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# VARIATIONAL PRINCIPLE FOR QUANTUM UNBOUNDED SPIN SYSTEMS

S. D. Choi, S. G. Jo, H. I. Kim, H. H. Lee, and H. J. Yoo

Abstract. We study the variational principle for quantum unbounded spin systems interacting via superstable and regular interactions. We show that the (weak) KMS state constructed via the thermodynamic limit of finite volume Green's functions satisfies the Gibbs variational equality.

## 1. Introduction

In this paper, we study the variational principle for quantum unbounded spin systems. Mostly, we aim to show that the (weak) KMS states satisfy the Gibbs variational equality.

In bounded spin systems (that is, the single spin states correspond to compact sets and finite dimensional Hilbert spaces for classical and quantum systems, respectively), it has been well-established that the equilibrium states for the systems can be investigated via the tangent functionals to the pressure which is a convex function of interactions, or via the Gibbs variational principle, or via the DLR (resp. KMS) conditions. These methods have turned out to be equivalent to each other (see the references [3-5, 10-11, 14] and the original papers cited therein). So, it would be worthwhile to extend the theory to the unbounded systems. The classical part of this problem has been investigated by Künsch [6], and so we restrict our attention only to quantum systems. The study of unbounded (continuous) spin systems, which we consider in this paper, draws also much attention from its close connection with Euclidean quantum field theory [1].

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The interactions which we consider in this paper satisfy the superstability and regularity in the sense of Ruelle [12]. The classical model for these interactions has been extensively investigated by Lebowitz and Presutti [7]. The quantum unbounded spin systems has been studied e.g., in [8] aiming at the survey of the existence of the thermodynamic limit theory of pressure and the existence and uniqueness of the equilibrium states satisfying the (weak) KMS condition. In [9], there has been proposed a characterization of Gibbs (equilibrium) states for quantum unbounded spin systems. In this paper, we will show the existence of mean entropy and show that the (weak) KMS state constructed in [8] via the thermodynamic limit of finite volume Green's functions satisfies the Gibbs variational principle.

We organize this paper as follows: In Section 2, we give necessary notations, preliminaries, and main results. Section 3 is devoted to show the proofs. In Appendix, we prove some technical inequalities needed in Section 3.

## 2. Notations, Preliminaries and Main Results

It is generally accepted that in quantum statistical mechanics, the equilibrium states are those of KMS states [2-3, 5, 14]. For quantum unbounded spin systems, by using the Green's function method [3], a state satisfying (weak) KMS condition has been constructed in [8]. In [9], there has been proposed a characterization of Gibbs states by using the concept of Gibbs measures and the conditional reduced density matrices. We will show that the (weak) KMS state constructed in [8] satisfies the Gibbs variational equality. Let us begin with presenting necessary notations. We refer to [9] for details.

Let $\mathbb{Z}^\nu$ be the $\nu$-dimensional integer lattice. At each site $i \in \mathbb{Z}^\nu$, there corresponds a vector spin variable $x_i \in \mathbb{R}^d$. For $x := (x^1, \cdots, x^d) \in \mathbb{R}^d$ and $i := (i_1, \cdots, i_\nu) \in \mathbb{Z}^\nu$, we write

$$|x| := \Big[\sum_{l=1}^{d} (x^l)^2\Big]^{1/2}, \qquad |i| := \max_{1\le l\le \nu} |i_\nu|.$$

When a subset $\Lambda \subset \mathbb{Z}^\nu$ is a finite set, we will write $\Lambda \subset\subset \mathbb{Z}^\nu$. We will consider both interactions between the spins at different sites as well as self interactions.

For each bounded region $\Lambda \subset\subset \mathbb{Z}^\nu$, the Hilbert space for the unbounded spin systems is given by

$$\begin{aligned} \mathcal{H}_\Lambda &:= \otimes_{i\in\Lambda} L^2(\mathbb{R}^d, dx_i) \\ &= L^2((\mathbb{R}^d)^\Lambda, dx_\Lambda), \end{aligned} \tag{2.1}$$

where $dx_i$ is the Lebesgue measure on $\mathbb{R}^d$ for each $i \in \mathbb{Z}^\nu$ and $dx_\Lambda := \times_{i\in\Lambda} dx_i$. The Hamiltonian operator for the region $\Lambda \subset\subset \mathbb{Z}^\nu$ is given by

$$H_\Lambda := -\frac{1}{2}\sum_{i\in\Lambda} \Delta_i + V(x_\Lambda), \tag{2.2}$$

where $\Delta_i$ is the Laplacian operator for the variable $x_i \in \mathbb{R}^d$ and $V(x_\Lambda)$ is the potential energy in the region $\Lambda$. Throughout this paper, we will impose the following conditions on the interaction:

Assumption 2.1. *The interaction $\Phi = (\Phi_\Delta)_{\Delta\subset\subset\mathbb{Z}^\nu}$ satisfies the following conditions:*

(a) *$\Phi_\Delta$ is a* Borel *measurable function on $(\mathbb{R}^d)^\Delta$ for each $\Delta \subset\subset \mathbb{Z}^\nu$.*

(b) *$\Phi_\Delta$ is invariant under translations of $\mathbb{Z}^\nu$, i.e., for any $i \in \mathbb{Z}^\nu$, $\Phi_{\Delta+i} = \tau_i\Phi_\Delta$, where $\tau_i$ is the natural translation of functions.*

(c) (Superstability) *There are $A > 0$ and $c \in \mathbb{R}$ such that for every $x_\Lambda \in (\mathbb{R}^d)^\Lambda$,*

$$V(x_\Lambda) := \sum_{\Delta\subset\Lambda} \Phi_\Delta(x_\Delta) \geq \sum_{i\in\Lambda}[Ax_i^2 - c].$$

(d) (Strong regularity) *There exists a decreasing positive function $\Psi$ on the natural numbers such that*

$$\Psi(r) \leq K r^{-\nu-\varepsilon} \text{ for some } K \text{ and } \varepsilon > 0 \text{ with } \sum_{i\in\mathbb{Z}^\nu} \Psi(|i|) < A.$$

*Furthermore, if $\Lambda_1$ and $\Lambda_2$ are disjoint finite subsets of $\mathbb{Z}^\nu$ and if one writes*

$$V(x_{\Lambda_1\cup\Lambda_2}) = V(x_{\Lambda_1}) + V(x_{\Lambda_2}) + W(x_{\Lambda_1}, x_{\Lambda_2}),$$

*then the bound*

$$|W(x_{\Lambda_1}, x_{\Lambda_2})| \leq \sum_{i \in \Lambda_1} \sum_{i \in \Lambda_2} \Psi(|i-j|)\frac{1}{2}(x_i^2 + x_j^2)$$

*holds.*

For each bounded region $\Lambda \subset\subset \mathbb{Z}^\nu$, the $C^*$-algebra for local observables is defined by

$$\mathcal{A}_\Lambda := \mathcal{L}(\mathcal{H}_\Lambda), \tag{2.3}$$

where $\mathcal{L}(\mathcal{H}_\Lambda)$ is the algebra of all bounded linear operators on $\mathcal{H}_\Lambda$. If $\Lambda_1 \cap \Lambda_2 = \emptyset$, then $\mathcal{H}_{\Lambda_1 \cup \Lambda_2} = \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2}$ and $\mathcal{A}_{\Lambda_1}$ is isomorphic to the $C^*$-algebra $\mathcal{A}_{\Lambda_1} \otimes \mathbf{1}_{\Lambda_2}$, where $\mathbf{1}_{\Lambda_2}$ denotes the identity operator on $\mathcal{H}_{\Lambda_2}$. In this way, we identify $\mathcal{A}_\Lambda$ as a sub-algebra of $\mathcal{A}_{\Lambda'}$ when $\Lambda \subset \Lambda'$. Let

$$\mathcal{A} := \overline{\cup_{\Lambda \subset\subset \mathbb{Z}^\nu} \mathcal{A}_\Lambda} \tag{2.4}$$

be the $C^*$-algebra of the quasilocal observables. Notice that $\mathcal{A}$ has an identity.

The partition function in a region $\Lambda \subset\subset \mathbb{Z}^\nu$ is given by

$$Z_\Lambda^\Phi := \mathrm{Tr}_\Lambda(e^{-H_\Lambda}), \tag{2.5}$$

where $\mathrm{Tr}_\Lambda$ means the trace on the Hilbert space $\mathcal{H}_\Lambda$. Notice that by the superstability condition of Assumption 2.1 (c), $e^{-H_\Lambda}$ belongs to the trace class and so $Z_\Lambda^\Phi$ is well defined as a finite number. The finite volume pressure is defined by

$$P_\Lambda^\Phi := \frac{1}{|\Lambda|} \log Z_\Lambda^\Phi. \tag{2.6}$$

We will suppress the superscript $\Phi$ from the notations whenever there is no confusion involved.

We notice that by the Feynman-Kac formula [13], the operator $e^{-H_\Lambda}$ has its integral kernel function

$$e^{-H_\Lambda}(x_\Lambda, y_\Lambda) = \int P_{x_\Lambda, y_\Lambda}(ds_\Lambda) \exp\Big[-\int_0^1 V(s_\Lambda(\tau))\, d\tau\Big], \tag{2.7}$$

where $x_\Lambda$ and $y_\Lambda$ are points in $(\mathbb{R}^d)^\Lambda$, $s_\Lambda \in (C([0,1];\mathbb{R}^d))^\Lambda$, and $P_{x_\Lambda, y_\Lambda}(ds_\Lambda)$ is the conditional Wiener measure on the path space

$$\Omega_{x_\Lambda, y_\Lambda} := \{s_\Lambda \in (C([0,1];\mathbb{R}^d))^\Lambda |\, s_\Lambda(0) = x_\Lambda,\ s_\Lambda(1) = y_\Lambda\}.$$

(See e.g. [13] for details.) We will simply write $V(s_\Lambda)$ for $\int_0^1 V(s_\Lambda(\tau))\,d\tau$. Thus, the partition function $Z_\Lambda$ can be rewritten as an integral over a path space:

$$Z_\Lambda = \int_{(\mathbb{R}^d)^\Lambda} dx_\Lambda \int_{\Omega_{x_\Lambda,x_\Lambda}} P_{x_\Lambda,x_\Lambda}(ds_\Lambda)e^{-V(s_\Lambda)}. \tag{2.8}$$

Since, by a translation $y_\Lambda \mapsto y_\Lambda - x_\Lambda$, the measure $P_{x_\Lambda,x_\Lambda}(ds_\Lambda)$ on $\Omega_{x_\Lambda,x_\Lambda}$ becomes the measure $P_{0;0}(ds_\Lambda) =: P(ds_\Lambda)$ on $\Omega_{0,0}$, by letting $\lambda(ds_\Lambda) := dx_\Lambda P(ds_\Lambda)$ on $(\mathbb{R}^d)^\Lambda \times \Omega_{0,0}$, the expression in (2.8) can be further simplified:

$$Z_\Lambda = \int \lambda(ds_\Lambda)e^{-V(s_\Lambda)}. \tag{2.9}$$

By using the above Wiener integral formalism and Ruelle-type probability estimates, Park has shown the existence of the thermodynamic limit of the pressure with free boundary condition [8]. Recall that a sequence $\{\Lambda_n\}$, $\Lambda_n \subset\subset \mathbb{Z}^\nu$, is said to be tending to $\mathbb{Z}^\nu$ in the sense of van Hove (we write $\lim_{\Lambda_n\to\mathbb{Z}^\nu \text{ (van Hove)}}$) if: (a) $\Lambda_{n+1} \supset \Lambda_n$, (b) $\Lambda_n \supset \Delta$ for any $\Delta \subset\subset \mathbb{Z}^\nu$ and some $n$, (c) given any parallelepiped $\Gamma$ and the partition $\pi(\Gamma)$ of $\mathbb{Z}^\nu$ generated by translations of $\Gamma$

$$\lim N_\Gamma^-(\Lambda_n) = \infty, \quad \lim \frac{N_\Gamma^-(\Lambda_n)}{N_\Gamma^+(\Lambda_n)} = 1,$$

where $N_\Gamma^-(\Lambda_n)$ is the number of sets of $\pi(\Gamma)$ contained in $\Lambda_n$ and $N_\Gamma^+(\Lambda_n)$ is the number of sets with non-void intersection with $\Lambda_n$.

THEOREM 2.2. ([8, Theorem 2.2]) *Suppose that the hypotheses of* Assumption 2.1 *hold. Then, the limit*

$$\lim_{\Lambda_n\to\mathbb{Z}^\nu \text{ (van Hove)}} P_{\Lambda_n}^\Phi = P^\Phi$$

*exists.*

We now consider the equilibrium states for quantum systems. Define the finite volume Green's functions [3] by

$$G_\Lambda(A,B;t) := \omega_\Lambda(A\alpha_t^\Lambda(B)), \quad A,\ B \in \mathcal{A}_\Lambda, \tag{2.10}$$

where $\omega_\Lambda$ is the local Gibbs state on $\mathcal{A}_\Lambda$:

$$\omega_\Lambda(A) := \frac{1}{Z_\Lambda}\mathrm{Tr}_\Lambda(e^{-H_\Lambda}A), \quad A \in \mathcal{A}_\Lambda, \tag{2.11}$$

and $\alpha_t^\Lambda$ is the time evolution automorphism on $\mathcal{A}_\Lambda$:

$$\alpha_t^\Lambda(B) := e^{itH_\Lambda} B e^{-itH_\Lambda}, \quad B \in \mathcal{A}_\Lambda. \tag{2.12}$$

By Hahn-Banach theorem, the state $\omega_\Lambda$ can be extended to a state on $\mathcal{A}$ and we use the same notation for the extension. Notice that $G_\Lambda$ is bounded as

$$|G_\Lambda(A, B; t)| \leq \|A\| \, \|B\|. \tag{2.13}$$

Thus by Tychonoff's theorem, there exists a subsequence $\{\Lambda_n\} \subset \mathbb{Z}^\nu$ such that

$$G_\Lambda(A, B; t) := \lim_{n\to\infty} G_{\Lambda_n}(A, B; t) \tag{2.14}$$

exists for all $A$, $B \in \mathcal{A}$, $t \in \mathbb{R}$ [3]. Clearly, the values

$$\omega(A) := G(A, B; t), \quad A \in \mathcal{A} \tag{2.15}$$

determine a state $\omega$ over the quasilocal algebra $\mathcal{A}$. That is, $\omega$ is a positive-definite linear functional with norm one on $\mathcal{A}$. Let us impose a further condition on the interaction:

ASSUMPTION 2.3. (Polynomial boundedness of interaction) *There exist a constant $D > 0$ and a natural number $n$ such that the one-body interaction $\Phi_{\{i\}}(x_i) \equiv P(x_i)$ satisfies the following bound:*

$$P(x_i) \leq D(|x_i|^{2n} + 1), \quad i \in \mathbb{Z}^\nu.$$

Under the conditions of Assumption 2.1 and Assumption 2.3, Park has shown that the state $\omega$ in (2.15) satisfies a (weak) KMS condition. We refer to [8] for the details. We take the state $\omega$ as a probe for our purpose.

The mean entropy for the states of quantum systems is defined as follows. Let $\rho$ be any state on $\mathcal{A}$ and $\rho_\Lambda$ the restriction of $\rho$ to $\mathcal{A}_\Lambda$, $\Lambda \subset\subset \mathbb{Z}^\nu$. Suppose that $\rho_\Lambda$ is a normal state on $\mathcal{A}_\Lambda$, i.e., there exists a density matrix $\rho^{(\Lambda)} \in \mathcal{A}_\Lambda$ such that

$$\rho_\Lambda(A) = \mathrm{Tr}_\Lambda(\rho^{(\Lambda)} A), \quad A \in \mathcal{A}_\Lambda. \tag{2.16}$$

The entropy of $\rho$ in $\Lambda$ is defined by [3, 5, 14]

$$S_\Lambda(\rho) := -\mathrm{Tr}_\Lambda(\rho^{(\Lambda)}\log\rho^{(\Lambda)}) = -\rho_\Lambda(\log\rho^{(\Lambda)}). \tag{2.17}$$

From the definition, we see that (we omit $\Phi$ from the notations)

$$S_\Lambda(\omega_\Lambda) - \omega_\Lambda(H_\Lambda) = |\Lambda| P_\Lambda. \tag{2.18}$$

In general, we have the following properties:

PROPOSITION 2.4. *Suppose that $\rho$ is a state on $\mathcal{A}$ such that the restrictions $\rho_\Lambda$ of $\rho$ to any $\mathcal{A}_\Lambda$, $\Lambda \subset\subset \mathbb{Z}^\nu$, are normal states. Then, the following properties hold:*

(a) *For any $\Lambda \subset\subset \mathbb{Z}^\nu$,*

$$S_\Lambda(\rho) - \rho(H_\Lambda) \leq |\Lambda| P_\Lambda. \tag{2.19}$$

(b) *The mean entropy*

$$s(\rho) := \lim_{a\to\infty} a^{-\nu} S_{C_a}(\rho) \tag{2.20}$$

*exists as a nonnegative finite number, where $C_a$ is a cube of sides $a$.*

We will give a proof of Proposition 2.4 in the next section. As a corollary to Proposition 2.4, we have the following variational inequality:

COROLLARY 2.5. *Suppose the hypotheses of* Proposition 2.4 *hold. Then, the inequality*

$$s(\rho) - \liminf_{a\to\infty} a^{-\nu} \rho(H_{C_a}) \leq P^\Phi \tag{2.21}$$

*holds.*

The main purpose of this paper is to show that for the state $\omega$ given in (2.15), the equality holds in (2.21):

THEOREM 2.6. *Suppose the hypotheses of* Assumption 2.1 *and* 2.3 *hold. Let $\omega$ be the equilibrium state given in* (2.15). *Then, the mean energy per unit volume $\lim_{a\to\infty} a^{-\nu}\omega(H_{C_a})$ exists and the equality*

$$s(\omega) - \lim_{a\to\infty} a^{-\nu}\omega(H_{C_a}) = P^\Phi$$

*holds.*

The proof of the above theorem will be given in the next section.

## 3. Proofs of Main Results

In this section, we provide with the proofs for the main results given in the last section. Let us begin with some lemmas that will be needed.

LEMMA 3.1. (Jensen's inequality) *Let $\rho$ be any state on $\mathcal{A}$. Then for any self-adjoint $B \in \mathcal{A}$, $e^{\rho(B)} \leq \rho(e^B)$.*

The proof of the above lemma can be found, e.g., in [5, Lemma I.3.1]. Recall the definition of entropy for states in (2.17).

LEMMA 3.2. *Suppose that $\rho$ is a state on $\mathcal{A}$ such that the restrictions $\rho_\Lambda$ of $\rho$ to any $\mathcal{A}_\Lambda$, $\Lambda \subset\subset \mathbb{Z}^\nu$, are normal states with density matrices $\rho^{(\Lambda)} \in \mathcal{A}_\Lambda$. Then the following properties hold:*

(a) $S_\Lambda(\rho) \geq 0$ (positivity).

(b) $\Lambda \cap \Lambda' = \emptyset \Longrightarrow S_{\Lambda\cup\Lambda'}(\rho) \leq S_\Lambda(\rho) + S_{\Lambda'}(\rho)$ (subadditivity).

*Proof.* (a) is obvious since $\mathrm{Tr}_\Lambda(\rho^{(\Lambda)}) = 1$. For (b), we notice that the bound

$$\mathrm{Tr}(A \log A) - \mathrm{Tr}(A \log B) \geq \mathrm{Tr}(A - B) \tag{3.1}$$

holds for any trace class non-singular positive operators $A$ and $B$. By letting $A := \rho^{(\Lambda\cup\Lambda')}$ and $B := \rho^{(\Lambda)} \otimes \rho^{(\Lambda')}$ in (3.1), we get

$$\begin{aligned} S_{\Lambda\cup\Lambda'}(\rho) &= -\mathrm{Tr}_{(\Lambda\cup\Lambda')}(A \log A) \\ &\leq -\mathrm{Tr}_{(\Lambda\cup\Lambda')}(A \log B) \\ &= -\mathrm{Tr}_\Lambda(\rho^{(\Lambda)} \log \rho^{(\Lambda)}) - \mathrm{Tr}_{\Lambda'}(\rho^{(\Lambda')} \log \rho^{(\Lambda')}) \\ &= S_\Lambda(\rho) + S_{\Lambda'}(\rho). \end{aligned}$$

□

**Proof of Proposition 2.4**. (b) is a simple consequence of the subadditivity of entropy in Lemma 3.2 (b). In order to prove (a), let $\{e_j\}$ be the normalized eigen-vectors of $\rho^{(\Lambda)}$ with $\rho^{(\Lambda)} e_j = \lambda_j e_j$. We notice that

$\sum_j \lambda_j = 1$. Then

$$\begin{aligned} e^{S_\Lambda(\rho)-\rho(H_\Lambda)} &= e^{\mathrm{Tr}_\Lambda(\rho^{(\Lambda)}(-\log\rho^{(\Lambda)}-H_\Lambda))} \\ &= \exp\big[\sum_j \lambda_j(-\log\lambda_j - <e_j, H_\Lambda e_j>)\big] \\ &\le \sum_j \exp[<e_j, -H_\Lambda e_j>] \quad \text{(by Jensen's inequality)} \\ &\le \sum_j <e_j, e^{-H_\Lambda} e_j> \quad \text{(by Lemma 3.1)} \\ &= \mathrm{Tr}_\Lambda(e^{-H_\Lambda}) = e^{|\Lambda| P_\Lambda}. \end{aligned}$$ □

We are now going to prove Theorem 2.6. For that purpose, we need to define some notations. Recall the local Gibbs states $\omega_\Lambda$ defined in (2.11). We simply write $\omega_n$ for $\omega_{\Lambda_n}$, where $\{\Lambda_n\}$ is a sequence in (2.14). We notice that the state $\omega$ in (2.15) is a weak limit of $\{\omega_n\}$. Let us write

$$K^{(n)} := \frac{1}{Z_{\Lambda_n}} \exp(-H_{\Lambda_n}) \tag{3.2}$$

and $\omega^{(\Lambda)}$ the density matrix of the restriction of $\omega$ to $\mathcal{A}_\Lambda$. When $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces and if $A$ is a trace class operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$, we denote by $\mathrm{Tr}_{(\mathcal{H}_1\otimes\mathcal{H}_2|\mathcal{H}_1)}(A)$ the partial trace of $A$ on $\mathcal{H}_2$, i.e., $\mathrm{Tr}_{(\mathcal{H}_1\otimes\mathcal{H}_2|\mathcal{H}_1)}(A)$ is a trace class operator on $\mathcal{H}_1$ such that

$$\mathrm{Tr}_{\mathcal{H}_1}(\mathrm{Tr}_{(\mathcal{H}_1\otimes\mathcal{H}_2|\mathcal{H}_1)}(A)) = \mathrm{Tr}_{\mathcal{H}_1\otimes\mathcal{H}_2}(A).$$

When $\Lambda \subset \Lambda'$, we simply write $\mathrm{Tr}_{(\Lambda'|\Lambda)}(A)$ for $\mathrm{Tr}_{(\mathcal{H}_{\Lambda'}|\mathcal{H}_\Lambda)}(A)$. One notes that

$$\omega^{(\Lambda)} = \text{w-}\lim_{n\to\infty} \mathrm{Tr}_{(\Lambda'|\Lambda)} K^{(n)}, \tag{3.3}$$

where w-limit means that

$$\mathrm{Tr}_\Lambda(\omega^{(\Lambda)}A) = \lim_{n\to\infty} \mathrm{Tr}_\Lambda((\mathrm{Tr}_{(\Lambda_n|\Lambda)}K^{(n)})A), \quad A \in \mathcal{A}_\Lambda. \tag{3.4}$$

We further simplify the notation $\mathrm{Tr}_{(\Lambda_n|\Lambda)}K^{(n)}$ by $K^{(n)}_\Lambda$ and $\mathrm{Tr}_{(\Lambda_n|\Lambda_n\setminus\Lambda)}K^{(n)}$ by $K^{(n)}_{\Lambda_n\setminus\Lambda}$.

**Proof of Theorem 2.6.** By Corollary 2.5, it is enough to show

$$s(\omega) \ge \limsup_{a\to\infty} a^{-\nu}\omega(H_{C_a}) + P^\Phi. \tag{3.5}$$

By the inequality (3.1), the subadditivity of entropy in Lemma 3.2 (b) and (2.18)-(2.19), we see that

(3.6)
$$\begin{aligned} S_\Lambda(\omega) &= -\mathrm{Tr}_\Lambda(\omega^{(\Lambda)}\log\omega^{(\Lambda)}) \\ &= -\lim_{n\to\infty}\mathrm{Tr}_\Lambda(K_\Lambda^{(n)}\log\omega^{(\Lambda)}) \\ &\geq -\limsup_{n\to\infty}\mathrm{Tr}_\Lambda(K_\Lambda^{(n)}\log K_\Lambda^{(n)}) \\ &= \limsup_{n\to\infty} S_\Lambda(\omega_n) \\ &\geq \limsup_{n\to\infty}[S_{\Lambda_n}(\omega_n) - S_{\Lambda_n\setminus\Lambda}(\omega_n)] \\ &\geq \limsup_{n\to\infty}[\omega_n(H_{\Lambda_n}) + |\Lambda_n|P_{\Lambda_n} - \omega_n(H_{\Lambda_n\setminus\Lambda}) - |\Lambda_n\setminus\Lambda|P_{\Lambda_n\setminus\Lambda}]. \end{aligned}$$

Recall the definition of the measure $d\lambda$ on the path spaces appeared in (2.9) and define for $0\leq\alpha\leq 1$,

$$Z_{\Lambda_n}(\alpha) := \int d\lambda(s_{\Lambda_n})\exp[-V(s_\Lambda) - V(s_{\Lambda_n\setminus\Lambda}) - \alpha W(s_\Lambda, s_{\Lambda_n\setminus\Lambda})]. \tag{3.7}$$

Notice that

$$\begin{aligned} &\log Z_{\Lambda_n}(1) = \log Z_{\Lambda_n} = |\Lambda_n|P_{\Lambda_n}, \\ &\log Z_{\Lambda_n}(0) = \log Z_\Lambda + \log Z_{\Lambda_n\setminus\Lambda} = |\Lambda|P_\Lambda + |\Lambda_n\setminus\Lambda|P_{\Lambda_n\setminus\Lambda}. \end{aligned} \tag{3.8}$$

Using (3.8) in the last expression of (3.6), we get

$$\begin{aligned} S_\Lambda(\omega) \geq \limsup_{n\to\infty}\Big[\omega_n(H_\Lambda) &+ |\Lambda|P_\Lambda + \omega_n(W(x_\Lambda, x_{\Lambda_n\setminus\Lambda})) \\ &+ \log Z_{\Lambda_n}(1) - \log Z_{\Lambda_n}(0)\Big]. \end{aligned} \tag{3.9}$$

In Appendix, we will show that there exists $c>0$ such that the bounds

$$\begin{aligned} \limsup_{n\to\infty}|\omega_n(W(x_\Lambda, x_{\Lambda_n\setminus\Lambda}))| &\leq c\sum_{i\in\Lambda}\sum_{j\in\Lambda^c}\Psi(|i-j|), \\ \limsup_{n\to\infty}|\log Z_{\Lambda_n}(1) - \log Z_{\Lambda_n}(0)| &\leq c\sum_{i\in\Lambda}\sum_{j\in\Lambda^c}\Psi(|i-j|) \end{aligned} \tag{3.10}$$

hold. Using (3.10) in (3.9), we get

$$S_\Lambda(\omega) \geq \omega(H_\Lambda) + |\Lambda|P_\Lambda - 2c\sum_{i\in\Lambda}\sum_{j\in\Lambda^c}\Psi(|i-j|). \tag{3.11}$$

We take $\Lambda$ to be the cubes $C_a$ of sides $a$ and divide both sides of (3.11) by $a^\nu$, the volume of $C_a$. Since $\Psi(r)$ decreases as $r$ increases and $\sum_{i\in\mathbb{Z}^\nu}\Psi(|i|) < \infty$, it is easy to show that

$$\lim_{a\to\infty} a^{-\nu}\sum_{i\in C_a}\sum_{j\in C_a{}^c}\Psi(|i-j|)=0.$$

Thus by using (2.20) and Theorem 2.2, we get the inequality (3.5). The proof is now completed. □

## 4. Appendix

We show the inequalities in (3.10). We first prove the second inequality of (3.10). By the mean value theorem, there exists $\alpha_1 \in (0,1)$ such that

$$(A.1)\quad \begin{aligned}&\log Z_{\Lambda_n}(1)-\log Z_{\Lambda_n}(0)\\ &=\frac{d}{d\alpha}\log Z_{\Lambda_n}(\alpha)\Big|_{\alpha=\alpha_1}\\ &=\frac{1}{Z_{\Lambda_n}(\alpha_1)}\int d\lambda(s_{\Lambda_n})\exp\Big[-V(s_\Lambda)-V(s_{\Lambda_n\setminus\Lambda})-\alpha_1 W(s_\Lambda,s_{\Lambda_n\setminus\Lambda})\Big]\\ &\qquad\qquad\Big(-W\big(s_\Lambda,s_{\Lambda_n\setminus\Lambda}\big)\Big)\\ &=:<-W(s_\Lambda,s_{\Lambda_n\setminus\Lambda})>_{\alpha_1},\end{aligned}$$

where the expectation $<\cdot>_\alpha$, $0\le\alpha\le 1$, is defined by

$$(A.2)\quad \begin{aligned}<A>_\alpha:=\tfrac{1}{Z_{\Lambda_n}(\alpha)}\int d\lambda(s_{\Lambda_n})\exp\Big[&-V(s_\Lambda)-V(s_{\Lambda_n\setminus\Lambda})\\ &-\alpha W(s_\Lambda,s_{\Lambda_n\setminus\Lambda})\Big]A(s_{\Lambda_n}).\end{aligned}$$

We then have

$$\frac{d^2}{d\alpha^2}\log Z_{\Lambda_n}(\alpha)=<W(s_\Lambda,s_{\Lambda_n\setminus\Lambda})^2>_\alpha-<W(s_\Lambda,s_{\Lambda_n\setminus\Lambda})>_\alpha^2\ge 0.$$

That is, the function $\alpha\mapsto\frac{d}{d\alpha}\log Z_{\Lambda_n}(\alpha)$ is a monotone function and hence we have

$$(A.3)\quad \begin{aligned}&|\log Z_{\Lambda_n}(1)-\log Z_{\Lambda_n}(0)|\\ &\le\max\Big\{\Big|\tfrac{d}{d\alpha}\log Z_{\Lambda_n}(\alpha)\big|_{\alpha=0}\Big|,\ \Big|\tfrac{d}{d\alpha}\log Z_{\Lambda_n}(\alpha)\big|_{\alpha=1}\Big|\Big\}.\end{aligned}$$

We calculate $\frac{d}{d\alpha}\log Z_{\Lambda_n}(\alpha)\big|_{\alpha=1}$. The other one can be done similarly. First, notice that by a superstable estimate established in [9, Proposition 2.5], there exist $A^* > 0$ and $\delta > 0$ such that for any $\Delta \subset \Lambda_n$,

$$\frac{1}{Z_{\Lambda_n}}\int d\lambda(s_{\Lambda_n\setminus\Delta})e^{-V(s_{\Lambda_n})} \le \exp\big[\sum_{i\in\Delta}(-A^* s_i^2+\delta)\big]. \tag{A.4}$$

Then, by the regularity of the interaction given in Assumption 2.1 (d), we see that

$$\begin{aligned}
&\big|\frac{d}{d\alpha}\log Z_{\Lambda_n}(\alpha)\big|_{\alpha=1}\big| \\
&= \big|\frac{1}{Z_{\Lambda_n}}\int d\lambda(s_{\Lambda_n})e^{-V(s_{\Lambda_n})}W(s_\Lambda, s_{\Lambda_n\setminus\Lambda})\big| \\
&\le \sum_{i\in\Lambda}\sum_{j\in\Lambda_n\setminus\Lambda}\frac{1}{2}\Psi(|i-j|)\frac{1}{Z_{\Lambda_n}}\int d\lambda(s_{\Lambda_n})e^{-V(s_{\Lambda_n})}(s_i^2+s_j^2) \\
&\le \sum_{i\in\Lambda}\sum_{j\in\Lambda_n\setminus\Lambda}\frac{1}{2}\Psi(|i-j|)\int d\lambda(s_i)d\lambda(s_j)e^{-A^*(s_i^2+s_j^2)+2\delta}(s_i^2+s_j^2) \\
&\le c\sum_{i\in\Lambda}\sum_{j\in\Lambda_n\setminus\Lambda}\Psi(|i-j|).
\end{aligned}$$

In order to prove the first inequality of (3.10), we notice, by using the regularity of the interaction and (A.4), that

(A.5)
$$\begin{aligned}
&|\omega_n(W(x_\Lambda, x_{\Lambda_n\setminus\Lambda}))| \\
&= \big|\frac{1}{Z_{\Lambda_n}}\int d\lambda(s_{\Lambda_n})e^{-V(s_{\Lambda_n})}W(x_\Lambda, x_{\Lambda_n\setminus\Lambda})\big| \\
&\le \sum_{i\in\Lambda}\sum_{j\in\Lambda_n\setminus\Lambda}\frac{1}{2}\Psi(|i-j|)\int d\lambda(s_i)d\lambda(s_j)e^{-A^*(s_i^2+s_j^2)+2\delta}(s_i(0)^2+s_j(0)^2).
\end{aligned}$$

Notice that

$$\int d\lambda(s_i)e^{-A^* s_i^2}s_i(0)^2 = \int dx_i\, x_i^2\int P_{x_i,x_i}(ds_i)e^{-A^* s_i^2}. \tag{A.6}$$

By using the same argument used in the proof of the bounds in (A.13)-(A.15) of [9], we get that the r.h.s. of (A.6) is bounded by

$$\int dx_i\, x_i^2 e^{-A' x_i^2+\delta'} \le c' < \infty, \tag{A.7}$$

for some constants $A'$, $\delta'$, and $c'$. Inserting (A.7) into (A.5), we finish the proof of (3.10).

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Sang Don Choi
Department of Physics
Kyungpook National University
Taegu 702-701, Korea
*E-mail*: sdchoi@kyungpook.ac.kr

Sang Gyu Jo
Department of Physics
Kyungpook National University
Taegu 702-701, Korea
*E-mail*: sgjo@kyungpook.ac.kr

Ho Il Kim
TGRC
Kyungpook National University
Taegu 702-701, Korea
*E-mail*: hikim@gauss.kyungpook.ac.kr

Hung Hwan Lee
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
*E-mail*: hhlee@kyungpook.ac.kr

Hyun Jae Yoo
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
*E-mail*: yoohj@gauss.kyungpook.ac.kr