J. Korean Math. Soc. **37** (2000), No. 4, pp. 503–519

NORMALIZING MAPPINGS OF AN ANALYTIC GENERIC CR MANIFOLD WITH ZERO LEVI FORM

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Abstract. It is well-known that an analytic generic CR submanifold M of codimension m in $\mathbb{C}^{n+m}$ is locally transformed by a biholomorphic mapping to a plane $\mathbb{C}^n \times \mathbb{R}^m \subset \mathbb{C}^n \times \mathbb{C}^m$ whenever the Levi form L on M vanishes identically. We obtain such a normalizing biholomorphic mapping of M in terms of the defining function of M. Then it is verified without Frobenius theorem that M is locally foliated into complex manifolds of dimension n.

0. Introduction

Let $\rho_1, \cdots, \rho_m$ be real-valued functions near the origin in $\mathbb{C}^{n+m}$ such that

$$\rho_1|_0 = \cdots = \rho_m|_0 = 0$$

and

$$\partial\rho_1 \wedge \cdots \wedge \partial\rho_m|_0 \neq 0.$$

Suppose that a generic CR submanifold M of codimension m in a sufficiently small domain $\Omega \ni 0$ is defined by the real-valued functions $\rho_1, \cdots, \rho_m$ as follows

$$\rho_1 = \cdots = \rho_m = 0.$$

Then there is a natural differential system D on M defined by

$$d\rho_1 = \cdots = d\rho_m = d^c\rho_1 = \cdots = d^c\rho_m = 0$$

where d^c is the imaginary part of ∂. The differential system D is indeed a subbundle of real dimension $2n$ in TM. Further, the complex structure of $\mathbb{C}^{n+m}$ induces a bundle automorphism I on D satisfying the following conditions

Received January 23, 1998. Revised April 15, 2000.
2000 Mathematics Subject Classification: Primary 32H99.
Key words and phrases: CR submanifold, Levi form, biholomorphic mapping.

$$(1) \qquad I^2U = -U$$
$$(2) \qquad [U,V] - [IU,IV], \quad [IU,V] + [U,IV] \in \Gamma D$$
$$(3) \qquad [U,V] - [IU,IV] + I([IU,V] + [U,IV]) = 0$$

for all $U, V \in \Gamma D$. By (1), we have the following decomposition

$$D \otimes \mathbb{C} = H \oplus \overline{H},$$

where

$$IW = iW \quad \text{for} \quad W \in \Gamma H.$$

Then (2) and (3) are equivalent to

$$[W, Z] \in \Gamma H \quad \text{for} \quad W, Z \in \Gamma H.$$

Then the Levi form L of the generic CR submanifold M is defined by the intrinsic objects (M, D, I) as the composition of the following sequence

$$D \otimes D \xrightarrow{b_1} TM \xrightarrow{b_2} TM/D,$$

where b_1 is the Lie bracket with the operation I as follows

$$b_1(U, V) = [U, IV]$$

and b_2 is the natural projection. Clearly, the Levi form L is also an intrinsic object of M. With (1) and (2), we obtain the following properties of the Levi form L

$$\begin{aligned} L(fU, V) &= L(U, fV) = fL(U, V) \\ L(U, V) &= L(V, U) \\ L(IU, IV) &= L(U, V) \end{aligned}$$

for $f \in \Gamma(M, \mathbb{R})$ and $U, V \in \Gamma D$. Hence we obtain

$$L(W, Z) = L(\overline{W}, \overline{Z}) = 0$$

for $W, Z \in \Gamma H$. Thus the Levi form L is completely determined by the value $L(W, \overline{Z})$.

Note that the operation I is an automorphism on D. Thus the Levi form L is faithfully represented by a two-form l obtained by composing the following sequence

$$\Lambda^2 D \xrightarrow{b_1'} TM \xrightarrow{b_2} TM/D \to TM/D \otimes (TM/D)^* \to M \times \mathbb{R},$$

where b_1^* is the Lie bracket. Since the generic CR submanifold M is defined by the real-valued functions $\rho_1, \cdots, \rho_m$ satisfying the condition $\partial\rho_1 \wedge \cdots \wedge \partial\rho_m \neq 0$, the one-forms $d^c\rho_1, \cdots, d^c\rho_m$ make a basis of $(TM/D)^*$. Then we define a two-form $l = (l_1, \cdots, l_m)$ as follows

$$\begin{aligned} l_1(U,V) &= -d^c\rho_1([U,V]) = 2dd^c\rho_1(U,V) = 2i\partial\overline{\partial}\rho_1(U,V) \\ &\cdots \\ l_m(U,V) &= -d^c\rho_m([U,V]) = 2dd^c\rho_m(U,V) = 2i\partial\overline{\partial}\rho_m(U,V) \end{aligned}$$

for $U, V \in \Gamma D$. Note that the differential system D on M is defined by the one-forms

$$d\rho_1, \cdots, d\rho_m, d^c\rho_1, \cdots, d^c\rho_m.$$

Thus the Levi form L is essentially equivalent to the information of the two-form

$$l = 2dd^c\rho = 2i\partial\overline{\partial}\rho$$

up to

$$\mod \quad d\rho_1, \cdots, d\rho_m, d^c\rho_1, \cdots, d^c\rho_m.$$

Then the zero Levi form is represented by the following condition

$$l \equiv 0 \quad \mod \quad d\rho_1, \cdots, d\rho_m, d^c\rho_1, \cdots, d^c\rho_m.$$

Since we have

$$\phi^* \circ \partial = \partial \circ \phi^*, \quad \phi^* \circ \overline{\partial} = \overline{\partial} \circ \phi^*$$

for any biholomorphic mapping ϕ, the zero Levi form leaves invariant under a biholomorphic mapping ϕ as follows

$$\begin{aligned} 2i\partial\overline{\partial}\phi^*\rho &= 2i\phi^*\partial\overline{\partial}\rho \\ &\equiv 0 \quad \mod \quad d\phi^*\rho_1, \cdots, d\phi^*\rho_m, d^c\phi^*\rho_1, \cdots, d^c\phi^*\rho_m. \end{aligned}$$

It is well-known that a generic CR submanifold M with zero Levi form is locally foliated into complex manifolds(cf. [1]). Further, an analytic generic CR submanifold M with zero Levi form is locally biholomorphic to a plane $\mathbb{C}^n \times \mathbb{R}^m \subset \mathbb{C}^n \times \mathbb{C}^m$. We shall obtain a biholomorphic mapping in terms of the defining functions $\rho_1, \cdots, \rho_m$ which transforms M to a plane $\mathbb{C}^n \times \mathbb{R}^m$. Thus it is verified that M is locally analytically foliated into complex manifolds of complex dimension n, which has been obtained within our knowledge under the assumption of Frobenius theorem for the existence of a foliation and Newlander-Nirenberg theorem/Levi-Civita theorem for its leaf to be a complex manifold(cf. [1]).

1. Straightening a totally real surface Γ

Let M be an analytic generic CR submanifold in $\Omega \subset \mathbb{C}^{n+m}$ near the origin defined by

$$\rho_1 = \cdots = \rho_m = 0,$$

where

$$\partial\rho_1 \wedge \cdots \wedge \partial\rho_m \neq 0.$$

Then we may take a coordinate $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$, if necessary, after a suitable linear change of coordinates such that

$$\rho = -v + F(z, \bar{z}, u), \quad F|_0 = dF|_0 = 0,$$

where $\rho = (\rho_1, \cdots, \rho_m)$, $u = \Re w$ and $v = \Im w$. Thus M is defined near the origin by the following equation

$$v = F(z, \bar{z}, u), \quad F|_0 = dF|_0 = 0.$$

Let Γ be an analytic real surface of dimension m on M, which is transversal to the complex tangent hyperplane at the origin 0. Then the equation of Γ is given near the origin as follows

$$\Gamma \begin{cases} z = p(\mu) \\ w = q(\mu), \end{cases}$$

where

$$p(0) = q(0) = 0, \quad \det q'(0) \neq 0.$$

By the condition $F|_0 = dF|_0 = 0$, we can take the $\mathbb{R}^m$-valued parameter μ such that

$$q'(0) = Id_{m\times m}, \quad \Re q(\mu) = \mu,$$

where $Id_{m\times m}$ is the identity matrix and $\Re q(\mu)$ is the real part of $q(\mu)$. Hence the real surface Γ on M determines a unique function $p(\mu)$ and Γ is uniquely described by the function $p(\mu)$ via the following equation

$$\Gamma \begin{cases} z = p(\mu) \\ u = \mu \\ v = F\big(p(\mu), \overline{p}(\mu), \mu\big). \end{cases} \tag{4}$$

Assume that the generic CR submanifold M and the surface Γ on M are both analytic so that the functions $F(z, \overline{z}, u)$ and $p(u)$ are both analytic. Then there is a unique holomorphic function $g(z, w)$, which is implicitly defined by the equations

(5)

$$\begin{aligned} g(z,w)-g(0,w) &= -2iF\big(p(w),\bar{p}(w),w\big) \\ &\quad +2iF\Big(z+p(w)\,,\bar{p}(w)\,,w+\tfrac{1}{2}\{g(z,w)-g(0,w)\}\Big), \\ g(0,w) &= iF\big(p(w)\,,\bar{p}(w)\,,w\big). \end{aligned}$$

The holomorphic function $g(z,w)$ is well defined because of the condition

$$F|_0 = dF|_0 = 0,$$

which implies

$$g|_0 = \left.\frac{\partial g}{\partial z}\right|_0 = \left.\frac{\partial g}{\partial w}\right|_0 = 0. \tag{6}$$

Then we consider a holomorphic mapping near the origin as follows

$$\begin{aligned} z &= z^* + p(w^*), \\ w &= w^* + g(z^*, w^*). \end{aligned} \tag{7}$$

By (6), the mapping (7) is biholomorphic near the origin for any analytic function $p(u)$. We claim that the generic CR submanifold M is transformed to a generic CR submanifold M' of the form

$$v = \sum_{s,t=1}^{\infty} F^*_{st}(z,\bar{z},u)$$

and the surface Γ on M via the equation (4) is mapped on the u-plane, $z = v = 0$, under the biholomorphic mapping (7).

Suppose that the generic CR submanifold M' is defined by

$$v^* = F^*(z^*\,,\bar{z}^*\,,u^*).$$

The mapping (7) yields the following equality

$$F(z\,,\bar{z}\,,u) = F^*(z^*\,,\bar{z}^*\,,u^*) + \frac{1}{2i}\Big\{g(z^*\,,u^*+iv^*) - \bar{g}(\bar{z}^*\,,u^*-iv^*)\Big\},$$

where

$$\begin{aligned} z &= z^* + p(u^*+iv^*), \\ \bar{z} &= \bar{z}^* + \bar{p}(u^*-iv^*), \\ u &= u^* + \frac{1}{2}\Big\{g(z^*\,,u^*+iv^*) + \bar{g}(\bar{z}^*\,,u^*-iv^*)\Big\}. \end{aligned}$$

Since F and F^* are both real-analytic, we can consider $z^*, \bar{z}^*$ and u^* as independent variables. Hence the condition of $F^*(z^*, 0, u^*) = v^* = 0$ is equivalent to the following equality

(8)

$$g(z, u) - \overline{g}(0, u) = 2iF\Big(z + p(u), \bar{p}(u), u + \tfrac{1}{2}\{g(z, u) + \overline{g}(0, u)\}\Big).$$

We obtain an equality by taking $z = 0$

$$g(0, u) - \overline{g}(0, u) = 2iF\Big(p(u), \bar{p}(u), u + \frac{1}{2}\{g(0, u) + \overline{g}(0, u)\}\Big),$$

which implies that

$$g(0, u) + \overline{g}(0, u) = 0$$

if and only if

$$g(0, u) = iF(p(u), \bar{p}(u), u).$$

Hence (8) reduces to

$$\begin{aligned} g(z, u) - g(0, u) &= -2iF\Big(p(u), \bar{p}(u), u\Big) \\ &\quad + 2iF\Big(z + p(u), \bar{p}(u), u + \frac{1}{2}\Big\{g(z, u) - g(0, u)\Big\}\Big). \end{aligned}$$

Thus the equality (8) is satisfied by the function $g(z, w)$ defined in the mapping (5). By putting

$$z^* = \overline{z}^* = v^* = 0$$

in (7), we obtain

$$\begin{aligned} z &= p(u^*), \\ u &= u^*, \\ v &= F(p(u^*), \overline{p}(u^*), u^*). \end{aligned}$$

Thus the surface Γ on M in (4) is mapped on the u-plane by the biholomorphic mapping (7).

From the equation (5), we obtain the holomorphic function $g(z,w)$ up to order 2 inclusive of the variable z as follows

$$
(9)\qquad
\begin{aligned}
g(z,w) &= iF\big(p(w)\,,\overline{p}(w)\,,w\big)\\
&+2i\big(Id-iF'\big)^{-1}\Big\{\textstyle\sum_{\alpha=1}^{n} z^{\alpha}\big(\frac{\partial F}{\partial z^{\alpha}}\big)\big(p(w),\overline{p}(w),w\big)\\
&\quad+\textstyle\sum_{\alpha,\beta=1}^{n}\frac{z^{\alpha}z^{\beta}}{2}\big(\frac{\partial^2 F}{\partial z^{\alpha}\partial z^{\beta}}\big)(p(w),\overline{p}(w),w)\Big\}\\
&-2(Id-iF')^{-1}\Big\{\textstyle\sum_{\alpha=1}^{n} z^{\alpha}\big(\frac{\partial F'}{\partial z^{\alpha}}\big)\Big\}\\
&\quad\times(Id-iF')^{-1}\Big\{\textstyle\sum_{\alpha=1}^{n} z^{\alpha}\big(\frac{\partial F}{\partial z^{\alpha}}\big)(p(w),\overline{p}(w),w)\Big\}\\
&-2i(Id-iF')^{-1}F''\\
&\quad\times\Big((Id-iF')^{-1}\big\{\textstyle\sum_{\alpha=1}^{n} z^{\alpha}\big(\frac{\partial F}{\partial z^{\alpha}}\big)\big(p(w),\overline{p}(w),w\big)\big\}\Big)^2\\
&+\textstyle\sum_{|I|=3} O(z^{I}),
\end{aligned}
$$

where

$$
\begin{aligned}
(F')_{ab} &= \left(\frac{\partial F^a}{\partial u^b}\right)(p(w)\,,\overline{p}(w)\,,w),\\
\left(\frac{\partial F'}{\partial z^{\alpha}}\right)_{ab} &= \left(\frac{\partial^2 F^a}{\partial z^{\alpha}\partial u^b}\right)(p(w)\,,\overline{p}(w)\,,w),\\
(F'')_{abc} &= \frac{1}{2}\left(\frac{\partial^2 F^a}{\partial u^b\partial u^c}\right)(p(w)\,,\overline{p}(w)\,,w).
\end{aligned}
$$

We shall examine the dependence of the function $F^*_{11}(z\,,\overline{z}\,,u)$ of the lowest type $(1,1)$ on the function $p(u)$ and its derivatives.

LEMMA 1. *Let M' be the generic CR submanifold obtained from M by the mapping (7) and defined by*

$$
v = F^*(z,\bar{z},u) = \sum_{s,t=1}^{\infty} F^*_{st}(z,\bar{z},u).
$$

Then the function $F^*_{11}(z,\overline{z},u)$ *depends on* $p(u)$ *and* $p'(u)$ *as follows*

$$
\begin{aligned}
&F^*_{11}(z,\bar{z},u)\\
&=\left\{Id - i(Id+iF')\sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial z^\alpha}\right)p'^\alpha\right.\\
&\qquad\qquad\left.+i(Id-iF')\sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial \overline{z}^\alpha}\right)\overline{p}'^\alpha+(F')^2\right\}^{-1}\\
&\quad\times\left\{\sum_{\alpha,\beta=1}^{n}\left(\frac{\partial^2 F}{\partial z^\alpha\partial\overline{z}^\beta}\right)z^\alpha\overline{z}^\beta\right.\\
&\qquad\qquad -i\sum_{\alpha,\beta=1}^{n}\left(\frac{\partial F'}{\partial z^\alpha}\right)(Id+iF')^{-1}z^\alpha\overline{z}^\beta\\
&\qquad\qquad +i\sum_{\alpha,\beta=1}^{n}\left(\frac{\partial F'}{\partial \overline{z}^\alpha}\right)(Id-iF')^{-1}\left(\frac{\partial F}{\partial z^\beta}\right)\overline{z}^\alpha z^\beta\\
&\qquad\qquad \left.-2\sum_{\alpha,\beta=1}^{n}F''(Id-iF')^{-1}\left(\frac{\partial F}{\partial z^\alpha}\right)z^\alpha(Id+iF')^{-1}\left(\frac{\partial F}{\partial \overline{z}^\beta}\right)\overline{z}^\beta\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\left(\frac{\partial F}{\partial z^\alpha}\right)_a &= \left(\frac{\partial F^a}{\partial z^\alpha}\right)\big(p(u)\,,\overline{p}(u)\,,u\big),\\
\left(\frac{\partial^2 F}{\partial z^\alpha\partial\overline{z}^\beta}\right)_a &= \left(\frac{\partial^2 F^a}{\partial z^\alpha\partial\overline{z}^\beta}\right)\big(p(u)\,,\overline{p}(u)\,,u\big),\\
\left\{\left(\frac{\partial F}{\partial z^\alpha}\right)p^{\alpha\prime}\right\}_{ab} &= \left(\frac{\partial F^a}{\partial z^\alpha}\right)\big(p(u)\,,\overline{p}(u)\,,u\big)\left(\frac{\partial p^\alpha}{\partial u^b}\right)(u),\\
\left(\frac{\partial F'}{\partial z^\alpha}\right)_{ab} &= \left(\frac{\partial^2 F^a}{\partial z^\alpha\partial u^b}\right)\big(p(u)\,,\overline{p}(u)\,,u\big),\\
(F')_{ab} &= \left(\frac{\partial F^a}{\partial u^b}\right)\big(p(u)\,,\overline{p}(u)\,,u\big),\\
(F'')_{abc} &= \frac{1}{2}\left(\frac{\partial^2 F^a}{\partial u^b\partial u^c}\right)\big(p(u)\,,\overline{p}(u)\,,u\big).
\end{aligned}
$$

Proof. The generic CR submanifold M' is defined by the following equation

$$\begin{aligned} v &= F\Big(z+p(u+iv), \bar z+\bar p(u-iv), \\ &\qquad u+\frac{1}{2}\{g(z,u+iv)+\bar g(\bar z,u-iv)\}\Big) \\ &\quad -\frac{1}{2i}\{g(z,u+iv)-\bar g(\bar z,u-iv)\} \\ &= A(z,\overline{z},u)+B(z,\overline{z},u)v+O(|v|^2), \end{aligned}$$

where

$$\begin{aligned} A(z,\overline{z},u) &= F\left(z+p(u),\bar z+\bar p(u),u+\frac{1}{2}\{g(z,u)+\bar g(\bar z,u)\}\right) \\ &\quad -\frac{1}{2i}\{g(z,u)-\bar g(\bar z,u)\} \end{aligned}$$

$$\begin{aligned} &B(z,\overline{z},u) \\ &= i\sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial z^\alpha}\right)\left(z+p(u),\bar z+\bar p(u),u+\frac{1}{2}\{g(z,u)+\bar g(\bar z,u)\}\right)p^{\alpha\prime}(u) \\ &\quad -i\sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial \overline{z}^\alpha}\right)\left(z+p(u),\bar z+\bar p(u),u+\frac{1}{2}\{g(z,u)+\bar g(\bar z,u)\}\right)\overline{p}^{\alpha\prime}(u) \\ &\quad -F'\left(z+p(u),\bar z+\bar p(u),u+\frac{1}{2}\{g(z,u)+\bar g(\bar z,u)\}\right) \\ &\qquad\times\frac{1}{2i}\{g'(z,u)-\bar g'(\bar z,u)\} \\ &\quad -\frac{1}{2}\{g'(z,u)+\bar g'(\bar z,u)\}. \end{aligned}$$

With the function $g(z,w)$ in (5), we can put

$$\begin{aligned} A(z,\overline{z},u) &= \sum_{s,t\geq 1} A_{st}(z,\overline{z},u), \\ B(z,\overline{z},u) &= \sum_{s,t\geq 0} B_{st}(z,\overline{z},u). \end{aligned}$$

By using the expansion (9) of $g(z,w)$, we obtain

$$v=\left\{Id-B_{00}(z,\overline{z},u)\right\}^{-1}A_{11}(z,\overline{z},u)+O(|z|^3),$$

where

$$\begin{aligned}
&\Big\{A_{11}(z,\overline{z},u)\Big\}_a \\
&= \sum_{\alpha,\beta=1}^{n} z^\alpha \overline{z}^\beta \left(\frac{\partial^2 F^a}{\partial z^\alpha \partial \overline{z}^\beta}\right) \big(p(u),\overline{p}(u),u\big) \\
&\quad + \sum_{\alpha,\beta=1}^{n} \sum_{b=1}^{m} \frac{z^\alpha \overline{z}^\beta}{2} \left(\frac{\partial^2 F^a}{\partial z^\alpha \partial u^b}\right) \big(p(u),\overline{p}(u),u\big) \left(\frac{\partial \overline{g}^b}{\partial \overline{z}^\beta}\right)(0,u) \\
&\quad + \sum_{\alpha,\beta=1}^{n} \sum_{b=1}^{m} \frac{z^\alpha \overline{z}^\beta}{2} \left(\frac{\partial^2 F^a}{\partial \overline{z}^\beta \partial u^b}\right) \big(p(u),\overline{p}(u),u\big) \left(\frac{\partial g^b}{\partial z^\alpha}\right)(0,u) \\
&\quad + \sum_{\alpha,\beta=1}^{n} \sum_{b,c=1}^{m} \frac{z^\alpha \overline{z}^\beta}{4} \left(\frac{\partial^2 F^a}{\partial u^b \partial u^c}\right) \big(p(u),\overline{p}(u),u\big) \\
&\qquad\qquad \times \left(\frac{\partial g^b}{\partial z^\alpha}\right)(0,u) \left(\frac{\partial \overline{g}^c}{\partial \overline{z}^\beta}\right)(0,u) \\
&\Big\{B_{00}(z,\overline{z},u)\Big\}_{ab} \\
&= i \sum_{\alpha=1}^{n} \left(\frac{\partial F^a}{\partial z^\alpha}\right) \big(p(u),\overline{p}(u),u\big) \left(\frac{\partial p^\alpha}{\partial u^b}\right)(u) \\
&\quad - i \sum_{\alpha=1}^{n} \left(\frac{\partial F^a}{\partial \overline{z}^\alpha}\right) \big(p(u),\overline{p}(u),u\big) \left(\frac{\partial \overline{p}^\alpha}{\partial u^b}\right)(u) \\
&\quad - \frac{1}{2i} \sum_{c=1}^{m} \left(\frac{\partial F^a}{\partial u^c}\right) \big(p(u),\overline{p}(u),u\big) \\
&\qquad\qquad \times \left\{ \left(\frac{\partial g^c}{\partial u^b}\right)(0,u) - \left(\frac{\partial \overline{g}^c}{\partial u^b}\right)(0,u) \right\}.
\end{aligned}$$

From the expansion (9), we obtain

$$\begin{aligned}
g(0,u) &= iF\big(p(u),\overline{p}(u),u\big), \\
\left(\frac{\partial g^b}{\partial z^\alpha}\right)(0,u) &= \left\{ 2i(Id - iF')^{-1} \left(\frac{\partial F}{\partial z^\alpha}\right) \big(p(u),\overline{p}(u),u\big) \right\}_b, \\
\left(\frac{\partial g^b}{\partial u^c}\right)(0,u) &= \left\{ i \sum_\alpha \left(\frac{\partial F}{\partial z^\alpha}\right) p^{\alpha\prime}(u) + i \sum_\alpha \left(\frac{\partial F}{\partial \overline{z}^\alpha}\right) \overline{p}^{\alpha\prime}(u) + iF' \right\}_{bc},
\end{aligned}$$

where

$$\left\{\left(\frac{\partial F}{\partial z^\alpha}\right)p^{\alpha\prime}(u)\right\}_{ab} = \left(\frac{\partial F^a}{\partial z^\alpha}\right)(p(u),\overline{p}(u),u)\left(\frac{\partial p^\alpha}{\partial u^b}\right)(u),$$
$$(F')_{ab} = \left(\frac{\partial F^a}{\partial u^b}\right)(p(u),\overline{p}(u),u).$$

This completes the proof. □

Note that the functions $F^*_{st}(z,\bar{z},u)$ in Lemma 1 are functionals of the function $p(u)$, i.e., functions of the function $p(u)$ and its derivatives. The highest order of the derivatives of the function $p(u)$ in $F^*_{st}(z,\overline{z},u)$ depends on the type (s,t) of $F^*_{st}(z,\overline{z},u)$.

2. Zero Levi form

We shall study a generic CR submanifold M with Levi form L vanishing identically on M.

LEMMA 2. *Suppose that a generic CR submanifold M is defined near the origin by*

$$v = F(z,\overline{z},u) = \sum_{s+t\geq 2} F_{st}(z,\overline{z},u).$$

Then the u-plane, $z=v=0$, is on M and

$$2dd^c\rho|_{z=v=0} = 2i\sum_{\alpha,\beta=1}^{n}\left(\frac{\partial^2 F}{\partial z^\alpha\partial\overline{z}^\beta}\right)(0,0,u)dz^\alpha\wedge d\overline{z}^\beta,$$

where

$$\rho = -v + F(z,\overline{z},u).$$

Proof. By the definition of d^c, we have

$$2id^c\rho = \sum_{\alpha=1}^{n}\left(\frac{\partial\rho}{\partial z^\alpha}dz^\alpha + \frac{\partial\rho}{\partial\overline{z}^\alpha}d\overline{z}^\alpha\right) + \sum_{a=1}^{m}\left(\frac{\partial\rho}{\partial w^a}dw^a + \frac{\partial\rho}{\partial\overline{w}^a}d\overline{w}^a\right).$$

Thus we obtain

$$
\begin{aligned}
& 2idd^c\rho \\
= & -2\left(\sum_{\alpha,\beta=1}^{n}\frac{\partial^2\rho}{\partial z^\alpha\partial\overline{z}^\beta}dz^\alpha\wedge d\overline{z}^\beta+\sum_{a=1}^{m}\frac{\partial^2\rho}{\partial w^a\partial\overline{w}^b}dw^a\wedge d\overline{w}^b\right) \\
& -2\sum_{\alpha=1}^{n}\sum_{a=1}^{m}\left(\frac{\partial^2\rho}{\partial z^\alpha\partial\overline{w}^a}dz^\alpha\wedge d\overline{w}^a-\frac{\partial^2\rho}{\partial\overline{z}^\alpha\partial w^a}d\overline{z}^\alpha\wedge dw^a\right) \\
= & -2\sum_{\alpha,\beta=1}^{n}\frac{\partial^2 F}{\partial z^\alpha\partial\overline{z}^\beta}dz^\alpha\wedge d\overline{z}^\beta-\frac{1}{2}\sum_{a=1}^{m}\frac{\partial^2 F}{\partial u^a\partial u^b}dw^a\wedge d\overline{w}^b \\
& -\sum_{\alpha=1}^{n}\sum_{a=1}^{m}\left(\frac{\partial^2 F}{\partial z^\alpha\partial u^a}dz^\alpha\wedge d\overline{w}^a-\frac{\partial^2 F}{\partial\overline{z}^\alpha\partial u^a}d\overline{z}^\alpha\wedge dw^a\right).
\end{aligned}
$$

Note that the generic CR submanifold M contains the u-plane, $z = v = 0$, since

$$F(0,0,u) = 0.$$

Further, the condition

$$F_{10}(z,\overline{z},u) = F_{01}(z,\overline{z},u) = 0$$

gives the following equality on the u-plane

$$2idd^c\rho|_{z=v=0} = -2\sum_{\alpha,\beta=1}^{n}\left(\frac{\partial^2 F}{\partial z^\alpha\partial\overline{z}^\beta}\right)(0,0,u)dz^\alpha\wedge d\overline{z}^\beta. \tag{10}$$

This completes the proof. □

Note that the differential system defined by

$$d\rho_1 = \cdots = d\rho_m = d^c\rho_1 = \cdots = d^c\rho_m = 0$$

along the u-plane on M in Lemma 2 is given by the complex tangent planes of the variable z in $\mathbb{C}^n \times \mathbb{C}^m$. Thus the Levi form L on M in Lemma 2 is faithfully represented on the u-plane by the two-form in (10).

Lemma 3. *Suppose that an analytic generic CR submanifold M is defined near the origin by*

$$v = F(z,\overline{z},u) = \sum_{s,t\geq 1} F_{st}(z,\overline{z},u)$$

and the Levi *form* L *on* M *vanishes identically. Then*

$$F(z,\overline{z},u)=0,$$

i.e., M *is a plane* $\mathbb{C}^n\times\mathbb{R}^m$ *defined by* $v=0$.

Proof. Let M' be the generic CR submanifold obtained from M by the biholomorphic mapping as in (7)

$$\begin{aligned} z &= z^*+p(w^*),\\ w &= w^*+g(z^*,w^*) \end{aligned}$$

for a given function $p(u)$. Then M' is given near the origin by

$$v=\sum_{s,t\geq 1}F^*_{st}(z,\overline{z},u),$$

where

$$F^*_{11}(z,\overline{z},u)=\left\{Id-B_{00}(0,0,u)\right\}^{-1}A_{11}(z,\overline{z},u).$$

Since the generic CR submanifold M is defined by

$$v=F(z,\overline{z},u)=\sum_{s,t\geq 1}F_{st}(z,\overline{z},u),$$

we have the following equalities

$$\begin{aligned} \sum_{\alpha=1}^{n} z^\alpha\left(\frac{\partial F}{\partial z^\alpha}\right)(z,\overline{z},u) &= \sum_{s,t\geq 1} sF_{st}(z,\overline{z},u),\\ \sum_{\alpha=1}^{n} \overline{z}^\alpha\left(\frac{\partial F}{\partial \overline{z}^\alpha}\right)(z,\overline{z},u) &= \sum_{s,t\geq 1} tF_{st}(z,\overline{z},u),\\ \sum_{\alpha,\beta=1}^{n} z^\alpha\overline{z}^\beta\left(\frac{\partial^2 F}{\partial z^\alpha\partial \overline{z}^\beta}\right)(z,\overline{z},u) &= \sum_{s,t\geq 1} stF_{st}(z,\overline{z},u). \end{aligned}$$

Then from Lemma 1, we obtain

$$
\begin{aligned}
&\textstyle\sum_{\alpha,\beta=1}^{n} p^\alpha \overline{p}^\beta \left(\frac{\partial^2 A_{11}}{\partial z^\alpha \partial \overline{z}^\beta}\right)(0,0,u) \\
&= \textstyle\sum_{s,t\geq 1} st F_{st}(p,\overline{p},u) \\
&\quad -i\Big\{ \textstyle\sum_{s,t\geq 1} s F'_{st}(p,\overline{p},u)\Big\}(Id+iF')^{-1}\Big\{ \textstyle\sum_{s,t\geq 1} t F_{st}(p,\overline{p},u)\Big\} \\
&\quad +i\Big\{ \textstyle\sum_{s,t\geq 1} t F'_{st}(p,\overline{p},u)\Big\}(Id-iF')^{-1}\Big\{ \textstyle\sum_{s,t\geq 1} s F_{st}(p,\overline{p},u)\Big\} \\
&\quad -2F''(Id-iF')^{-1}\Big\{ \textstyle\sum_{s,t\geq 1} s F_{st}(p,\overline{p},u)\Big\} \\
&\qquad\qquad \times(1+iF')^{-1}\Big\{ \textstyle\sum_{s,t\geq 1} t F_{st}(p,\overline{p},u)\Big\},
\end{aligned} \tag{11}
$$

where

$$\Big\{F'_{st}(p,\overline{p},u)\Big\}_{ab} = \left(\frac{\partial F^a_{st}}{\partial u^b}\right)(p,\overline{p},u).$$

Note that the Levi form L' on M' vanishes identically whenever the Levi form L on M vanishes identically. By Lemma 2, the function $F^*_{11}(z,\overline{z},u)$ vanishes identically for any function $p(u)$. Thus the equality (11) yields the following identity

$$
\begin{aligned}
&\textstyle\sum_{s,t\geq 1} st F_{st}(z,\overline{z},u) \\
&= i\left\{\textstyle\sum_{s,t\geq 1} s F'_{st}(z,\overline{z},u)\right\}(Id+iF')^{-1}\left\{\textstyle\sum_{s,t\geq 1} t F_{st}(z,\overline{z},u)\right\} \\
&\quad -i\left\{\textstyle\sum_{s,t\geq 1} t F'_{st}(z,\overline{z},u)\right\}(Id-iF')^{-1}\left\{\textstyle\sum_{s,t\geq 1} s F_{st}(z,\overline{z},u)\right\} \\
&\quad +2F''(Id-iF')^{-1}\left\{\textstyle\sum_{s,t\geq 1} s F_{st}(z,\overline{z},u)\right\} \\
&\qquad\qquad \times(Id+iF')^{-1}\left\{\textstyle\sum_{s,t\geq 1} t F_{st}(z,\overline{z},u)\right\}.
\end{aligned} \tag{12}
$$

In the identity (12), we expand the right hand side with respect to z and $\overline{z}$. Then we observe that the function

$$\sum_{s,t\geq 1, s+t=k} st F_{st}(z,\overline{z},u)$$

is represented by a linear combination of products of the following functions

$$\begin{aligned} F_{st}(z,\overline{z},u) &\quad \text{for } s+t \leq k-2\,, \\ F'_{st}(z,\overline{z},u) &\quad \text{for } s+t \leq k-2\,, \\ F''_{st}(z,\overline{z},u) &\quad \text{for } s+t \leq k-4\,, \end{aligned}$$

where

$$\begin{aligned} \left\{F'_{st}(z,\overline{z},u)\right\}_{ab} &= \left(\frac{\partial F^a_{st}}{\partial u^b}\right)(z,\overline{z},u), \\ \left\{F''_{st}(z,\overline{z},u)\right\}_{abc} &= \frac{1}{2}\left(\frac{\partial^2 F^a_{st}}{\partial u^b \partial u^c}\right)(z,\overline{z},u). \end{aligned}$$

We easily see that

$$\sum_{s,t\geq 1, s+t=2,3} stF_{st}(z,\overline{z},u) = 0$$

so that

$$F_{st}(z,\overline{z},u) = 0 \quad \text{for } s+t = 2,3.$$

As inductive hypothesis, we suppose that

$$F_{st}(z,\overline{z},u) = 0$$

for $s+t = k \geq 4$. Then we obtain

$$\sum_{s,t\geq 1, s+t\leq k+2} stF_{st}(z,\overline{z},u) = 0$$

so that

$$F_{st}(z,\overline{z},u) = 0 \quad \text{for } s+t \leq k+2.$$

Therefore we conclude that $F(z,\overline{z},u) = 0$. This completes the proof. □

Hence we have proved the following theorem

THEOREM 4. *Let M be an analytic generic CR submanifold of codimension m with zero* Levi *form defined by*

$$v = F(z,\overline{z},u), \quad F| = dF| = 0.$$

Then M is locally transformed to a plane $\mathbb{C}^n \times \mathbb{R}^m$ defined by

$$v = 0$$

by the following biholomorphic mapping

$$z = z^*, \quad w = w^* + g(z^*, w^*), \tag{13}$$

where the function $g(z, w)$ *is implicitly defined by*

$$g(z,w) = -iF(0,0,w) + 2iF\Big(z, 0, w - \tfrac{i}{2}F(0,0,w) + \tfrac{1}{2}g(z,w)\Big). \tag{14}$$

Let ϕ be a biholomorphic mapping near the origin, which transforms the generic CR submanifold M in Theorem 4 to the plane $v = 0$. Then the mapping ϕ is factorized to the mapping (13) and an element of the pseudo-group of the local biholomorphic automorphisms of the plane $v = 0$ such that

$$\begin{aligned} z^* &= f(z,w), \\ w^* &= q(w) \end{aligned}$$

where

$$\det(f_z|_0) \neq 0, \quad \Im q(u) = 0 \quad \text{and} \quad \det q'(0) \neq 0.$$

Note that the biholomorphic mapping (13) is a local trivialization of a family of complex manifolds of complex dimension n parametrized by a subset of $\mathbb{R}^m$. Thus the analytic generic CR submanifold M with zero Levi form is locally foliated into complex manifolds. Further, the leaves of the complex foliation on M are locally given by the complex submanifold near the origin as follows

$$w = \tau + g(z, \tau)$$

for $\tau \in \mathbb{R}^m$, where the function $g(z, \tau)$ is defined by the equation (14).

Corollary 5. *Let M be an analytic generic CR submanifold of CR dimension n with zero* Levi *form in a complex manifold. Then there is a open neighborhood U of each point of M such that $M \cap U$ is an analytic foliation of complex manifolds of complex dimension n.*

This corollary is a well-known special case of a general result(cf. [1]). The significance of this article is that we do not require Frobenius theorem and Newlander-Nirenberg theorem/Levi-Civita theorem in the proof(cf. [1]).

References

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