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INTEGRAL KERNEL OPERATORS ON REGULAR GENERALIZED WHITE NOISE FUNCTIONS

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ABSTRACT. Let $\mathcal{G}$ (and $\mathcal{G}^*$) be the space of regular test (and generalized, resp.) white noise functions. The integral kernel operators acting on $\mathcal{G}$ and transformation groups of operators on $\mathcal{G}$ are studied, and then every integral kernel operator acting on $\mathcal{G}$ can be extended to continuous linear operator on $\mathcal{G}^*$. The existence and uniqueness of solutions of Cauchy problems associated with certain integral kernel operators with initial date in $\mathcal{G}^*$ are investigated.

1. Introduction

The white noise analysis initiated by Hida [11] has been considerably developed to an infinite dimensional Schwartz type distribution theory with many applications (see [13], [18], [20], [21] and references cited therein). The mathematical framework of white noise analysis is the Gel'fand triple

$$(E) \subset (L^2) \equiv L^2(E^*, \mu) \subset (E)^*,$$

where E^* is the space of tempered distributions and μ is the standard Gaussian measure on E^*. Recently, the following triplet:

$$\mathcal{G} \subset (L^2) \subset \mathcal{G}^*$$

has been studied by several authors (e.g. [1], [10], [23]) with applications to theory of generalized martingales, where $\mathcal{G}$ and $\mathcal{G}^*$ are the spaces of regular test white noise functions and regular generalized white noise functions, respectively.

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Gross [9] and Piech [22] initiated the study of the infinite dimensional Laplacians (the Gross Laplacian Δ_G and the number operator N, resp.), as infinite dimensional analogue of a finite dimensional Laplacian, in connection with the Cauchy problems in infinite dimensional abstract Wiener space. In the white noise analysis, Kuo [16] reformulated the Gross Laplacian Δ_G and N as continuous linear operators acting on the test white noise function space (E). In previous works [3]-[6], the existence and uniqueness of solutions of Cauchy problems (with initial date in the space of test white noise functions) associated with the Gross Laplacian, the number operator and certain integral kernel operators have been investigated. In [8] and [24], a C_0-group and Cauchy problem associated with the Lévy Laplacian were studied.

On the other hand, in the theory of white noise operators which was rigorously studied in [14] and [20], the integral kernel operators play an important role. In fact, every continuous linear operator from (E) into $(E)^*$ admits a unique representation into a sum of integral kernel operators (see [20]).

In this paper, we shall study the integral kernel operators $\Xi_{l,m}(K)$, $K \in \mathcal{L}(H_{\mathbf{C}}^{\otimes m}, H_{\mathbf{C}}^{\otimes l})$ acting on $\mathcal{G}$ and then $\Xi_{l,m}(K)$ can be extended to a continuous linear operator on $\mathcal{G}^*$. It is of interest to study the existence and uniqueness of solutions of the Cauchy problems associated with integral kernel operators with initial date in $\mathcal{G}^*$. Then we shall study the existence and uniqueness of (explicit) solution of the Cauchy problem:

$$\frac{du(\theta)}{d\theta} = (\Xi_{0,2}(K) + d\Gamma(B))u(\theta), \qquad u(0) = \phi \in \mathcal{G}^*, \quad -\delta < \theta < \delta, \tag{1.1}$$

where $\delta > 0$ (depends on ϕ), $K \in H_{\mathbf{C}}^{\widehat{\otimes} 2}$ and $B \in \mathcal{L}(H_{\mathbf{C}}, H_{\mathbf{C}})$ satisfying certain condition and $d\Gamma(B)$ is the differential second quantization operator of B. An application to finite dimensional partial differential equations is now in progress.

The paper is organized as follows. In Section 2, we recall some basic definitions and results in the white noise analysis. In Section 3, we study the integral kernel operators on regular white noise functions. In Section 4, the one-parameter transformation groups of operators on regular test white noise functions and their infinitesimal generators are discussed. In

Section 5, the existence and the uniqueness of solutions of Cauchy problems associated with certain integral kernel operators with initial date in the space of regular generalized white noise functions are investigated.

2. The spaces of white noise functions

Let $H = L^2(\mathbb{R}, dt)$ be the real Hilbert space of square integrable functions with respect to the Lebesgue measure dt on $\mathbb{R}$ and the norm denoted by $|\cdot|_0$. From H and the positive self-adjoint operator $A = -d^2/(dt^2) + t^2 + 1$ on H, a Gel'fand triple $E \subset H \subset E^*$ is constructed in the standard manner (see [13], [18], [20]). Note that E is a nuclear space equipped with the Hilbertian norms $|\xi|_p = |A^p\xi|_0$, $p \in \mathbb{R}$. The canonical bilinear form on $E^* \times E$ is denoted by $< \cdot, \cdot >$. Let (E^*, μ) be the standard Gaussian space with Gaussian measure μ whose characteristic function is given by

$$\int_{E^*} e^{i\langle x,\xi\rangle}\mu(dx) = \exp\left(-\frac{1}{2}|\xi|_0^2\right), \qquad \xi \in E.$$

Let $(L^2) = L^2(E^*, \mu; \mathbb{C})$ be the complex Hilbert space of square integrable functions with respect to μ on E^* and the norm denoted by $\|\cdot\|_0$. Then by the Wiener-Itô decomposition theorem, we have the following unitary isomorphism between (L^2) and the Boson Fock space $\Gamma(H_{\mathbb{C}})$:

(2.1)

$$(L^2) \ni \phi(x) = \sum_{n=0}^{\infty}\langle :x^{\otimes n}:, f_n\rangle \quad \longleftrightarrow \quad (f_n) \sim \phi \in \Gamma(H_{\mathbb{C}}), \qquad f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n},$$

where $:x^{\otimes n}:$ denotes the Wick ordering of $x^{\otimes n}$ and $H_{\mathbb{C}}^{\widehat{\otimes} n}$ is the n-fold symmetric tensor product of the complexification $H_{\mathbb{C}}$ of H. Moreover, the (L^2)-norm $\|\phi\|_0$ of $\phi \in (L^2)$ is given by

$$\|\phi\|_0^2 = \sum_{n=0}^{\infty} n!|f_n|_0^2, \qquad \phi \sim (f_n),$$

where $|\cdot|_0$ denotes also the $H_{\mathbb{C}}^{\otimes n}$-norm for any n.

Let $\Gamma(A)$ denote the second quantization operator of A defined by

$$\Gamma(A)\phi \sim (A^{\otimes n}f_n), \qquad (f_n) \sim \phi \in (L^2).$$

Then $\Gamma(A)$ becomes a positive self-adjoint operator on (L^2) with Hilbert-Schmidt inverse and $\|\Gamma(A)^{-1}\|_{OP} < 1$. From (L^2) and $\Gamma(A)$, a complex Gel'fand triple

$$(E) \subset (L^2) \subset (E)^* \tag{2.2}$$

is constructed in the standard manner (see [13], [18], [20]). The Gel'fand triple (2.2) is called the *Hida-Kubo-Takenaka space.* We note that (E) is a nuclear space equipped with the Hilbertian norms:

$$\|\phi\|_p = \|\Gamma(A)^p \phi\|_0, \qquad \phi \in (E), \quad p \in \mathbb{R}.$$

An element in (E) (and in $(E)^*$) is called a *test (and generalized, respectively) white noise function.* We denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the canonical $\mathbb{C}$-bilinear form on $(E)^* \times (E)$. For each $\Phi \in (E)^*$, there exists a unique sequence $\{F_n\}_{n=0}^{\infty}$, $F_n \in (E_{\mathbb{C}}^{\otimes n})^*_{sym}$ such that

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \qquad (f_n) \sim \phi \in (E). \tag{2.3}$$

Let N be the number operator and let $\mathcal{G}_p$ be the (L^2)-domain of e^{pN} for each $p \geq 0$. Then $\mathcal{G}_p$ is a Hilbert space with norm $|\!|\!|\cdot|\!|\!|_p := \|e^{pN} \cdot \|_0$. More precisely, for any $p \geq 0$

$$|\!|\!|\phi|\!|\!|_p^2 = \sum_{n=0}^{\infty} n! e^{2pn} |f_n|_0^2, \qquad \phi \sim (f_n) \in \mathcal{G}_p.$$

Let $\mathcal{G}$ be the projective limit of $\{\mathcal{G}_p; p \geq 0\}$ and $\mathcal{G}^*$ be the topological dual space of $\mathcal{G}$. Then $\mathcal{G}^*$ is isomorphic to the inductive limit of $\{\mathcal{G}_{-p}; p \geq 0\}$, where $\mathcal{G}_{-p} = \mathcal{G}_p^*$ is the Hilbert space with norm $|\!|\!|\cdot|\!|\!|_{-p}$. In fact, for each $p \geq 0$, $\mathcal{G}_{-p}$ is the completion of (L^2) with respect to the norm $|\!|\!|\cdot|\!|\!|_{-p}$. Then we have a triplet:

$$\mathcal{G} \subset (L^2) \subset \mathcal{G}^*.$$

An element in $\mathcal{G}$ (and in $\mathcal{G}^*$) is called a *regular test (and generalized, respectively) white noise function* (see [1], [10], [23]). Then we have natural inclusions:

$$(E) \subset \mathcal{G} \subset (L^2) \subset \mathcal{G}^* \subset (E)^*. \tag{2.4}$$

The canonical bilinear form on $\mathcal{G}^* \times \mathcal{G}$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$ again.

3. Integral kernel operators

Let $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ denote the space of all continuous linear operators from a locally convex space $\mathfrak{X}$ into another locally convex space $\mathfrak{Y}$. From (2.4), we have the following natural inclusions:

$$\mathcal{L}(\mathcal{G}, \mathcal{G}) \subset \mathcal{L}(\mathcal{G}, \mathcal{G}^*) \subset \mathcal{L}((E), (E)^*). \tag{3.1}$$

In this section, we shall study the integral kernel operators on $\mathcal{G}$. Let l, m be non-negative integers. Then for each $\kappa_{l,m} \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$, the *integral kernel operator* $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((E), (E)^*)$ with kernel distribution $\kappa_{l,m}$ (see [14], [20]) is defined by

$$\Xi_{l,m}(\kappa_{l,m})\phi \sim \left(\frac{(n+m)!}{n!} \kappa_{l,m} \widehat{\otimes}_m f_{n+m} \right), \qquad \phi \sim (f_n) \in (E), \tag{3.2}$$

where $\kappa_{l,m} \widehat{\otimes}_m f_{n+m}$ is the right symmetric contraction [20]. Note that by the kernel theorem, for each $\kappa_{l,m} \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$ there exists a continuous linear operator $K_{l,m}$ from $E_{\mathbb{C}}^{\otimes m}$ into $(E_{\mathbb{C}}^{\otimes l})^*$ such that

$$\langle K_{l,m} f, g \rangle = \langle \kappa_{l,m}, g \otimes f \rangle, \qquad f \in E_{\mathbb{C}}^{\otimes m}; \quad g \in E_{\mathbb{C}}^{\otimes l}.$$

Then for any $f_{n+m} \in E_{\mathbb{C}}^{\widehat{\otimes}(n+m)}$ and $g_{l+n} \in E_{\mathbb{C}}^{\widehat{\otimes}(l+n)}$, we have

$$\langle \kappa_{l,m} \widehat{\otimes}_m f_{n+m}, g_{l+n} \rangle = \langle (K_{l,m} \otimes I^{\otimes n} f_{n+m})_{\mathrm{sym}}, g_{l+n} \rangle, \tag{3.3}$$

where $(f)_{\mathrm{sym}}$ is the symmetrization of $f \in (E_{\mathbb{C}}^{\otimes l})^*$. By (3.2) and (3.3), we have

(3.4)
$$\Xi_{l,m}(\kappa_{l,m})\phi \sim \left(\frac{(n+m)!}{n!} (K_{l,m} \otimes I^{\otimes n} f_{n+m})_{\mathrm{sym}} \right), \qquad \phi \sim (f_n) \in (E).$$

In order to distinguish the expression of integral kernel operator in (3.2) from the one in (3.4), we use the notation $\Xi_{l,m}(K_{l,m})$ for (3.4) instead of $\Xi_{l,m}(\kappa_{l,m})$. The kernel distribution $K_{l,m}$ is unique as an element in the subspace

$$\begin{aligned} &\mathcal{L}_{\mathrm{s}(l,m)}(E_{\mathbb{C}}^{\otimes m}, (E_{\mathbb{C}}^{\otimes l})^*) \\ = &\left\{ K \in \mathcal{L}(E_{\mathbb{C}}^{\otimes m}, (E_{\mathbb{C}}^{\otimes l})^*); \begin{array}{l} K(f) = K((f)_{\mathrm{sym}}) \\ K(f) = (K(f))_{\mathrm{sym}} \end{array}, \ f \in E_{\mathbb{C}}^{\otimes m} \right\}. \end{aligned}$$

It is easy to see that $\mathcal{L}(H_{\mathbb{C}}^{\otimes m}, H_{\mathbb{C}}^{\otimes l}) \subset \mathcal{L}(E_{\mathbb{C}}^{\otimes m}, (E_{\mathbb{C}}^{\otimes l})^*)$.

THEOREM 3.1. *Let $K_{l,m} \in \mathcal{L}(E_{\mathbf{C}}^{\otimes m}, (E_{\mathbf{C}}^{\otimes l})^*)$. Then the followings hold:*

(i) *if $K_{l,m} \in \mathcal{L}(H_{\mathbf{C}}^{\otimes m}, H_{\mathbf{C}}^{\otimes l})$, then $\Xi_{l,m}(K_{l,m}) \in \mathcal{L}(\mathcal{G}, \mathcal{G})$. In this case, for any p, $q \in \mathbb{R}$ with $q-p>0$ and $\phi \in \mathcal{G}$,*

$$\|\Xi_{l,m}(K_{l,m})\phi\|_p \le C\left(e^{(pl-qm)+(q-p)/2}\right) l^{l/2} m^{m/2} D_{q-p}^{(l+m)/2} \|\phi\|_q,$$

where D_{q-p} is given as in (3.5) and $C \ge 0$ satisfies that $|K_{l,m}f|_0 \le C|f|_0$ for any $f \in H_{\mathbf{C}}^{\otimes m}$.

(ii) *if $\Xi_{l,m}(K_{l,m}) \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$, then $K_{l,m} \in \mathcal{L}(H_{\mathbf{C}}^{\otimes m}, H_{\mathbf{C}}^{\otimes l})$.*

Proof. (i) Suppose that $K_{l,m} \in \mathcal{L}(H_{\mathbf{C}}^{\otimes m}, H_{\mathbf{C}}^{\otimes l})$. Then there exists $C \ge 0$ such that for any $f \in H_{\mathbf{C}}^{\otimes m}$, $|K_{l,m}f|_0 \le C|f|_0$. Since $|(f)_{\mathrm{sym}}|_0 \le |f|_0$,

$$|(K_{l,m} \otimes I^{\otimes n} f_{n+m})_{\mathrm{sym}}|_0 \le |K_{l,m} \otimes I^{\otimes n} f_{n+m}|_0 \le C|f_{n+m}|_0.$$

Hence by (3.4) and the Schwartz's inequality, we obtain that for any $\phi \sim (f_n) \in \mathcal{G}$

$$\begin{aligned}\|\Xi_{l,m}(K_{l,m})\phi\|_p^2 &= \sum_{n=0}^{\infty}(l+n)!e^{2p(l+n)}\left|\frac{(n+m)!}{n!}(K_{l,m}\otimes I^{\otimes n}f_{n+m})_{\mathrm{sym}}\right|_0^2 \\ &\le C^2\sum_{n=0}^{\infty}(n+m)!\,\frac{(l+n)!}{n!}\frac{(n+m)!}{n!}e^{2p(l+n)}|f_{n+m}|_0^2 \\ &\le C^2 e^{2(pl-qm)}\max_{n\ge 0}\left\{\frac{(l+n)!}{n!}\frac{(n+m)!}{n!}e^{-2(q-p)n}\right\}\|\phi\|_q^2.\end{aligned}$$

Note that for any $s>0$ and any integer $m \ge 0$,

$$\max_{x\ge 0}(x+m)\cdots(x+1)e^{-sx} \le e^{s/2}m^m D_s^m, \qquad D_s = \frac{e^{s/2}}{se}. \tag{3.5}$$

Therefore, we have that for any p, $q \in \mathbb{R}$ with $q-p>0$

$$\|\Xi_{l,m}(K_{l,m})\phi\|_p^2 \le C^2\left(e^{2(pl-qm)+q-p}\right) l^l m^m D_{q-p}^{l+m}\|\phi\|_q^2.$$

(ii) Choose $p \ge 0$ and $C \ge 0$ such that

$$\|\Xi_{l,m}(K_{l,m})\phi\|_{-p} \le C\|\phi\|_p, \qquad \phi \in \mathcal{G}. \tag{3.6}$$

For $f \in H_{\mathbf{C}}^{\otimes m}$ and $g \in H_{\mathbf{C}}^{\otimes l}$ we put

$$\phi_m \sim (0,\cdots,0,f,0,\cdots) \in \mathcal{G} \quad \text{and} \quad \psi_l \sim (0,\cdots,0,g,0,\cdots) \in \mathcal{G}.$$

Then by definition, we have that $\|\phi_m\|_p^2 = m!e^{2pm}|f|_0^2$, $\|\psi_l\|_p^2 = l!e^{2pl}|g|_0^2$ and

$$\langle\langle \Xi_{l,m}(K_{l,m})\phi_m, \psi_l\rangle\rangle = l!m!\langle K_{l,m}f, g\rangle.$$

Therefore, in view of (3.6), we obtain that

$$|\langle K_{l,m}f, g\rangle| \le \frac{C}{l!m!}\|\phi_m\|_p\|\psi_l\|_p = C\frac{e^{p(l+m)}}{\sqrt{l!m!}}|g|_0|f|_0.$$

Hence we have

$$|K_{l,m}f|_0 \le C\frac{e^{p(l+m)}}{\sqrt{l!m!}}|f|_0.$$

It follows that $K_{l,m} \in \mathcal{L}(H_{\mathbf{C}}^{\otimes m}, H_{\mathbf{C}}^{\otimes l})$. □

By the dual property and (i) in Theorem 3.1, the following corollary is obvious.

COROLLARY 3.2. *Let $K_{l,m} \in \mathcal{L}(H_{\mathbf{C}}^{\otimes m}, H_{\mathbf{C}}^{\otimes l})$. Then for any $q > p \ge 0$ and $\phi \in \mathcal{G}$, we have*

$$\|\Xi_{l,m}(K_{l,m})\phi\|_{-q} \le C\left(e^{(pm-ql)+(q-p)/2}\right) l^{l/2}m^{m/2}D_{q-p}^{(l+m)/2}\|\phi\|_{-p}$$

for some constant C satisfying that for any $f \in H_{\mathbf{C}}^{\otimes m}$, $|K_{l,m}f|_0 \le C|f|_0$.

THEOREM 3.3. *Let $K \in \mathcal{L}(H_{\mathbf{C}}^{\otimes m}, H_{\mathbf{C}}^{\otimes l})$. Then the integral kernel operator $\Xi_{l,m}(K) \in \mathcal{L}(\mathcal{G},\mathcal{G})$ has a unique extension to a continuous linear operator $\overline{\Xi}_{l,m}(K)$ from $\mathcal{G}^*$ into itself. We denote $\overline{\Xi}_{l,m}(K)$ by the same notation $\Xi_{l,m}(K)$ from now on.*

Proof. Since $\mathcal{G}$ is dense in $\mathcal{G}^*$, for any $\Phi \in \mathcal{G}^*$ there exists a sequence $\{\phi_n\}_{n=1}^{\infty} \subset \mathcal{G}$ such that ϕ_n converges to Φ in the strong topology of $\mathcal{G}^*$, i.e., there exists a $p \ge 0$ such that $\Phi \in \mathcal{G}_{-p}$ and $\lim_{n\to\infty}\|\phi_n - \Phi\|_{-p} = 0$. By Corollary 3.2, $\{\Xi_{l,m}(K)\phi_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathcal{G}_{-q}$ for any $q > p$. Define

$$\overline{\Xi}_{l,m}(K)\Phi = \lim_{n\to\infty}\Xi_{l,m}(K)\phi_n \quad \text{in } \mathcal{G}_{-q}.$$

It is obvious that $\overline{\Xi}_{l,m}(K)\Phi$ is uniquely well-defined and we have

$$\|\overline{\Xi}_{l,m}(K)\Phi\|_{-q} \le C\left(e^{(pm-ql)+(q-p)/2}\right) l^{l/2}m^{m/2}D_{q-p}^{(l+m)/2}\|\Phi\|_{-p}.$$

Hence $\overline{\Xi}_{l,m}(K)$ is the unique continuous extension of $\Xi_{l,m}(K)$ to $\mathcal{G}^*$. □

EXAMPLE 3.4. For each $\xi \in \mathcal{L}(H_{\mathbf{C}}, \mathbb{C}) \cong H_{\mathbf{C}}$, by Theorem 3.3, $D_\xi \equiv \Xi_{0,1}(\xi)$ and $D_\xi^* \equiv \Xi_{1,0}(\xi^*)$ can be extended to continuous linear operators from $\mathcal{G}^*$ into itself, where $\xi^* \in \mathcal{L}(\mathbb{C}, H_{\mathbf{C}})$ is the multiplication operator by ξ. The operator D_ξ and D_ξ^* are called respectively the *annihilation operator* and the *creation operator*.

EXAMPLE 3.5. Let $K \in \mathcal{L}(H_{\mathbb{C}}^{\otimes 2}, \mathbb{C}) \cong H_{\mathbb{C}}^{\otimes 2}$. Then by Theorem 3.3, $\Xi_{0,2}(K) \in \mathcal{L}(\mathcal{G}, \mathcal{G})$ can be extended to a continuous linear operator from $\mathcal{G}^*$ into itself. Moreover, for any $\Phi \sim (f_n) \in \mathcal{G}^*$,

$$\Xi_{0,2}(K) \sim ((n+2)(n+1)(K \otimes I^{\otimes n})f_{n+2}) \in \mathcal{G}^*.$$

For simple notation, we use the notation $K\widehat{\otimes}_2 f_{n+2}$ instead of $(K \otimes I^{\otimes n})f_{n+2}$.

EXAMPLE 3.6. For each $B \in \mathcal{L}(H_{\mathbb{C}}, H_{\mathbb{C}})$, the *differential second quantization operator* $d\Gamma(B)$ of B is defined by $d\Gamma(B) = \Xi_{1,1}(B)$ and hence $d\Gamma(B)$ can be extended to a continuous linear operator from $\mathcal{G}^*$ into itself. Moreover, we have

$$d\Gamma(B)\Phi \sim ((n+1)(B \otimes I^{\otimes n} f_{n+1})_{\mathrm{sym}}), \qquad \phi \sim (f_n) \in \mathcal{G}^*.$$

The *number operator* N is defined by $N = d\Gamma(I)$.

4. Transformations on regular test white noise functions

For any $K \in (E_{\mathbb{C}}^{\otimes 2})^*$ and $B \in \mathcal{L}(E_{\mathbb{C}}, E_{\mathbb{C}})$, there exists a unique operator $\mathcal{G}_{K,B} \in \mathcal{L}((E), (E))$ (see [5]) such that

$$\mathcal{G}_{K,B}\phi_\xi = \exp\left\{\langle K, \xi^{\otimes 2}\rangle\right\}\phi_{B\xi}, \qquad \xi \in E_{\mathbb{C}}. \tag{4.1}$$

Moreover, for any $\phi \sim (f_n) \in (E)$, $\mathcal{G}_{K,B}\phi$ is given by

$$\mathcal{G}_{K,B}\phi \sim \left(\sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!} B^{\otimes n}(K^{\otimes m}\widehat{\otimes}_{2m} f_{n+2m})\right).$$

LEMMA 4.1. *Let $K \in H_{\mathbb{C}}^{\otimes 2}$ and $B \in \mathcal{L}(H_{\mathbb{C}}, H_{\mathbb{C}})$. Then for any $p, q \in \mathbb{R}$ with $3e^{-2p}|K|_0 < 1$ and $3e^{-2q}\|B\|^2 < 1$, the series*

$$M \equiv \sum_{n=0}^{\infty} n! e^{2(p-q)n} \left|\sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!} B^{\otimes n}\left(K^{\otimes m}\widehat{\otimes}_{2m} f_{n+2m}\right)\right|_0^2, \quad \phi \sim (f_n) \in \mathcal{G}$$

converges, where $\|B\|$ is the operator norm of B. In this case,

$$M \le M_{K,B}(p,q)^2 \|\phi\|_p^2, \tag{4.2}$$

$$M_{K,B}(p,q)^2 = \left(\frac{1}{1-(3e^{-2p}|K|_0)^2}\right)\left(\frac{1}{1-3e^{-2q}\|B\|^2}\right).$$

Proof. Since $K \in H_{\mathbb{C}}^{\otimes 2}$ and $B \in \mathcal{L}(H_{\mathbb{C}}, H_{\mathbb{C}})$, we have

$$\left|B^{\otimes n}\left(K^{\otimes m}\widehat{\otimes}_{2m} f_{n+2m}\right)\right|_0 \le \|B\|^n |K|_0^m |f_{n+2m}|_0.$$

Hence, by the Schwartz's inequality, we obtain that for any $p, q \in \mathbb{R}$

$$M \le \sum_{n=0}^{\infty} n! e^{2(p-q)n} \left(\sum_{m=0}^{\infty} \frac{(n+2m)!}{n!^2 m!^2} \|B\|^{2n} |K|_0^{2m} e^{-2p(n+2m)}\right) \|\phi\|_p^2.$$

By using the fact $(n+2m)! \le n! m!^2 3^{n+2m}$, we have

$$M \le \left(\sum_{n=0}^{\infty} \left(3e^{-2q}\|B\|^2\right)^n\right) \left(\sum_{m=0}^{\infty} \left(3e^{-2p}|K|_0\right)^{2m}\right) \|\phi\|_p^2.$$

Thus $M < \infty$ for any $p, q \in \mathbb{R}$ with $3e^{-2p}|K|_0 < 1$ and $3e^{-2q}\|B\|^2 < 1$. □

THEOREM 4.2. *Let $K \in H_{\mathbb{C}}^{\otimes 2}$ and $B \in \mathcal{L}(H_{\mathbb{C}}, H_{\mathbb{C}})$. Then $\mathcal{G}_{K,B}$ is well-defined as a continuous linear operator from $\mathcal{G}$ into itself satisfying* (4.1). *Moreover, for any $p \ge 0$ there exists $q \ge 0$ with $3e^{-2q}\|B\|^2 < 1$ and $3e^{-2(p+q)}|K|_0 < 1$ such that for any $\phi \in \mathcal{G}$*

$$\|\mathcal{G}_{K,B}\phi\|_p \le M_{K,B}(p+q, q)\|\phi\|_{p+q},$$

where $M_{K,B}(p+q,q)$ is given as in (4.2).

The proof is immediate from Lemma 4.1. On the other hand, if $K = 0$ and $B \neq 0$, then

$$\mathcal{G}_{0,B}\phi \sim (B^{\otimes n} f_n), \qquad \phi \sim (f_n) \in \mathcal{G}.$$

It follows that $\mathcal{G}_{0,B} = \Gamma(B)$ and we can easily show that $\|\mathcal{G}_{0,B}\Phi\|_{p-\log\|B\|} \le \|\Phi\|_p$ for any $p \in \mathbb{R}$ and $\Phi \in \mathcal{G}_p$. Therefore, for any $\Phi \in \mathcal{G}^*$, $\mathcal{G}_{0,B}\Phi$ is well-defined. Moreover, $\mathcal{G}_{0,B}$ is a continuous linear operator from $\mathcal{G}^*$ into itself.

From now on, we fix $K \in H_{\mathbb{C}}^{\widehat{\otimes} 2}$ and $B \in \mathcal{L}(H_{\mathbb{C}}, H_{\mathbb{C}})$ satisfying $K \circ (B \otimes I) = \alpha K$ for some $\alpha \in \mathbb{C}$, where $K \circ (B \otimes I) \in \mathcal{L}(H_{\mathbb{C}}^{\otimes 2}, \mathbb{C}) \cong H_{\mathbb{C}}^{\otimes 2}$ defined by

$$\langle K \circ (B \otimes I), f\rangle = \langle K, (B \otimes I) f\rangle, \qquad f \in H_{\mathbb{C}}^{\otimes 2}.$$

Since $e^B \in \mathcal{L}(H_{\mathbb{C}}, H_{\mathbb{C}})$ is well-defined as series expansion, we define

$$\mathcal{G}_{K,B;\theta} = \begin{cases} \mathcal{G}_{\theta K, e^{\theta B}} & \text{if } \alpha = 0 \\ \mathcal{G}_{1/(2\alpha) K \circ ((e^{\theta B})^{\otimes 2} - 1), e^{\theta B}} & \text{if } \alpha \neq 0. \end{cases} \tag{4.3}$$

Then we can easily show that $\{\mathcal{G}_{K,B;\theta}\}_{\theta\in\mathbb{R}} \subset \mathcal{L}(\mathcal{G},\mathcal{G})$ is a one–parameter transformation group, i.e.,

$$\mathcal{G}_{K,B;0} = I, \qquad \mathcal{G}_{K,B;\theta_2}\mathcal{G}_{K,B;\theta_1} = \mathcal{G}_{K,B;\theta_2+\theta_1}, \qquad \theta_1,\theta_2 \in \mathbb{R}.$$

THEOREM 4.3. *$\{\mathcal{G}_{K,B;\theta}\}_{\theta\in\mathbb{R}}$ is a differentiable one-parameter transformation group with the infinitesimal generator $\Xi_{0,2}(K) + d\Gamma(B)$, i.e., for any $p \geq 0$ and $\phi \in \mathcal{G}$, we have*

$$\lim_{\theta\to 0}\left\|\left\|\frac{\mathcal{G}_{K,B;\theta}\phi - \phi}{\theta} - (\Xi_{0,2}(K) + d\Gamma(B))\phi\right\|\right\|_p = 0.$$

Proof. Let $p \in \mathbb{R}$ and $\phi \sim (f_n) \in \mathcal{G}$ be given. Then we have

$$(4.4)\qquad \begin{aligned} &\frac{\mathcal{G}_{K,B;\theta}\phi - \phi}{\theta} - \Big(\Xi_{0,2}(K) + d\Gamma(B)\Big)\phi \\ &\sim \left(\left[\tfrac{(e^{\theta B})^{\otimes n} - I^{\otimes n}}{\theta} - \gamma_n(B)\right] f_n\right) \\ &\quad + \left((n+2)(n+1)\left[(e^{\theta B})^{\otimes n}\tfrac{K_\theta}{\theta}\widehat{\otimes}_2 f_{n+2} - K\widehat{\otimes}_2 f_{n+2}\right]\right) \\ &\quad + \left(\textstyle\sum_{m=2}^{\infty}\tfrac{(n+2m)!}{n!m!}(e^{\theta B})^{\otimes n}\left(\tfrac{K_\theta^{\otimes m}}{\theta}\widehat{\otimes}_{2m} f_{n+2m}\right)\right), \end{aligned}$$

where

$$(4.5)\qquad \begin{cases} \gamma_n(B) = \sum_{k=0}^{n-1} I^{\otimes k}\otimes B\otimes I^{(n-1-k)}, & n \geq 1, \\ \gamma_0(B) = 0 \end{cases}$$

and

$$(4.6)\qquad K_\theta = \begin{cases} 1/(2\alpha)K\circ((e^{\theta B})^{\otimes 2} - 1) & \text{if } \alpha \neq 0, \\ \theta K & \text{if } \alpha = 0. \end{cases}$$

The proof of the case $\alpha = 0$ is similar to the proof of the case $\alpha \neq 0$. We now prove only the case $\alpha \neq 0$. By (4.4), we have

$$\left\|\left\|\frac{\mathcal{G}_{K,B;\theta}\phi - \phi}{\theta} - (\Xi_{0,2}(K) + d\Gamma(B))\phi\right\|\right\|_p^2 \leq 3(I_1(\theta) + I_2(\theta) + I_3(\theta)),$$

where

$$I_1(\theta) = \sum_{n=0}^{\infty} n! e^{2pn}\left|\left[\frac{(e^{\theta B})^{\otimes n} - I^{\otimes n}}{\theta} - \gamma_n(B)\right] f_n\right|_0^2,$$

$$I_2(\theta) = \sum_{n=0}^{\infty} n! e^{2pn}\left|(n+2)(n+1)\left[(e^{\theta B})^{\otimes n}\frac{K_\theta}{\theta}\widehat{\otimes}_2 f_{n+2} - K\widehat{\otimes}_2 f_{n+2}\right]\right|_0^2$$

and

$$I_3(\theta) = \sum_{n=0}^{\infty} n! e^{2pn} \left| \sum_{m=2}^{\infty} \frac{(n+2m)!}{n!m!} (e^{\theta B})^{\otimes n} \left(\frac{K_\theta^{\otimes m}}{\theta} \widehat{\otimes}_{2m} f_{n+2m} \right) \right|_0^2 .$$

Now, we shall prove that $\lim_{\theta\to 0}(I_1(\theta) + I_2(\theta) + I_3(\theta)) = 0$. We first prove that $\lim_{\theta\to 0} I_1(\theta) = 0$ (for the proof, we use the similar arguments in the proof of Proposition 5.4.5 in [20]). Note that

$$\begin{aligned}\frac{(e^{\theta B})^{\otimes n} - I^{\otimes n}}{\theta} - \gamma_n(B) &= \sum_{k=0}^{n-1} I^{\otimes k} \otimes \left[\frac{e^{\theta B} - I}{\theta} - B \right] \otimes (e^{\theta B})^{\otimes(n-1-k)} \\ &\quad + \sum_{k=0}^{n-1} I^{\otimes k} \otimes B \otimes \left[(e^{\theta B})^{\otimes(n-1-k)} - I^{\otimes(n-1-k)} \right]\end{aligned}$$

and hence, for any $f_n \in H_{\mathbf{C}}^{\widehat{\otimes} n}$ we have

$$\begin{aligned}&\left| \left[\frac{(e^{\theta B})^{\otimes n} - I^{\otimes n}}{\theta} - \gamma_n(B) \right] f_n \right|_0 \\ \le & \sum_{k=0}^{n-1} \left| \left(I^{\otimes k} \otimes \left[\frac{e^{\theta B} - I}{\theta} - B \right] \otimes (e^{\theta B})^{\otimes(n-1-k)} \right) f_n \right|_0 \\ + & \sum_{k=0}^{n-1} \left| \left(I^{\otimes k} \otimes B \otimes \left[(e^{\theta B})^{\otimes(n-1-k)} - I^{\otimes(n-1-k)} \right] \right) f_n \right|_0 .\end{aligned}$$

Suppose that $\epsilon > 0$ be given. Then there exists $\theta_0 > 0$ such that for any $\xi \in H_{\mathbf{C}}$

$$\left| \frac{e^{\theta B}\xi - \xi}{\theta} - B\xi \right|_0 \le \epsilon |\xi|_0, \qquad |\theta| < \theta_0.$$

Therefore, we obtain that for any $|\theta| < \theta_0$

$$\left| \left(I^{\otimes k} \otimes \left[\frac{e^{\theta B} - I}{\theta} - B \right] \otimes (e^{\theta B})^{\otimes(n-1-k)} \right) f_n \right|_0 \le \epsilon e^{|\theta| \|B\| (n-1-k)} |f_n|_0 .$$

Hence, for any $|\theta| < \theta_0$ and $q > 0$

$$\begin{aligned}
&\sum_{k=0}^{n-1} \left| \left(I^{\otimes k} \otimes \left[\frac{e^{\theta B} - I}{\theta} - B \right] \otimes (e^{\theta B})^{\otimes(n-1-k)} \right) f_n \right| \\
\leq\ & \epsilon \sum_{k=0}^{n-1} e^{|\theta| \|B\| (n-1-k)} \, |f_n|_0 \\
\leq\ & \epsilon C_q e^{(|\theta|\|B\|+q)n} \, |f_n|_0
\end{aligned}$$

for some $C_q \geq 0$ satisfying $n \leq C_q e^{qn}$, $n \in \mathbb{N}$. On the other hand, for any $\theta \in \mathbb{R}$ and $q > 0$ we have

(4.7)

$$\begin{aligned}
\left| \left((e^{\theta B})^{\otimes n} - I^{\otimes n} \right) f_n \right|_0 &\leq \sum_{k=0}^{n-1} \left| \left((e^{\theta B})^{\otimes(n-1-k)} \otimes (e^{\theta B} - I) \otimes I^{\otimes k} \right) f_n \right|_0 \\
&\leq |\theta| \|B\| e^{|\theta|\|B\|} \left(\textstyle\sum_{k=0}^{n-1} e^{|\theta|\|B\|(n-1-k)} \right) |f_n|_0 \\
&\leq |\theta| \|B\| e^{|\theta|\|B\|} C_q e^{(|\theta|\|B\|+q)n} |f_n|_0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left| \left(I^{\otimes k} \otimes B \otimes \left[(e^{\theta B})^{\otimes(n-1-k)} - I^{\otimes(n-1-k)} \right] \right) f_n \right|_0 \\
&\qquad \leq |\theta| \|B\|^2 e^{|\theta|\|B\|} C_q e^{(|\theta|\|B\|+q)(n-1-k)} |f_n|_0.
\end{aligned}$$

It follows that for any $\theta \in \mathbb{R}$ and $q > 0$

$$\begin{aligned}
&\sum_{k=0}^{n-1} \left| \left(I^{\otimes k} \otimes B \otimes \left[(e^{\theta B})^{\otimes(n-1-k)} - I^{\otimes(n-1-k)} \right] \right) f_n \right|_0 \\
&\qquad \leq |\theta| \|B\|^2 e^{|\theta|\|B\|} C_q^2 e^{(|\theta|\|B\|+2q)n} |f_n|_0.
\end{aligned}$$

Hence for any $|\theta| < \theta_0$ and $q > 0$ we have

$$\begin{aligned}
&\left| \left[\frac{(e^{\theta B})^{\otimes n} - I^{\otimes n}}{\theta} - \gamma_n(B) \right] f_n \right|_0 \\
&\qquad \leq \left(\epsilon + |\theta| \|B\|^2 e^{|\theta|\|B\|} C_q \right) C_q e^{(|\theta|\|B\|+2q)n} |f_n|_0.
\end{aligned}$$

Therefore, for any $|\theta| < \theta_0$ and $q > 0$, we obtain that

$$
\begin{aligned}
I_1(\theta) &\leq \left(\epsilon + |\theta| \|B\|^2 e^{|\theta|\|B\|} C_q\right)^2 C_q^2 \sum_{n=0}^{\infty} n! e^{2(|\theta|\|B\|+p+2q)n} |f_n|_0^2 \\
&= \left(\epsilon + |\theta| \|B\|^2 e^{|\theta|\|B\|} C_q\right)^2 C_q^2 \|\phi\|_{|\theta|\|B\|+p+2q}^2 .
\end{aligned}
$$

It implies that $\lim_{\theta \to 0} I_1(\theta) = 0$. To prove $\lim_{\theta \to 0} I_2(\theta) = 0$, we observe that

$$
\begin{aligned}
\left| (e^{\theta B})^{\otimes n} \frac{K_\theta}{\theta} \widehat{\otimes}_2 f_{n+2} - K \widehat{\otimes}_2 f_{n+2} \right|_0 \leq & \left| (e^{\theta B})^{\otimes n} \left[\frac{K_\theta}{\theta} \widehat{\otimes}_2 f_{n+2} - K \widehat{\otimes}_2 f_{n+2} \right] \right|_0 \\
& + \left| \left[(e^{\theta B})^{\otimes n} - I^{\otimes n} \right] K \widehat{\otimes}_2 f_{n+2} \right|_0 .
\end{aligned}
$$

By direct computation using $K \circ (B \otimes I) = \alpha K$, we have that for any $\theta \in \mathbb{R}$

$$
\left| (e^{\theta B})^{\otimes n} \left[\frac{K_\theta}{\theta} \widehat{\otimes}_2 f_{n+2} - K \widehat{\otimes}_2 f_{n+2} \right] \right|_0 \leq 2 |\theta| |K|_0 \|B\| e^{2|\theta|\|B\|} e^{|\theta|\|B\| n} |f_{n+2}|_0 .
$$

On the other hand, by (4.7) we have that for any $\theta \in \mathbb{R}$ and $q > 0$

$$
\left| \left[(e^{\theta B})^{\otimes n} - I^{\otimes n} \right] K \widehat{\otimes}_2 f_{n+2} \right|_0 \leq |\theta| \|B\| |K|_0 e^{|\theta|\|B\|} C_q e^{(|\theta|\|B\|+q)n} |f_{n+2}|_0 .
$$

Hence for any $\theta \in \mathbb{R}$ and $q > 0$ we have

$$
\begin{aligned}
& \left| (e^{\theta B})^{\otimes n} \frac{K_\theta}{\theta} \widehat{\otimes}_2 f_{n+2} - K \widehat{\otimes}_2 f_{n+2} \right|_0 \\
& \leq |\theta| |K|_0 \|B\| \left(2 e^{2|\theta|\|B\|} e^{|\theta|\|B\| n} + e^{|\theta|\|B\|} C_q e^{(|\theta|\|B\|+q)n} \right) |f_{n+2}|_0 \\
& \leq |\theta| L_{K,B} e^{(|\theta|\|B\|+q)n} |f_{n+2}|_0 ,
\end{aligned}
$$

where $L_{K,B} = |K|_0 \|B\| \left(2 + C_q\right) e^{2|\theta|\|B\|}$. Therefore, for any $\theta \in \mathbb{R}$ and $q, r > 0$ we have

$$
\begin{aligned}
I_2(\theta) &\leq |\theta|^2 L_{K,B}^2 C_r' \sum_{n=0}^{\infty} (n+2)! e^{2(|\theta|\|B\|+p+q+r)n} |f_{n+2}|_0^2 \\
&\leq |\theta|^2 L_{K,B}^2 C_r' \|\phi\|_{|\theta|\|B\|+p+q+r}^2 ,
\end{aligned}
$$

where $C_r' \geq 0$ satisfies the inequality $(n+2)(n+1) \leq C_r' e^{2rn}$. This implies that $\lim_{\theta \to 0} I_2(\theta) = 0$. Finally, we shall prove that $\lim_{\theta \to 0} I_3(\theta) = 0$. By using the Schwartz's inequality, we obtain that for any $|\theta| < 1$ and $q \geq 0$

$$\begin{aligned}
&\left|\sum_{m=2}^{\infty} \frac{(n+2m)!}{n!m!}(e^{\theta B})^{\otimes n}(K_\theta^{\otimes m}\widehat{\otimes}_{2m} f_{n+2m})\right|_0^2 \\
&\leq \left(\sum_{m=2}^{\infty} \frac{(n+2m)!}{n!m!} e^{\|B\| n} |K_\theta|_0^m \, |f_{n+2m}|_0\right)^2 \\
&\leq \frac{1}{n!}\left(3e^{2(\|B\|-(p+q))}\right)^n \left(\sum_{m=2}^{\infty}\left(3|K_\theta|_0 e^{-2(p+q)}\right)^{2m}\right) \|\phi\|_{p+q}^2.
\end{aligned}$$

Hence for any $|\theta| < 1$ and $q \geq 0$

$$I_3(\theta) \leq \left(\sum_{n=0}^{\infty}\left(3e^{2(\|B\|-q)}\right)^n\right)\left(\sum_{m=2}^{\infty}\frac{1}{|\theta|^2}\left(3|K_\theta|_0 e^{-2(p+q)}\right)^{2m}\right)\|\phi\|_{p+q}^2.$$

Note that $|K_\theta|_0 \leq |\theta||K|_0 e^{2|\theta|\|B\|}$ for any $\theta \in \mathbb{R}$ since $K \circ (B^m \otimes B^l) = \alpha K \circ (B^{m-1} \otimes B^l)$ for any $m \geq 1$ and $l \geq 0$. Therefore, there exists $q \geq 0$ such that for any $|\theta| < 1$

$$\left(\sum_{n=0}^{\infty}\left(3e^{2(\|B\|-q)}\right)^n\right)\left(\sum_{m=2}^{\infty}\left(3|K_\theta|_0 e^{-2(p+q)}\right)^{2m}\right) < \infty.$$

Then by the dominated convergence theorem, we prove that $\lim_{\theta\to 0} I_3(\theta) = 0$. Consequently, we complete the proof. □

COROLLARY 4.4. *Let $K \in H_{\mathbb{C}}^{\widehat{\otimes}2}$. Then $\{\mathcal{G}_{\theta K,I}\}_{\theta\in\mathbb{R}}$ is a differentiable one-parameter transformation group with the infinitesimal generator $\Xi_{0,2}(K)$.*

5. Cauchy problems

Let $\Xi \in \mathcal{L}(\mathcal{G},\mathcal{G})$ be the infinitesimal generator of a differentiable one-parameter group $\{\Omega_\theta\}_{\theta\in\mathbb{R}} \subset \mathcal{L}(\mathcal{G},\mathcal{G})$ and let ϕ be given in $\mathcal{G}$. Then the unique solution of the Cauchy problem of the following type:

$$\frac{du(\theta)}{d\theta} = \Xi u(\theta), \qquad u(0) = \phi, \qquad \theta \in \mathbb{R} \tag{5.1}$$

is immediately obtained by $u(\theta) = \Omega_\theta \phi \in \mathcal{G}$, $\theta \in \mathbb{R}$.

Let $K \in H_{\mathbf{C}}^{\widehat{\otimes}2}$ and $B \in \mathcal{L}(H_{\mathbf{C}}, H_{\mathbf{C}})$ satisfying $K \circ (B \otimes I) = \alpha K$ for some $\alpha \in \mathbb{C}$. Then by Theorem 4.3, $u(\theta) = \mathcal{G}_{K,B;\theta}\phi \in \mathcal{G}$ is the unique solution of (5.1) with $\Xi = \Xi_{0,2}(K) + d\Gamma(B)$ and $\phi \in \mathcal{G}$.

Now, we consider the Cauchy problem (1.1). Let I be a given bounded open interval containing 0. For each $p \geq 0$, put

(5.2)

$$S_p = \{\theta \in \mathbb{R}; 3e^{2p}|K_\theta|_0 < 1\} \cap I, \qquad M_p = \sup\{e^{|\theta|\|B\|}; \theta \in S_p\} < \infty,$$

where K_θ is given as in (4.6). Obviously, S_p is an open subset of $\mathbb{R}$ for any $p \geq 0$ and there exists an open ball $B_\epsilon(0) = (-\epsilon, \epsilon)$ such that $B_\epsilon(0) \subset S_p$. Moreover, for any $q \geq p$, $S_q \subset S_p$ and $M_q \leq M_p$.

PROPOSITION 5.1. *Let $p \geq 0$ be given. Then for any $\theta \in S_p$ and any $q > \log(\sqrt{3}M_p)$, $\mathcal{G}_{K,B;\theta} \in \mathcal{L}(\mathcal{G}_{-p}, \mathcal{G}_{-(p+q)})$. Moreover, for any $\phi \in \mathcal{G}_{-p}$ we have*

$$|\!|\!|\mathcal{G}_{K,B;\theta}\phi|\!|\!|_{-(p+q)} \leq \left(\frac{1}{1 - 3\|e^{\theta B}\|^2 e^{-2q}}\right)^{1/2} \left(\frac{1}{1 - (3e^{2p}|K_\theta|_0)^2}\right)^{1/2} |\!|\!|\phi|\!|\!|_{-p}.$$

The proof is similar to the proof of Lemma 4.1.

Let $p \geq 0$ be given. Then by Proposition 5.1, for any $q > \log(\sqrt{3}M_p)$ $(> \log(\sqrt{3}M_{p+q}))$ and for any $\theta_1 \in S_p$, $\theta_2 \in S_{p+q}$ we have

$$\mathcal{G}_{K,B;\theta_2}\mathcal{G}_{K,B;\theta_1} \in \mathcal{L}(\mathcal{G}_{-p}, \mathcal{G}_{-(p+2q)}).$$

Take $\theta \in S_p$. Then for any $h \in S_{p+q}$ such that $\theta + h \in S_p$, $\mathcal{G}_{K,B;\theta+h}$ coincide with $\mathcal{G}_{K,B;h}\mathcal{G}_{K,B;\theta}$ as a operator in $\mathcal{L}(\mathcal{G}_{-p}, \mathcal{G}_{-(p+2q)})$. To prove this, we used the equality:

$$\langle K_{\theta_1} + K_{\theta_2} \circ (e^{\theta_1 B})^{\otimes 2}, \xi^{\otimes 2}\rangle = \langle K_{\theta_2+\theta_1}, \xi^{\otimes 2}\rangle, \qquad \theta_1, \theta_2 \in \mathbb{R}, \qquad \xi \in H_{\mathbf{C}}.$$

Therefore, by using the similar arguments in the proof of Theorem 4.3, we can prove that for each $\phi \in \mathcal{G}_{-p}$ and $\theta \in S_p$,

(5.3)

$$\lim_{h \to 0} \left|\!\left|\!\left| \frac{\mathcal{G}_{K,B;\theta+h}\phi - \mathcal{G}_{K,B;\theta}\phi}{h} - (\Xi_{0,2}(K) + d\Gamma(B))\mathcal{G}_{K,B;\theta}\phi \right|\!\right|\!\right|_{-(p+2q)} = 0.$$

It follows that for each $\phi \in \mathcal{G}_{-p}$, $u(\theta) = \mathcal{G}_{K,B;\theta}\phi \in \mathcal{G}_{-(p+q)}$, $\theta \in S_p$ satisfies the initial-value problem:

$$(5.4) \qquad \frac{du(\theta)}{d\theta} = (\Xi_{0,2}(K) + d\Gamma(B))u, \quad u(0) = \phi, \qquad \theta \in S_p.$$

Now, we consider the uniqueness of solution of (5.4). Suppose that $v(\theta) \in \mathcal{G}_{-(p+q)}$, $\theta \in S_p$ is another solution of (5.4) satisfying (5.3). Take $\delta > 0$ such that $B_\delta(0) \subset S_{p+2q}$, where $q > \log(\sqrt{3}M_p)$. Take any $\theta \in B_{\delta/2}(0)$. Then for any $\epsilon \in B_{\delta/2}(0)$, $\theta - \epsilon \in B_\delta(0) \subset S_{p+2q} \subset S_p \cap S_{p+q}$. Therefore, by Proposition 5.1, we can define a $\mathcal{G}_{-(p+2q)}$-valued function on $B_{\delta/2}(0)$ by $w(\epsilon) = \mathcal{G}_{K,B;\theta-\epsilon}v(\epsilon)$. Moreover, by the similar arguments in the proof of Theorem 4.3, we obtain that

$$\begin{aligned} \frac{d}{d\epsilon}w(\epsilon) &= -(\Xi_{0,2}(K) + d\Gamma(B))\mathcal{G}_{K,B;\theta-\epsilon}v(\epsilon) + \mathcal{G}_{K,B;\theta-\epsilon}\frac{d}{d\epsilon}v(\epsilon) \\ &= -\mathcal{G}_{K,B;\theta-\epsilon}(\Xi_{0,2}(K) + d\Gamma(B))v(\epsilon) + \mathcal{G}_{K,B;\theta-\epsilon}(\Xi_{0,2}(K) \\ &\qquad + d\Gamma(B))v(\epsilon) \\ &= 0, \end{aligned}$$

where the equalities hold in the topology of $\mathcal{G}_{-(p+3q)}$. This implies that $w(\epsilon) = C$ (a constant) for all $\epsilon \in B_{\delta/2}(0)$. Set $\epsilon = 0$ and $\epsilon = \theta$, then $w(\theta) = w(0) = \mathcal{G}_{K,B;\theta}v(0) = \mathcal{G}_{K,B;\theta}\phi$. On the other hand, $w(\theta) = \mathcal{G}_{K,B;0}v(\theta) = v(\theta)$. Hence we have $v(\theta) = \mathcal{G}_{K,B;\theta}\phi$ for any $\theta \in B_{\delta/2}(0)$.

Finally, we summarize the results in the following statement.

THEOREM 5.2. *Let $K \in H_{\mathbb{C}}^{\widehat{\otimes}2}$ and $B \in \mathcal{L}(H_{\mathbb{C}}, H_{\mathbb{C}})$ satisfying $K \circ (B \otimes I) = \alpha K$ for some $\alpha \in \mathbb{C}$. Then $u(\theta) = \mathcal{G}_{K,B;\theta}\phi \in \mathcal{G}$, $\theta \in \mathbb{R}$ is the unique solution of the initial-value problem:*

$$\frac{du(\theta)}{d\theta} = (\Xi_{0,2}(K) + d\Gamma(B))u(\theta), \quad u(0) = \phi \in \mathcal{G}, \qquad \theta \in \mathbb{R}.$$

Let $p \geq 0$ and $\phi \in \mathcal{G}_{-p}$ be given. Then $u(\theta) = \mathcal{G}_{K,B;\theta}\phi \in \mathcal{G}_{-(p+q)}$ defined on S_p satisfies the initial-value problem:

$$\frac{du(\theta)}{d\theta} = (\Xi_{0,2}(K) + d\Gamma(B))u(\theta), \quad u(0) = \phi, \qquad \theta \in S_p, \tag{5.5}$$

where $q > \log(\sqrt{3}M_p)$. Moreover, if $v(\theta) \in \mathcal{G}_{-(p+q)}$, $\theta \in S_p$ is another solution of (5.5) satisfying (5.3), then there exists $\delta > 0$ such that v equals to u on $B_\delta(0)$.

In (5.2), if $K = 0$ and I is an unbounded interval, then S_p is an unbounded interval and $M_p = \infty$. But if $B = 0$, then we can take $I = \mathbb{R}$, i.e., $S_p = \{\theta \in \mathbb{R}; 3e^{2p}|\theta||K|_0 < 1\}$. Then by the same procedure as in above discussion, the following result is obvious.

THEOREM 5.3. *Let $K \in H_{\mathbb{C}}^{\widehat{\otimes}2}$ and let $p \geq 0$, $\phi \in \mathcal{G}_{-p}$ be given. Then $u(\theta) = \mathcal{G}_{\theta K,I}\phi \in \mathcal{G}_{-(p+q)}$, $\theta \in S_p$ satisfies the initial-value problem:*

$$\frac{du(\theta)}{d\theta} = \Xi_{0,2}(K)u(\theta), \qquad u(0) = \phi, \quad \theta \in S_p, \tag{5.6}$$

where $S_p = \{\theta \in \mathbb{R}; 3e^{2p}|\theta||K|_0 < 1\}$ and $q > \log(\sqrt{3})$. Moreover, if $v(\theta) \in \mathcal{G}_{-(p+q)}$, $\theta \in S_p$ is another solution of (5.6) satisfying (5.3) with $B = 0$, then there exists $\delta > 0$ such that v equals to u on $B_\delta(0)$.

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