(\|A^{-1}\|_{\mathrm{HS}}^{2p_0}) < \infty$. On the other hand, we see that

$$f'(0)(\xi,\eta) = \langle K\mathcal{S}_\theta(K)\xi,\eta\rangle e^{\langle \mathcal{C}_\theta(K)\xi,\eta\rangle}, \qquad \xi,\eta \in \mathcal{E}_{\mathbb{C}}$$

satisfies the conditions (S1) and (S2′) in Theorem 2.3. Therefore, there exists $\Gamma(\mathcal{C}_\theta(K))' \in \mathcal{L}(\mathcal{W})$ such that $\widehat{\Gamma(\mathcal{C}_\theta(K))'} = f'(0)$. Hence we have

$$\Xi_h = \Gamma(\mathcal{C}_{\theta+h}(K)) - \Gamma(\mathcal{C}_\theta(K)) - h\Gamma(\mathcal{C}_\theta(K))'.$$

From (5.2), it follows that

$$\sup_{\|\phi\|_{p+q-r,-}\leq 1}\left\|\frac{\Gamma(\mathcal{C}_{\theta+h}(K))\phi-\Gamma(\mathcal{C}_\theta(K))\phi}{h}-\Gamma(\mathcal{C}_\theta(K))'\phi\right\|_{p-p_0,+}$$
$$\leq\frac{|h|}{2}C\widetilde{G}_\alpha(\|A^{-1}\|_{\mathrm{HS}}^{2r})\widetilde{G}_{1/\alpha}(\|A^{-1}\|_{\mathrm{HS}}^{2p_0})\to 0$$

as $h\to 0$. Similarly, we can prove that for each $\phi\in\mathcal{W}$, $\Gamma(\mathcal{C}_\theta(K))\phi$ is twice differentiable and

$$\begin{aligned}&\langle\langle\Gamma(\mathcal{C}_\theta(K))''\phi_\xi,\phi_\eta\rangle\rangle\\ =\ &\left(\langle K\mathcal{S}_\theta(K)\xi,\eta\rangle^2+\langle K\mathcal{C}_\theta(K)\xi,\eta\rangle\right)e^{\langle\mathcal{C}_\theta(K)\xi,\eta\rangle},\quad \xi,\eta\in\mathcal{E}_{\mathbb{C}}.\end{aligned}$$

Since $\langle K\mathcal{S}_0(K)\xi,\eta\rangle=0$,

$$\langle\langle\Gamma(\mathcal{C}_0(K))''\phi_\xi,\phi_\eta\rangle\rangle=\langle K\xi,\eta\rangle e^{\langle\xi,\eta\rangle},\qquad \xi,\eta\in\mathcal{E}_{\mathbb{C}}.$$

Hence for each $\phi\in\mathcal{W}$, $\Gamma(\mathcal{C}_\theta(K))\phi$ is twice differentiable in $\theta\in\mathbb{R}$ and $\Gamma(\mathcal{C}_0(K))''\phi=d\Gamma(K)\phi$.

The proof of the converse is obvious since

$$\int_0^\theta\Gamma(\mathcal{C}_u(K))\phi du=\Gamma(\int_0^\theta\mathcal{C}_u(K)du)\phi,\qquad \phi\in\mathcal{W}.$$

Thus the proof is completed. □

THEOREM 5.3. *Let $K\in\mathcal{L}(\mathcal{E}_{\mathbb{C}})$ be the infinitesimal generator of a differentiable cosine family $\{\mathcal{C}_\theta(K)\}_{\theta\in\mathbb{R}}$ and let $\kappa\in(\mathcal{E}_{\mathbb{C}}^{\otimes m})^*_{\mathrm{sym}}$ satisfy $(I^{\otimes(m-1)}\otimes K^*)\kappa=\alpha\kappa$, $\alpha\in\mathbb{C}$ with $\alpha\neq 0$. Then $\Xi_{0,m}(\kappa)+d\Gamma(K)$ is the infinitesimal generator of a differentiable cosine family $\{\mathcal{G}_\kappa^{-1}\Gamma(\mathcal{C}_\theta(K))\mathcal{G}_\kappa\}_{\theta\in\mathbb{R}}$.*

Proof. By Theorem 5.2, $d\Gamma(K)$ is the infinitesimal generator of a differentiable cosine family $\{\Gamma(\mathcal{C}_\theta(K))\}_{\theta\in\mathbb{R}}$. Note that if $\Xi\in\mathcal{L}(\mathcal{W})$ is the infinitesimal generator of a differentiable cosine family $\{\mathcal{C}_\theta(\Xi)\}_{\theta\in\mathbb{R}}$ and $\Xi'\in\mathcal{L}(\mathcal{W})$ satisfies that there exists $\mathcal{G}\in GL(\mathcal{W})$ such that $\Xi'=\mathcal{G}^{-1}\Xi\mathcal{G}$, then Ξ' is the infinitesimal generator of a differentiable cosine family $\{\mathcal{C}_\theta(\Xi')\}_{\theta\in\mathbb{R}}$, where $\mathcal{C}_\theta(\Xi')=\mathcal{G}^{-1}\mathcal{C}_\theta(\Xi)\mathcal{G}$, $\theta\in\mathbb{R}$. Thus, by (iii) in Theorem 3.2, the proof is obvious. □

6. Cauchy problems

Let $\Xi\in\mathcal{L}(\mathcal{W})$ be the infinitesimal generator of a differentiable one-parameter subgroup $\{\Omega_\theta\}_{\theta\in\mathbb{R}}$ of $GL(\mathcal{W})$ and let ϕ be given in $\mathcal{W}$. Then the unique solution of the Cauchy problem for the first order differential equation of the following type:

$$\frac{\partial u}{\partial \theta} = \Xi u, \qquad u(0,x) = \phi(x), \qquad \theta \in \mathbb{R}, \quad x \in \mathcal{E}^* \tag{6.1}$$

is immediately obtained by $u(\theta, x) = \Omega_\theta \phi(x)$.

Now we shall discuss the Cauchy problem for the second order differential equation of the following type:

(6.2)
$$\frac{\partial^2 u}{\partial \theta^2} = \Xi u, \qquad u(0,x) = \phi(x), \quad \frac{\partial u(0,x)}{\partial \theta} = \psi(x), \qquad \theta \in \mathbb{R}, \quad x \in \mathcal{E}^*.$$

where ϕ, ψ are given in $\mathcal{W}$ and $\Xi \in \mathcal{L}(\mathcal{W})$. We easily see that the Cauchy problem (6.2) is equivalent to the Cauchy problem for the first order differential equation:

(6.3)
$$\frac{\partial}{\partial \theta}\begin{pmatrix} u \\ \frac{\partial u}{\partial \theta} \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Xi & 0 \end{pmatrix}\begin{pmatrix} u \\ \frac{\partial u}{\partial \theta} \end{pmatrix}, \begin{pmatrix} u(0,x) \\ \frac{\partial u(0,x)}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \theta \in \mathbb{R}, \ x \in \mathcal{E}^*.$$

Put

$$\mathcal{W} \times \mathcal{W} = \left\{ \begin{pmatrix} \phi \\ \psi \end{pmatrix} : \phi, \psi \in \mathcal{W} \right\}.$$

For each $p \geq 0$, define a norm on $\mathcal{W} \times \mathcal{W}$ by

$$\left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_{p,+} = \left(\|\phi\|_{p,+}^2 + \|\psi\|_{p,+}^2 \right)^{1/2}.$$

Then $\mathcal{W} \times \mathcal{W}$ becomes a nuclear Frechet space with the topology induced by the family of norms $\{\|(\cdot)\|_{p,+}\}_{p \geq 0}$.

Let $\{\mathcal{C}_\theta(\Xi)\}_{\theta \in \mathbb{R}} \subset \mathcal{L}(\mathcal{W})$ be a differentiable cosine family with the infinitesimal generator $\Xi \in \mathcal{L}(\mathcal{W})$ and let $\{\mathcal{S}_\theta(\Xi)\}_{\theta \in \mathbb{R}} \subset \mathcal{L}(\mathcal{W})$ be the associated sine family. Then by direct computation, we obtain that for any $\theta_1, \theta_2 \in \mathbb{R}$

$$\mathcal{S}_{\theta_1+\theta_2}(\Xi) + \mathcal{S}_{\theta_1-\theta_2}(\Xi) = 2\mathcal{S}_{\theta_1}(\Xi)\mathcal{C}_{\theta_2}(\Xi) \tag{6.4}$$

and

$$\mathcal{S}_{\theta_1+\theta_2}(\Xi) - \mathcal{S}_{\theta_1-\theta_2}(\Xi) = 2\mathcal{C}_{\theta_1}(\Xi)\mathcal{S}_{\theta_2}(\Xi). \tag{6.5}$$

By using (6.4), (6.5), and (i) in Definition 5.1, we can prove that $\{\mathcal{K}_\theta(\Xi)\}_{\theta \in \mathbb{R}}$ is a differentiable one-parameter group of continuous linear operators on $\mathcal{W} \times \mathcal{W}$ with the infinitesimal generator $\begin{pmatrix} 0 & I \\ \Xi & 0 \end{pmatrix}$, where for each $\theta \in \mathbb{R}$, $\mathcal{K}_\theta(\Xi) \in \mathcal{L}(\mathcal{W} \times \mathcal{W})$ is defined by

$$\mathcal{K}_\theta(\Xi) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \mathcal{C}_\theta(\Xi) & \mathcal{S}_\theta(\Xi) \\ \mathcal{C}'_\theta(\Xi) & \mathcal{S}'_\theta(\Xi) \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

Therefore, the following theorem is straightforward.

THEOREM 6.1. *Let $\phi, \psi \in \mathcal{W}$ and let $\Xi \in \mathcal{L}(\mathcal{W})$ be the infinitesimal generator of a differentiable cosine family $\{\mathcal{C}_\theta(\Xi)\}_{\theta\in\mathbb{R}} \subset \mathcal{L}(\mathcal{W})$ and let $\{\mathcal{S}_\theta(\Xi)\}_{\theta\in\mathbb{R}}$ be the associated sine family. Then there exists a unique solution $u(\theta, x) \in \mathcal{W}$ of the Cauchy problem (6.2) which is given by*

$$u(x,\theta) = \mathcal{C}_\theta(\Xi)\phi(x) + \mathcal{S}_\theta(\Xi)\psi(x), \qquad x \in \mathcal{E}^*, \quad \theta \in \mathbb{R}. \tag{6.6}$$

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