Bull. Korean Math. Soc. **37** (2000), No. 3, pp. 537–542

SPATIAL NUMERICAL RANGES OF ELEMENTS OF C^*-ALGEBRAS

SIN-EI TAKAHASI

Dedicated to Professor Kôzô Yabuta on his sixtieth birthday.

ABSTRACT. When A is a subalgebra of a C^*-algebra, the spatial numerical range of element of A can be described in terms of positive linear functionals on the C^*-algebra.

1. Introduction and results

Let A be a complex Banach algebra and A^* its dual space. Let $a \in A$. If A is unital, then $V(A,a) \equiv \{f(a) : f \in A^*, \|f\| = f(1) = 1\}$ is called the (algebra) numerical range of a and it is a non-void compact convex subset of the complex plane $\boldsymbol{C}$ (see [1, p.52]).

However if A is non-unital, then the above definition is not meaningful. In this case, we consider the following two sets :

$$V_1(A,a)=\{f(xa) : \exists f \in A^* \text{ and } \exists x \in A \text{ such that } \|f\|=\|x\|=f(x)=1\}$$

and

$$V_2(A,a)=\{f(ax) : \exists f \in A^* \text{ and } \exists x \in A \text{ such that } \|f\|=\|x\|=f(x)=1\}.$$

It is easy to see that $V(A,a) = V_1(A,a) = V_2(A,a)$ for the unital case. A. K. Gaur and T. Husain([3]) especially called the spatial numerical range $V_2(A,a)$ for non-unital case and investigated this situation. In particular, they showed that if A is a commutative C*-algebra with maximal ideal space Φ_A, then

$$\mathrm{co}\{\hat{a}(\phi) : \phi \in \Phi_A\} \subseteq V_1(A,a) \subseteq \overline{\mathrm{co}}\{\hat{a}(\phi) : \phi \in \Phi_A\},$$

2000 Mathematics Subject Classification: Primary 47A12.

Key words and phrases: C^*-algebra, pure state, spatial numerical range.

The author is partly supported by the Grants-in-Aid for Scientific Research, The Ministry of Education, Science, Sports and Culture, Japan.

where co, $\overline{\text{co}}$ and $\hat{a}$ denote the convex hull, the closed convex hull and the Gelfand transform of $a \in A$, respectively (see [3, Theorem 4.1]).

The purpose of this paper is to investigate the spatial numerical ranges for C*-algebras and obtain an extension of their result.

Our main result is the following.

THEOREM. *Let A be a C^*-algebra and B a subalgebra of A. Let $b \in B$. Then*

$$V_1(B,b)=\{|f|(b) : \exists f \in A^* \text{ and } \exists x \in B \text{ such that } \|f\|=\|x\|=f(x)=1\}$$

and

$$V_2(B,b) = \{|f|(b) : \exists f \in A^* \text{ and } \exists x \in B \text{ such that } \|f\|=\|x\|=f(x)=1\}$$

where $|f|$ denotes the absolute value of f (cf. [2, Definition 12.2.8]).

If B is a *-subalgebra, then $V_1(B,b) = V_2(B,b)$.

REMARK 1. The more detail for the commutative C^*-algebra case will be appeared in ([5]).

As a corollary of the main theorem, we have the following result which extends [3, Theorem 4.1].

COROLLARY. *Let A be a C^*-algebra and $a \in A$. Then*

$$\text{co}\{f(a) : f \in P(A)\} \subseteq V_1(A,a) = V_2(A,a) \subseteq \overline{\text{co}}\{f(a) : f \in P(A)\},$$

where $P(A)$ denotes the set of all pure states of A.

REMARK 2. We don't know conditions under which $\text{co}\{f(a) : f \in P(A)\} = V_1(A,a)(= V_2(A,a))$holds. Similarly for $\overline{\text{co}}\{f(a) : f \in P(A)\} = V_1(A,a)(= V_2(A,a))$.

2. Proof of results

Proof of Theorem. Set

$$W_1 = \{|f|(b) : \exists f \in A^* \text{ and } \exists x \in B \text{ such that } \|f\| = \|x\| = f(x) = 1\}$$

and let $\lambda \in V_1(B,b)$. Then there exist $g \in B^*$ and $x \in B$ such that $\lambda = g(xb)$ and $\|g\| = \|x\| = g(x) = 1$. Take a functional $f \in A^*$ such that $\|f\| = \|g\|$ and $f(b) = g(b)$ for each $b \in B$, and let $f = u \cdot |f|$ be the enveloping polar decomposition of f (cf. [2 , Definition 12.2.8]). Then

$$(1) \qquad 1 = f(x) = |f|(ux) = (x|u^*)_{|f|} \leq \|x\|_{|f|}\|u^*\|_{|f|} \leq 1 \cdot 1 = 1$$

so that we can find a scalar α satisfying

$$(2) \qquad \|u^* - \alpha x\|_{|f|} = 0$$

since the equality of the Cauchy-Schwarz inequality in (1) holds. Note that (1) implies

$$(3) \qquad (u^*|x)_{|f|} = (x|u^*)_{|f|} = (u^*|u^*)_{|f|} = (x|x)_{|f|} = 1$$

and hence $1 - \overline{\alpha} - \alpha + |\alpha|^2 = 0$ by (2). Therefore, α must be equal to 1, and so $\|u^* - x\|_{|f|} = 0$, that is $u^* - x$ belongs to the left kernel (in the enveloping von Neumann algebra of A) $N_{|f|} = \{x \in A^{**} : |f|(x^*x) = 0\}$ of $|f|$. Also since $|f|(x^*x) = (x|x)_{|f|} = \|x\|^2_{|f|} = 1$ by (1), it follows that $1 - x^*x \in N_{|f|}$, where 1 denotes the identity element of A^{**}. Therefore we have

$$\lambda = f(xb) = |f|(uxb) = (xb|u^*)_{|f|} = (xb|x)_{|f|} = |f|(x^*xb) = |f|(b)$$

(the 4^{th}-equality follows from $u^* - x \in N_{|f|}$ and the 6^{th}-equality follows from $1 - x^*x \in N_{|f|}$) and so $\lambda \in W_1$, hence $V_1(B,b) \subseteq W_1$.

Conversely suppose $\lambda \in W_1$. Then there exist $f \in A^*$ and $x \in B$ such that $\lambda = |f|(b)$ and $\|f\| = \|x\| = f(x) = 1$. Let $f = u \cdot |f|$ be the enveloping polar decomposition of f. Then we can apply directly the above arguments for f, x and u. Consequently, we have $f(xb) = |f|(b)$ and hence $\lambda \in V_1(B,b)$, so $W_1 \subseteq V_1(B,b)$. We thus obtain $V_1(B,b) = W_1$.

We next set

$$W_2 = \{|f|(b) : \exists f \in A^* \text{ and } \exists x \in B \text{ such that } \|f\| = \|x\| = f(x^*) = 1\},$$

and let $\lambda \in V_2(B,b)$. Then there exist $g \in B^*$ and $x \in B$ such that $\lambda = g(bx)$ and $\|g\| = \|x\| = g(x) = 1$. Take a functional $f \in A^*$ such that $\|f\| = \|g\|$ and $f(b) = g(b)$ for each $b \in B$. Then

$$\|f^*\| = \|f\| = \|x\| = \|x^*\| \text{ and } 1 = f(x) = f^*(x^*),$$

so that $\overline{\lambda} = \overline{f(bx)} = f^*(x^*b^*)$, $\|f^*\| = \|f\| = \|x\| = \|x^*\|$ and $1 = f(x) = f^*(x^*)$, and hence $\overline{\lambda} \in V_1(\overline{B}, b^*)$, where $\overline{B} = \{x \in A : x^* \in B\}$. Therefore by the preceding argument, we can find $h \in A^*$ and $y \in B$ such that $\overline{\lambda} = |h|(b^*)$ and $\|h\| = \|y\| = h(y^*) = 1$. This means that $\lambda \in W_2$, so we have $V_2(B,b) \subseteq W_2$.

The inverse inclusion $W_2 \subseteq V_2(B,b)$ can be easily obtained by tracing the converse of the above argument.

Set

$$A^*_{1,B} = \{f \in A^* : \|f\| = 1 \text{ and } \exists x \in B \text{ such that } \|x\| = f(x) = 1\}$$

and

$$A^*_{2,B} = \{f \in A^* : \|f\| = 1 \text{ and } \exists x \in B \text{ such that } \|x\| = f(x^*) = 1\}.$$

If B is a *-subalgebra, then $f \to f^*$ is a bijection of $A^*_{1,B}$ onto $A^*_{2,B}$ and hence we have

$$V_1(B,b) = \{|f|(b) : f \in A^*_{1,B}\} = \{|f|(b) : f \in A^*_{2,B}\} = V_2(B,b) \quad \square$$

Proof of Corollary. Let A be a C*-algebra and $a \in A$. Then we have $V_1(A,a) = V_2(A,a)$ by Theorem. We next show that $\mathrm{co}\{f(a) : f \in P(A)\} \subseteq V_1(A,a)$. To do this, let $\alpha \in \mathrm{co}\{f(a) : f \in P(A)\}$. Then there exist $f_{11}, \cdots, f_{1m_1}, \cdots, f_{n1}, \cdots, f_{nm_n} \in P(A)$ and $\lambda_{11}, \cdots, \lambda_{1m_1}, \cdots, \lambda_{n1}, \cdots, \lambda_{nm_n} \geq 0$ such that

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i} \lambda_{ij} = 1, \sum_{i=1}^{n}\sum_{j=1}^{m_i} \lambda_{ij} f_{ij}(a) = \alpha,$$

$$\pi_{f_{11}} \cong \cdots \cong \pi_{f_{1m_1}}, \cdots, \pi_{f_{n1}} \cong \cdots \cong \pi_{f_{nm_n}} \text{ and } \pi_{f_{i1}} \not\cong \pi_{f_{j1}} (i \neq j).$$

Let $\pi_1 \cong \pi_{f_{11}} \cong \cdots \cong \pi_{f_{1m_1}}, \cdots, \pi_n \cong \pi_{f_{n1}} \cong \cdots \cong \pi_{f_{nm_n}}$. For each $i, j (1 \leq i \leq n, 1 \leq j \leq m_i)$, choose an isomorphism U_{ij} of the Hilbert space H_{π_i} onto the Hilbert space $H_{\pi_{f_{ij}}}$ which transforms $\pi_i(x)$ into $\pi_{f_{ij}}(x)$ for every $x \in A$, and set $\xi_{ij} = U^*_{ij}(\xi_{f_{ij}})$. Also set $f = \sum_{i=1}^{n}\sum_{j=1}^{m_i} \lambda_{ij} f_{ij}$. Then we have $\|f\| = 1$, $f = |f|$, $\alpha = f(a)$ and

$$(4) \qquad f(x) = \sum_{i=1}^{n}\sum_{j=1}^{m_i} \lambda_{ij}(\pi_{f_{ij}}(x)\xi_{f_{ij}}|\xi_{f_{ij}}) = \sum_{i=1}^{n}\sum_{j=1}^{m_i} \lambda_{ij}(\pi_i(x)\xi_{ij}|\xi_{ij})$$

for every $x \in A$. Furthermore since $\pi_1, \cdots, \pi_n$ are mutually inequivalent, it follows that there exists a hermitian element $y \in A$ such that $\pi_i(y)\xi_{ij} = \xi_{ij} (1 \leq i \leq n, 1 \leq j \leq m_i)$ by ([2, Theorem 2.8.3, (i)]). Now consider the continuous function $h(t)$ on $[0, \infty)$ defined by

$$h(t) = \begin{cases} t, \text{if } 0 \leq t \leq 1 \\ 1, \text{if } t > 1 \end{cases}$$

and set $z = h(y^2)$. Then z is a positive element of A with $\|z\| \leq 1$. Moreover, we assert that

$$\pi_i(z)\xi_{ij} = \xi_{ij}(1 \leq i \leq n, 1 \leq j \leq m_i). \tag{5}$$

In fact, let $\varepsilon > 0$ be arbitrary and take a polynomial $p(t)$ such that $p(0) = 0$ and $\sup\{|p(t) - h(t)| : 0 \leq t \leq \|z\|\} < \varepsilon/2$. Let $1 \leq i \leq n$ and $1 \leq j \leq m_i$. Then

$$\begin{aligned}\|\pi_i(z)\xi_{ij} - \xi_{ij}\| &\leq \|\pi_i(h(y^2))\xi_{ij} - \pi_i(p(y^2))\xi_{ij}\| + \|p(\pi_i(y^2))\xi_{ij} - \xi_{ij}\| \\ &\leq \|h(y^2) - p(y^2)\| + |p(1) - 1| \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

and hence we obtain (5) since ε is arbitrary. By (4) and (5), we have

$$f(z) = \sum_{i=1}^{n}\sum_{j=1}^{m_i} \lambda_{ij}(\pi_i(z)\xi_{ij}|\xi_{ij}) = \sum_{i=1}^{n}\sum_{j=1}^{m_i} \lambda_{ij} = 1.$$

Consequently we have $\alpha \in V_1(A, a)$ and hence $\mathrm{co}\{f(a) : f \in P(A)\} \subseteq V_1(A, a)$.

We next show that $V_1(A, a) \subseteq \overline{\mathrm{co}}\{f(a) : f \in P(A)\}$. To do this, let $\alpha \in V_1(A, a)$ and so there exist $f \in A^*$ and $x \in A$ such that $\alpha = |f|(a)$ and $\|f\| = \|x\| = f(x) = 1$. Note that $|f|(x^*x) = 1$ as observed in the proof of the main theorem and consider the following set:

$$S = \{g \in A^* : g \geq 0 \text{ and } \|g\| = g(x^*x) = 1\}.$$

Then $|f| \in S$ and S is weak*-closed. Moreover, we can easily see that any extreme point of S is also an extreme point of $\{g \in A^* : g \geq 0$ and $\|g\| \leq 1\}$. But since the extreme points of $\{g \in A^* : g \geq 0$ and $\|g\| \leq 1\}$ consist of 0 and $P(A)$ (cf. [2, Proposition 2.5.5]), it follows by the Krein-Milman theorem that $S \subseteq \overline{\mathrm{co}}P(A)$. Then $\alpha = |f|(a) = \lim_\lambda g_\lambda(a)$ for some net $\{g_\lambda\}$ in $\mathrm{co}P(A)$, and hence $\alpha \in \overline{\mathrm{co}}\{f(a) : f \in P(A)\}$. □

References

[1] F. F. Bonsal and J. Dancan, *Complete Normed Algebras*, Springer-Verlag, Berlin/Heidelberg/ New York, 1973.

[2] J. Dixmier, *C^*-algberas*, North-Holland, New York, 1977.

[3] A. K. Gaur and T. Husain, *Spatial numerical ranges of elements of Banach algebras*, Internat. J. Math. Math. Sci. **12-4** (1989), 633–640.

[4] A. K. Gaur and Z. V. Kovarik, *Norms on unitizations of Banach algebras*, Proc. Amer. Math. Soc. **117-1** (1993), 111–113.

[5] S. -E. Takahasi, *Spatial numerical ranges of elements of subalgebras of* $C_0(X)$, to appear in Internat. J. Math. Math. Sci.

DEPARTMENT OF BASIC TECHNOLOGY, APPLIED MATHEMATICS AND PHYSICS, YAMAGATA UNIVERSITY, YONEZAWA 992-8510, JAPAN