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BRATTELI DIAGRAM ISOMORPHIC TO CHACON HOMEOMORPHISM

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ABSTRACT. We construct a stationary, properly ordered Bratteli diagram $B = (V, E, \geq)$ so that its Vershik map is isomorphic to Chacon's homeomorphism. It is the simplest stationary, properly ordered Bratteli diagram isomorphic to (X_C, T_C). We also find a primitive and proper substitution of Chacon's transformation.

1. Introduction

The present paper explores substitution minimal systems and their relations to stationary Bratteli diagrams. We construct a stationary, properly ordered Bratteli diagram $(V, E, \geq)$ so that the Vershik map is isomorphic to the Chacon homeomorphism (X_C, T_C), where T_C is a shift homeomorphism. In 1972, O. Bratteli introduced Bratteli diagrams in the context of special infinite graphs[1]. From a different direction came the extremely fruitful ideas of A. M. Vershik[Ve] to associate dynamics with Bratteli diagrams by introducing a lexicographic ordering on the infinite paths of the diagram. Through a careful investigation of Vershik's construction, R. H. Herman, I. F. Putnam and C. F. Skau[6] succeeded in showing that every Cantor minimal system is isomorphic to a Bratteli-Vershik system, i.e. the dynamical system of a properly Bratteli diagram with Vershik map.

One of the simplest ways of constructing minimal symbolic dynamical systems is by means of substitutions. We have the following Theorem[3]:

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The system associated to any primitive and aperiodic substitution is isomorphic to the system associated to some stationary, properly ordered Bratteli diagram.

One of the key methods of the construction in the proof of the above Theorem is a sequence of Kakutani-Rohlin partitions of the system associated to a primitive and aperiodic substitution[6]. That is, we construct a sequence of Kakutani-Rohlin partitions so that it satisfies some necessary conditions. Then we can construct Bratteli diagram isomorphic to the given primitive and aperiodic substitution. It is well known that the topological Chacon transformation is a substitution system via $0 \to 0010$ and $1 \to 1$. Clearly this substitution is not primitive. Therefore, we cannot directly apply the above method to construct a Bratteli diagram which is isomorphic to Chacon transformation.

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2. An Ordered Bratteli diagram

We shall explain here some notations and definitions concerning an ordered Bratteli diagram.

DEFINITION 1. A Bratteli diagram consists of a vertex set V and an edge set E satisfying the following conditions: We have a decomposition of V as a disjoint union $V_0 \cup V_1 \cup \cdots$, where each V_n is finite and non-empty and V_0 has exactly one element, v_0. Similarly, E is decomposed as a disjoint union $E_1 \cup E_2 \cup \cdots$, where each E_n is finite and non-empty. Moreover, we have maps $r, s : E \to V$ so that $r(E_n) \subset V_n$ and $s(E_n) \subset V_{n-1}$, $n = 1, 2, \cdots$ (r=range, s=source). We also assume that $s^{-1}(v)$ is non-empty for all v in V and $r^{-1}(v)$ is non-empty for all v in $V \setminus V_0$.

There is a natural and obvious notion of isomorphism between Bratteli diagrams (V, E) and (V', E'); namely, there exist a pair of bijections between V and V' and between E and E' respectively, preserving the grading and the intertwining of respective source and range maps. We denote the vertices at horizontal level n by V_n and the edges connecting the vertices at level $n-1$ with those at level n by E_n. If $|V_{n-1}| = t_{n-1}$ and $|V_n| = t_n$ then E_n determines the $t_n \times t_{n-1}$ incidence matrix. For a given Bratteli diagram (V, E) and non-negative integers $k < l$, let $P_{k,l}$

denote the set of all paths from V_k to V_l. Specifically,

$$P_{k,l} = \{(e_{k+1}, \cdots, e_l) | e_i \in E_i \text{ for } i = k+1, \cdots, l \text{ and } r(e_i) = s(e_{i+1}) \text{ for } i = k+1, \cdots, l-1\}$$

Note that we may identify $P_{k,k+1}$ with E_{k+1}. We define $r : P_{k,l} \to V_l$ by

$$r(e_{k+1}, \cdots, e_l) = r(e_l)$$

and $s : P_{k,l} \to V_k$ by

$$s(e_{k+1}, \cdots, e_l) = s(e_{k+1}).$$

DEFINITION 2. Given a Bratteli diagram (V, E) and a sequence $m_0 = 0 < m_1 < m_2 < \cdots$ in Z^+, we define the contraction of (V, E) to $\{m_n\}_0^\infty$ as (V', E') where $V'_n = V_{m_n}$, for $n \geq 0$ and $E'_n = P_{m_{n-1}, m_n}$ and $r' = r$, $s' = s$ as above.

It is routine to check that (V', E') in Definition 2 is again a Bratteli diagram. We let $\sim$ denote the equivalence relation of Bratteli diagrams generated by isomorphism and write $(V, E) \sim (V', E')$, whenever (V', E') is obtained from (V, E) by contraction.

We say that (V, E) is a simple Bratteli diagram if there exists a contraction (V', E') of (V, E) so that the incidence matrix of (V', E') at each level has only non-zero entries.

DEFINITION 3. An ordered Bratteli diagram $(V, E, \geq)$ is a Bratteli diagram (V, E) together with a partial order $\geq$ on E so that edges e, e' in E are comparable if and only if $r(e) = r(e')$. That is, we have a linear order on each set $r^{-1}(\{v\})$, $v \in V \setminus V_0$.

For a given ordered Bratteli diagram, we let E_{max} and E_{min} denote the maximal and minimal edges of E, respectively.

Note that if $(V, E, \geq)$ is an ordered Bratteli diagram and $k < l$ in Z^+, then the set $P_{k,l}$ of paths from V_k to V_l may be given an induced(lexicographic) order as follows : $(e_{k+1}, e_{k+2}, \cdots, e_l) > (f_{k+1}, f_{k+2}, \cdots, f_l)$ if and only if for some i with $k+1 \leq i \leq l$.

$$e_j = f_j \text{ for } i < j \leq l \text{ and } e_i > f_i.$$

It is a simple observation that if $(V, E, \geq)$ is an ordered Bratteli diagram and $(V', E', \leq')$ is a contraction of $(V, E, \leq)$ as defined above, then $(V', E', \geq')$ is again an ordered Bratteli diagram.

DEFINITION 4. The ordered Bratteli diagram $B = (V, E, \geq)$ is properly ordered if the following holds:

1. (V, E) is a simple Bratteli diagram
2. There is only one sequence $(e_1, e_2, \cdots)$ with each e_i in E_{max} (E_{min} respectively) and $s(e_{i+1}) = r(e_i)$ for all $i \geq 1$.

Let $B = (V, E, \geq)$ be an ordered Bratteli diagram. We denote by X_B the set of all infinite paths in (V, E), i.e.,

$$X_B = \{(e_1, e_2, \cdots) | e_i \in E \text{ and } r(e_i) = s(e_{i+1}) \text{ for all } i \geq 1\}.$$

To topologize, we define open sets $U(f_1, \cdots, f_k)$ for each $(f_1, \cdots, f_k)$ in $P_{0,k}$ as

$$U(f_1, f_2, \cdots . f_k) = \{(e_1, e_2, \cdots) \in X_B | e_i = f_i, 1 \leq i \leq k\}.$$

Note that $U(f_1, \cdots, f_k)$ is also closed. We call X_B with the above topology the Bratteli compactum associated to $B = (V, E, \geq)$.

In this paper, we assume that our Bratteli diagrams are properly ordered unless otherwise stated. We define a minimal homeomorphism $V_B : X_B \to X_B$, which is called the Vershik map, as follows: We let x_{max} be the unique infinite path in E_{max}. We define $V_B(x_{max}) = x_{min}$, where x_{min} is the unique infinite path in E_{min}. Let $x = (e_1, e_2, \cdots) \neq x_{max}$. Since $x = (e_1, e_2, \cdots) \neq x_{max}$, there exists the smallest number k such that e_k is not a maximal edge. Let f_k be the successor of e_k in E and let $(f_1, f_2, \cdots, f_{k-1})$ be the unique path in E_{min} from v_0 to $s(f_k)$. That is, $(f_1, f_2, \cdots, f_{k-1}, f_k)$ is the successor of $(e_1, e_2, \cdots, e_k)$ in $P_{0,k}$. Then we define

$$V_B(e_1, e_2, \cdots,) = (f_1, f_2, \cdots, f_k, e_{k+1} \cdots)$$

noting that since f_k is comparable to e_k,

$$r(f_k) = r(e_k) = s(e_{k+1}).$$

It is known that V_B is a minimal homeomorphism(See [6]).

We say that (X, T, x) is essentially minimal if for every neighborhood U of x, $\cup_{n \in Z} T^n(U) = X$. We say that (X_1, T_1, x_1) is pointedly topologically conjugate to (X_2, T_2, x_2) if there exists a homeomorphism $\phi : X_1 \to X_2$ such that $\phi(x_1) = x_2$ and $\phi \circ T_1 = T_2 \circ \phi$. The relation between a Cantor system and a properly ordered Bratteli diagram is characterized by the following theorem of Herman, Putnam and Skau.

THEOREM 1. [6] *Let* (X, T, x) *be an essentially minimal dynamical system, where* X *is Cantor set. Then there exists a properly ordered* Bratteli *diagram* $B = (V, E, \geq)$ *such that* (X, T, x) *is pointedly topological conjugate to* (X_B, V_B, x_{min}), *where* x_{min} *is the unique minimal path of* X_B.

Moreover, this correspondence established a bijection between equivalence classes of properly ordered Bratteli diagrams and pointedly topological conjugacy classes of essentially minimal systems.

The proof of the above theorem is given via Kakutani-Rohlin partitions, whose definition is as follows.

DEFINITION 5. A Kakutani-Rohlin partition of the Cantor minimal system (X, T) is a clopen partition $\mathcal{P}$ of the kind :

$$\mathcal{P} = \{T^j Z_k : k \in A \text{ and } 0 \leq j < h_k\}$$

where A is a finite set and h_k is a positive integer. The k'th tower of $\mathcal{P}$ is $\{T^j Z_k : 0 \leq j < h_k\}$ and the base of $\mathcal{P}$ is the set $Z = \cup_{k \in A} Z_k$.

Let (X, T, x) be a fixed minimal system. Then there exists a sequence of Kakutani-Rohlin partitions $\mathcal{P}_n$ with

$$\mathcal{P}_n = \{T^j Z_{n,k} | k \in A_n \text{ and } 0 \leq j < h_{n.k}\} \; \mathcal{P}_0 = \{X\}$$

and with base $Z_n = \cup_{k \in A_n} Z_{n,k}$. It satisfies the following conditions:
(1) $Z_{n+1} \subset Z_n$.
(2) $\mathcal{P}_{n+1} \succ \mathcal{P}_n$ as partition.
(3) The intersection of the bases $(Z_n : n \in N)$ of the partitions $(\mathcal{P}_n \, ; n \in N)$ consists of only one point x.
(4) The sequence of partitions spans the topology of X.

We construct the properly ordered Bratteli diagram $B = (V, E, \geq)$ using a sequence of Kakutani-Rohlin partitions satisfying the above conditions, so that the systems (X, T, x) and (X_B, T_B, x_{min}) are pointedly isomorphic. Its Bratteli diagram depends on the choice of Kakutani-Rohlin partitions. From Theorem 1, we see that equivalence class of the ordered Bratteli diagram constructed from (X, T, x) does not depend on the choice of Kakutani-Rohlin partitions.

A Bratteli diagram (V, E) is stationary if V_n has the same number of vertices and the incidence matrices between level n and $n+1$ are the same matrix for all $n = 1, 2, \cdots$. Also, we say that $(V, E, \geq)$ is a stationary ordered Bratteli diagram if (V, E) is stationary and the ordering on the edges with range $V(n, a_i)$ is the same as the ordering on the edges with range $V(m, a_i)$ for $n, m = 2, \cdots$ and $i = 1, \cdots, |V_n|$.

REMARK 1. If Bratteli diagrams $\mathcal{A}$ and $\mathcal{B}$ have the same graph from V_k onward for some k, then we say that two graphs are finitely related. It is easy to see that this defines an equivalence relation.

3. A Substitution

Let A be a alphabet. We denote by A^+ the set of finite words over the alphabet A. A substitution on the alphabet A is a map $\sigma : A \to A^+$. Using the extension to words by concatenation, σ can be iterated: for each integer $n > 0$, the n^{th} iteration $\sigma^n : A \to A^+$ is again a substitution.

DEFINITION 6. A substitution is primitive if there is a positive integer n such that for each $a, b \in A$, b occurs in $\sigma^n(a)$, and for some $a \in A$, $\lim_{n\to\infty} |\sigma^n(a)| = +\infty$.

In this section, we only consider primitive substitutions on the alphabet A. Note that if a substitution is primitive , then for each a, b any word in $\{\sigma^k(b)\}$ appears in $\sigma^n(a)$ for some n. We denote by $\mathcal{L}(\sigma)$ the language of σ i.e. the set of words on A which occurs in $\sigma^n(a)$ for some $a \in A$ and some $n \geq 1$. We let $X_\sigma \subset A^Z$ consist of all those sequences $x \in A^Z$ whose every finite segment of x belongs to $\mathcal{L}(\sigma)$. Clearly X_σ is closed subset of A^Z and invariant under the shift. We denote by T_σ the shift restricted to X_σ. The dynamical system (X_σ, T_σ) is called the substitution dynamical system associated to σ. It is well known that every primitive substitution dynamical system is minimal and uniquely ergodic[7].

Substitution dynamical systems are often defined by a different method using a fixed point as follows. For every integer $p > 0$, the substitution σ^p defines the same language, thus the same system, as σ does. Substituting σ^p for σ if needed, we can assume that there exist two letters $r, l \in A$ such that :
(1) r is the last letter of $\sigma(r)$
(2) l is the first letter of $\sigma(l)$
(3) $rl \in \mathcal{L}(\sigma)$

Whenever r and l satisfy the conditions (1) and (2), it is easy to check that there exists a unique $\omega \in A^Z$ such that

$$\omega_{-1} = r : \omega_0 = l \text{ and } \sigma(\omega) = \omega.$$

Such an ω is called a fixed point of σ. If r and l satisfy also (3), we say that ω is an admissible fixed point of σ.

The subshift spanned by a sequence $x \in A^Z$ is the $\overline{\text{orb}\{x, T\}}$ endowed with the restriction of the shift to $\overline{\text{orb}\{x, T\}}$. We have that $\overline{\text{orb}\{x, T\}}$ is minimal if and only if x is uniformly recurrent point i.e., for every word $x \in \mathcal{L}(x)$ there exists $n \geq 1$ such that for all words $v \in \mathcal{L}(x)$ with $|v| \geq n$, u occurs in v[7]. By minimality of X_σ, if w is an admissible fixed point of σ, then X_σ is the closure of the orbit of w for the shift.

DEFINITION 7. A substitution σ on the alphabet A is proper if there exists an integer $p > 0$ and two letters $r, l \in A$ such that :
(1) For every $a \in A$, r is the last letter of $\sigma^p(a)$.
(2) For every $a \in A$, l is the first letter of $\sigma^p(a)$.

It is easy to see that a proper substitution has only one fixed point. For every letter $a \in A$, we write $[a] = \{x \in X_\sigma; x_0 = a\}$. Then there exists a clopen partition $\mathcal{P}_n$ of X_σ with

$$\mathcal{P}_n = \{T_\sigma^k(\sigma^n([a]) | a \in A, 0 \leq k < |\sigma^n(a)|\}$$

for every $n > 0$. The base of this partition is $\bigcup_{a \in A} \sigma^n([a]) = \sigma^n(X_\sigma)$. It satisfies the following properties.

THEOREM 2([3]). *The sequence of partitions $\{\mathcal{P}_n\}$ satisfies the following conditions ;*

1. *The sequence of bases $\{(\sigma^n(X_\sigma) : n \in N)\}$ is decreasing.*
2. *For every n, $\mathcal{P}_{n+1} \succeq \mathcal{P}_n$ as partitions.*
   Moreover, if the substitution σ is proper, then
3. *The intersection of the bases consists of only one point.*
4. *The sequence of partition spans the topology of X_σ.*

Now, we describe a return word in substitution dynamical system. It was used to prove that every substitution dynamical system is isomorphic to the system associated to some stationary properly ordered Bratteli diagram.

Let (X, T) be a minimal subshift on the alphabet A and x a given fixed point of X. Let u be a word $x_{[-n,-1]}$ of $x_{(-\infty,-1]}$ and v a word $x_{[0,m)}$

of $x_{[0,+\infty)}$ for $n \geq 1$ and $m \geq 0$. We call the word $x_{[-n,-1]}$ ($x_{[0,m)}$) suffix (prefix), respectively. We define an occurrence of $u.v$ in x to be an integer n such that $x_{[n-|u|,n+|v|)} = uv$.

DEFINITION 8. A word w on A^+ is a return word to $u.v$ in x if there exist two consecutive occurrences j, k of $u.v$ in x such that $w = x_{[j,k)}$.

As x is uniformly recurrent, the difference between two consecutive occurrences of $u.v$ in x is bounded, hence the set $\mathcal{R}_{u.v}$ of the return words to u, v is finite.

Let $u.v$ be as above. The cylinder sets $[u.wv]$ for $w \in \mathcal{R}_{u.v}$ are obviously pairwise disjoint: they are included in the cylinder set $[u.v]$. Let $y \in [u.v]$ and n be the smallest positive occurrence of $u.v$ in y, then $w = y_{[0,n)}$ is a return word and $y \in [u.wv]$. Thus $\{[u.wv] : w \in \mathcal{R}_{u.v}\}$ is a partition of $[u.v]$. Moreover, if $w \in \mathcal{R}_{u.v}$ and $y \in [u.wv]$ the first return time of y to $[u.v]$ is $|w|$. It follows that

$$\mathcal{Q} = \{T^j[u.wv]; w \in \mathcal{R}_{u.v} \text{ and } 0 \leq j < |w|\}$$

is a Kakutani-Rohlin partition of X with base $[u.v]$.

We now consider return words for longer and longer prefixes of $x_{[0,+\infty)}$ and suffixes of $x_{(-\infty,-1]}$. To avoid unnecessary heavy notations, we write, for all $n \geq 1$;

$$\mathcal{R}_n = \mathcal{R}_{x_{[-n,-1]}.x_{[0,n)}}.$$

We get a sequence of Kakutani-Rohlin partitions $\mathcal{P}_n$ where

$$\mathcal{P}_n = \{T^j[x_{[-n,-1]}.wx_{[0,n)}]; w \in \mathcal{R}_n \text{ and } 0 \leq j < |w|\}.$$

The base of $\mathcal{P}_n$ is $B_n = [x_{[-n,-1]}.x_{[0,n)}]$. It is not hard to see that the sequence $\{\mathcal{P}_n\}$ of partitions is a nested sequence satisfying all the hypotheses (1), (2), (3) and (4) in Definition 5. Hence it can be used to construct a properly ordered Bratteli diagram B with (X_B, V_B) isomorphic to (X, T).

THEOREM 3. *Let B be a stationary, properly ordered* Bratteli *diagram with a single edge from top vertex to each vertex in first level. Then there exists a primitive and proper substitution, called a substitution read on B.*

Proof. Define the substitution as follows. For each $n \geq 1$, we consider the labeling map which assigns to each v_n in V_n a label $v_{n,a}$ from the finite alphabet A. That is, $V_n = \{v_{n,a} : a \in A\}$ for all $n \geq 1$. Fix an integer

$n > 1$, we read the label of source vertex in the order of the edges which end at $v_{n,a}$, say $(a_1, a_2, \cdots, a_k)$. The map $a \to a_1 a_2 \cdots a_k$ from A to A^+ defines the substitution τ on the alphabet A.

We now prove that τ is primitive. Since B is proper, for any $a, b \in A$, there exists n such that $v_{1,a}$ is connected to $v_{n,b}$. By the definition of τ, a occurs in $\tau^{n-1}(b)$. Therefore τ is primitive.

Suppose τ is not proper. Without loss of generality, we may assume that for each $n \geq 1$, the set $\{\tau^n(a) : a \in A\}$ has at least two elements with distinct first letter. Let $i(\tau^n(a))$ denote a first letter of $\tau^n(a)$, for $a \in A$ and $n \geq 1$. If for each pair (a, b), there exists k such that $i(\tau^k(a)) = i(\tau^k(b))$, then there exists N such that for $n \geq N$, τ^n has only one first letter. Thus We note that there exist at least one pair (a, b), $a, b \in A$ such that $i(\tau^n(a)) \neq i(\tau^n(b))$, for all $n \geq 1$. Therefore, there exist two sequences $\{a_n\}$ and $\{b_n\}$ such that $i(\tau(a_{n+1}))$ is a_n and $i(\tau(b_{n+1}))$ is b_n. Since for each $n \geq 1$, the edge from v_{n,a_n} (v_{n,b_n}) to $v_{n+1,a_{n+1}}$ $(v_{n+1,b_{n+1}})$ is minimal, there exist two minimal paths $x, y \in X_B$ such that for every $n \geq 1$, x goes through $V(n, a_n)$ and y through $V(n, b_n)$. This is a contradiction to the fact that B is proper. □

4. A Bratteli diagram isomorphic to Chacon substitution

We will find a sequence of Kakutani Rohlin partitions that satisfy required conditions given in Definition 5. We will construct a stationary ordered Bratteli diagram $B = (V, E, \leq)$ and show that (X_B, V_B) is isomorphic to (X_C, T_C).

The Chacon transformation is a substitution system via $0 \to 0010$ and $1 \to 1$. Another way to define the Chacon transformation is via successive construction of blocks. First we define the B_k block inductively as follows

$$B_0 = 0, \ B_1 = 0010, \ B_{k+1} = B_k B_k 1 B_k.$$

We let $X_C \subset \{0,1\}^Z$ consist of all those biinfinite sequences x such that each finite segment of x is a segment of B_k for some k. Clearly X_C is a closed shift-invariant subset of $\{0,1\}^Z$. We denote by T_C the shift homeomorphism on X_C.

There is a natural way to produce sequences for x's by a nesting procedure as follows[2]. Choose a sequence $\xi \in \{1,2,3\}^N$ which we may call the nesting instruction. Starting with a B_0 consider it as the $\xi(1)$th B_0

of its B_1, which in turn is considered as the $\xi(2)$th B_1 of its B_2 and so on. In this way the B_k's expand to define an infinite sequence ξ^*, which is well defined only up to a shift. ξ^* will be a doubly infinite sequence unless $\xi(i) = 3$ eventually in which case ξ^* is a left sequence we denote by $B_{-\infty}$, or $\xi(i) = 1$ eventually which yields a right infinite sequence denoted by B_∞. We can now give a precise description of X_C.

THEOREM 4([2]). *The set X_C consists(up to shifts) of all doubly infinite sequence $\xi^*, \xi \in \{1,2,3\}^N$ and the sequences $B_{-\infty}B_\infty$ and $B_{-\infty}1B_\infty$.*

REMARK 2. We note that $B_{-\infty}.B_\infty$ and $B_{-\infty}.1B_\infty$ are fixed points of the Chacon substitution.

PROPOSITION 1. *There is a* Kakutani-Rohlin *partition $\mathcal{P}_n, n = 1, 2, \cdots$ with base $[B_n.B_n]$ of the system (X_C, T_C).*

Proof. Let R_n be the set of return words to $B_n.B_n$ in $B_{-\infty}.B_\infty$ for $n \geq 0$, $n = 1, 2, \cdots$. It is immediate to find that

$$R_n = \{B_n, B_n1B_n, B_n1B_n1B_n\}.$$

To get easier notation, we put;

$$w_{n,1} = B_n, w_{n,2} = B_n1B_n, w_{n,3} = B_n1B_n1B_n$$

The cylinder sets $[B_n.w_{n,k}B_n]$ for $w_{n,k} \in R_n$ are obviously pairwise disjoint. They are included in the cylinder set $[B_n.B_n]$. Let $y \in [B_n.B_n]$ and l be the smallest positive occurrence of $B_n.B_n$ in y. There exists integer k such that $y_{[0,l)} = w_{n,k}$. Thus $\{[B_n.w_{n,k}B_n] : w_{n,k} \in R_n\}$ is a partition of $[B_n.B_n]$. Moreover, if $w_{n,k} \in R_n$ and $y \in [B_n.w_{n,k}B_n]$, the first return time of y to $[B_n.B_n]$ is $|w_{n.k}|$. It follows that

$$\mathcal{P}_n = \{T_C^j[B_n.w_{n,k}B_n] : w_{n,k} \in R_n \quad \text{and} \quad 0 \leq j < |w_{n,k}|\}$$

is a Kakutani-Rohlin partition of X_C. It is easy to proved that $[B_n.B_n]$ is the base of $\mathcal{P}_n$. □

Let $\mathcal{P}_{n,k} = \{T_C^j[B_n.w_{n,k}B_n] : 0 \leq j < |w_{n,k}|\}$, $k = 1, 2, 3$ be the k^{th} tower of $\mathcal{P}_n$ with a base $Z_{n,k} = [B_n.w_{n,k}B_n]$. If $F_{n,k}$ be a subset of $Z_{n.k}$, then we call the set $\{T_C^j F_{n,k} : 0 \leq j \leq |w_{n,k}|\}$ a subcolumn of $\mathcal{P}_{n,k}$. Therefore, we get a sequence of Kakutani-Rohlin partition $\mathcal{P}_n$ where

$$\mathcal{P}_n = \{\mathcal{P}_{n,1}, \mathcal{P}_{n,2}, \mathcal{P}_{n,3}\}$$

with the base $Z_n = [B_n.B_n]$ (See Figure 1).

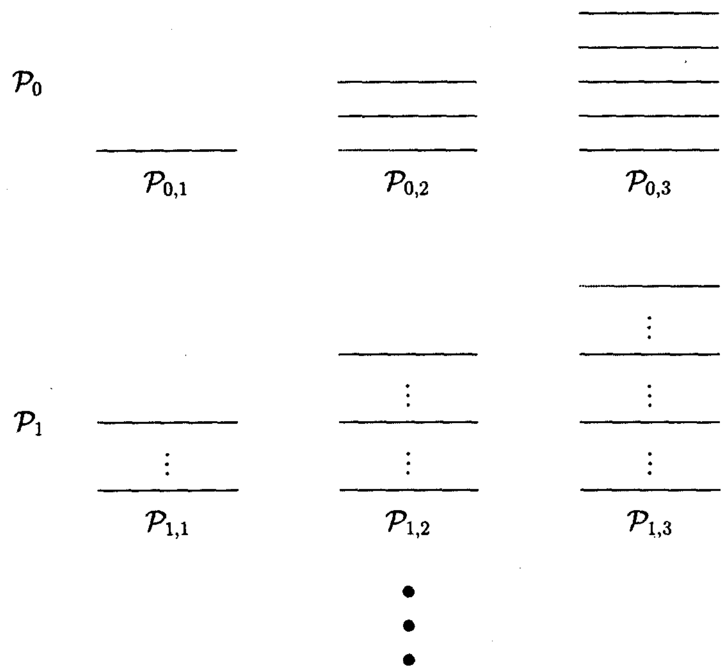

Figure 1.

PROPOSITION 2. *The sequence $\{\mathcal{P}_n\}$ is satisfies the following conditions:*

(i) $Z_{n+1} \subset Z_n$, *for all* $n \geq 0$.
(ii) $\mathcal{P}_{n+1} \succ \mathcal{P}_n$, *for all* $n \geq 0$.
(iii) $\cap Z_n = B_{-\infty}.B_{\infty}$.
(iv) *The sequence of partitions spans the topology of* X_C.

Proof. (i) is obvious. Every word in R_{n+1} is made up of the words in R_n. The word $w_{n+1,1}$ belonging to R_{n+1} consists of words B_n and $B_n 1 B_n$

in order. Also $w_{n+1,2}$ can be written as a concatenation of words B_n, $B_n 1 B_n 1 B_n$ and $B_n 1 B_n$ and $w_{n+1,3}$ is a concatenation of B_n, $B_n 1 B_n 1 B_n$, $B_n 1 B_n 1 B_n$ and $B_n 1 B_n$. Hence $\mathcal{P}_{n+1,1}$ consists of a subcolumn of $\mathcal{P}_{n,1}$ and a subcolumn of $\mathcal{P}_{n,2}$ in order and $\mathcal{P}_{n+1,2}$ consists of a subcolumn of $\mathcal{P}_{n,1}$, a subcolumn of $\mathcal{P}_{n,3}$ and a subcolumn of $\mathcal{P}_{n,2}$. Also $\mathcal{P}_{n+1,3}$ consists of a subcolumn of $\mathcal{P}_{n,1}$, a subcolumn of $\mathcal{P}_{n,3}$, subcolumn of $\mathcal{P}_{n,3}$ and a subcolumn of $\mathcal{P}_{n,2}$. We proved (ii). Since $Z_n = [B_n.B_n]$, (iii) is obvious.

Given an integer $m > 0$, we claim that any cylinder set $x_{[-m,m]}$ in X_C is a union of levels in some tower of partitions $\mathcal{P}_n$. Choose n such that $|B_n| > m$. Fix $w_{0,k}$ and $0 \le l < |w_{n,k}|$. For each $x \in T_C^l \sigma^n([w_{0,k}])$, there exists $y \in X_C$ with $y_{[0,|w_{0,k}|)} = w_{0,k}$ such that $x = T_C^l \sigma^n(y)$. The word B_n is a prefix of $\sigma^n(y)_{[0,\infty)}$ and suffix of $\sigma^n(y)_{(\infty,-1]}$. Hence

$$\sigma^n(y)_{[-|B_n|,|w_{n,k}|+|B_n|)} = \sigma^n(B_0).\sigma^n(w_{0,k})\sigma^n(B_0).$$

We get

$$x_{[-m,m]} = \sigma^n(B_0).\sigma^n(w_{0,k})\sigma^n(B_0)_{[l-m,l+m]}.$$

Therefore, any cylinder set with length $2m+1$ is the union of levels in $\mathcal{P}_n$ for some large enough n and (iv) follows. □

THEOREM 5. *There exists a stationary and properly ordered* Bratteli *diagram (X_B, V_B) isomorphic to (X_C, T_C).*

Proof. We will construct an ordered Bratteli diagram $\mathcal{B}$. Let V_{-1} consist of one vertex. For each $n = 0, 1, 2\cdots$, the set V_n denotes the set of vertices of the n^{th} step and E_n be the set of edges connecting the vertices of $(n-1)^{th}$ step with those of the step n. A vertex in V_n represents a tower in the Kakutani-Rohlin partition $\mathcal{P}_n$. Let $v_{n,k} \in V_n$ correspond to the tower $\mathcal{P}_{n,k}$, $k = 1, 2, 3$. That is,

$$V_n = \{v_{n,1}, v_{n,2}, v_{n,3}\}$$

For $n = 0, 1, 2, \cdots$, edges appear between $v_{n,k}$ and $v_{n+1,l}$ when the tower $\mathcal{P}_{n+1,l}$ contains a subcolumn of the tower $\mathcal{P}_{n,k}$. We order the edges so that it represents the order of the subcolumn of the level n appearing in the tower $\mathcal{P}_{n+1,l}$. For $n = 0$, E_0 consists of edges from V_{-1} to each vertex in V_0, representing level sets of each tower $\mathcal{P}_{0,k}$ $k = 1, 2, 3$(See Figure 2). It is clear from the Kakutani-Rohlin partitions that $\mathcal{B} = (V, E, \ge)$ is a stationary and properly ordered Bratteli diagram.

To define the conjugacy $G : X_C \to X_{\mathcal{B}}$, we note that it is enough to specify the edges, $e_0e_1e_2\cdots$. For any $x \in X_C$, if x is located on some level of $\mathcal{P}_{0,k}$, $k = 1, 2, 3$, then we choose the edge e_0 in E_0 which corresponds to that level of $\mathcal{P}_{0,k}$. We find a tower in $\mathcal{P}_1$ which contains x. We choose the edge e_1 such that $s(e_1) = r(e_0)$ and the range vertex of the e_1 is the vertex in V_1, say $v_{1,x}$ corresponding to the tower $\mathcal{P}_{1,x}$ which contains x. At each step, we find the edge e_n such that $s(e_n) = r(e_{n-1})$ and the range of e_n is the vertex $v_{n,x}$ where $\mathcal{P}_{n,x}$ contains x. Moreover, we choose the i^{th} edge among the edges with the same range vertex if x lies in the i^{th} subcolumn of $\mathcal{P}_{n-1}$ in a tower of $\mathcal{P}_n$. Hence $G(x)$ consists the edges through $v_{n,x}$'s where the corresponding $\mathcal{P}_{n,x}$ contains x. It is easily seen that $G(B_{-\infty}.B_{\infty}) = x_{min}$, $G(B_{-\infty}.1B_{\infty}) = 5333\cdots$ and $G(T_C^{-1}(B_{-\infty}.B_{\infty})) = x_{max}$.

We claim that G is an isomorphism between $(X_C, T_C)$and $(X_{\mathcal{B}}, V_{\mathcal{B}})$. It is well known from the construction that G is one-to-one. It is also easy to show that $G \circ T_C = V_{\mathcal{B}} \circ G$.

We now want to show that the map G is continuous and onto. For $x, y \in X_C$, suppose $x_{[-l,l]} = y_{[-l,l]}$ for sufficiently large l. We find $k > 0$ such that some concatenation of the block B_k and 1 is located around the 0^{th}-coordinate in $x_{[-l,l]}$. So, there exists $k' \leq k$ such that one of the names of the bases of the towers in $\mathcal{P}_{k'}$ appears in the concatenation of the block B_k and 1, and the name of the base is located around the 0^{th}-coordinate. By the construction of the sequence $\{\mathcal{P}_n\}$, it is clear that points x and y are on the same level in the partition $\mathcal{P}_{k'}$. Thus $G(x)$ and $G(y)$ share the same finite path from top vertex to some vertex v in $V_{k'}$. Hence $G(x)$ and $G(y)$ are close.

Suppose $x_{\mathcal{B}} \in X_{\mathcal{B}}$. Let $x_{\mathcal{B}} = e_0e_1\cdots e_n\cdots$. We read a vertex name of $v = r(e_n)$ in V_n and find the minimal finite path x_n from the top vertex to v. We choose for some l such that $V_{\mathcal{B}}^l(x_n) = e_0e_1\cdots e_n$. Then we know that the finite path $e_0e_1\cdots e_n$ corresponds to the T_C^l (the base set $w_{n,k}$ of the vertex name of v) which is a some level set in $\mathcal{P}_n$. Since the level sets generate the σ-algebra, there exists a unique point x_C belonging to all the level sets. Clearly $G(x_C) = x_{\mathcal{B}}$. Our claim is proved. □

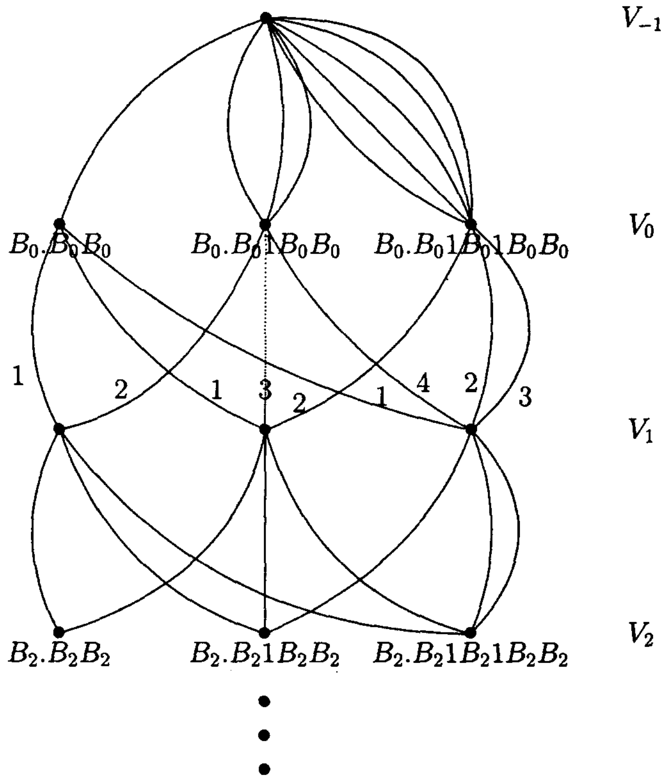

Figure 2. ($\mathcal{B}$)

We have constructed a stationary and properly ordered Bratteli diagram so that its Bratteli-Vershik system is isomorphic to the substitution system associated to neither primitive nor proper substitution.

We know the following facts using the results in [5] and [3]. From the stationary and properly ordered Bratteli diagram $\mathcal{B}$, we get a stationary and properly ordered Bratteli diagram C, so that there is one edge between the top level and each vertex in first level and (X_B, V_B) is isomorphic to (X_C, V_C). We find the primitive substitution of Chacon's transformation.

REMARK 3. We note that the minimal path corresponds to $B_{-\infty}.B_{\infty}$ in X_C.

REMARK 4. We note that for $n \geq 0$ and $k = 1, 2, 3$, a minimal finite path from the top vertex V_{-1} to each vertex $v_{n,k}$ in V_n corresponds to the base $w_{n,k}$ of the name of a vertex $v_{n,k}$.

REMARK 5. For each $n \geq 0$, a finite maximal path $33 \cdots 33$ with length n corresponds to a cylinder set $T_C^{(|B_n 1 B_n|-1)}[B_n.B_n 1 B_n B_n]$. Hence it is clear that the maximal path corresponds to $T_C^{-1}(B_{-\infty}.B_\infty)$

REMARK 6 The point $B_{-\infty}.1B_\infty$ in X_C corresponds to the infinite path $5333 \cdots$ in $X_{\mathcal{B}}$.

5. Construction of other Bratteli diagrams isomorphic to Chacon substitution

5.1 Construction of $(X_{\mathcal{B}'}, V_{\mathcal{B}'})$

We define τ by $\tau 0 = 0012$, $\tau 1 = 12$ and $\tau 2 = 012$. It is known that the shift map generated by τ is isomorphic to the shift of the Chacon substitution([4]). Since we have two letters 0 and 1 each of whose substitution begins with itself, there exist exactly two fixed points.

Since τ is not proper, we look for proper substitution to find the Bratteli diagram. We note that $\{0012, 001212, 012, 01212\}$ is the set of return words to the word 2.0 in the fixed point $\tau^{-\infty}(2).\tau^{\infty}(0)$. We let $a = 0012$, $b = 001212$, $c = 012$ and $d = 01212$. Then we find the substitution τ' such that $\tau'(a) = abc$, $\tau'(b) = abdc$, $\tau'(c) = bc$ and $\tau'(d) = bdc$. Here τ' is a proper and primitive substitution. We construct the ordered stationary Bratteli diagram $\mathcal{B}'$ corresponding to τ' as follows. Let $A = \{a, b, c, d\}$ be alphabet. Each letter $t \in A$ corresponds to a vertex v_t in V_n in $\mathcal{B}'$ for all $n = 0, 1, 2, \cdots$. The edges with range vertex v_t in each V_n are ordered so that the names of the source vertices in the order of the edges are the same as $\tau'(t)$. For each $t \in A$, the top vertex V_{-1} is joined by $|t|$ edges to the each vertex t in the 0^{th} level V_0. Then (X_τ, T_τ) is isomorphic to $(X_{\mathcal{B}'}, V_{\mathcal{B}'})$(See Figure 3).

REMARK 7. Let $\mathcal{A}'$ be the stationary properly ordered Bratteli diagram which is identical to $\mathcal{B}'$ except that there is one edge between the top vertex and each vertex in the 0^{th} level V_0. Then $(X_{\mathcal{A}'}, V_{\mathcal{A}'})$ is isomorphic to $(X_{\tau'}, T_{\tau'})$.

REMARK 8. We note that the minimal path in $\mathcal{B}'$ corresponds to $B_{-\infty}.B_{\infty}$ in X_C.

REMARK 9. The point $B_{-\infty}.1B_{\infty}$ in X_C corresponds to the infinite path $5222\cdots$ in $X_{\mathcal{B}'}$.

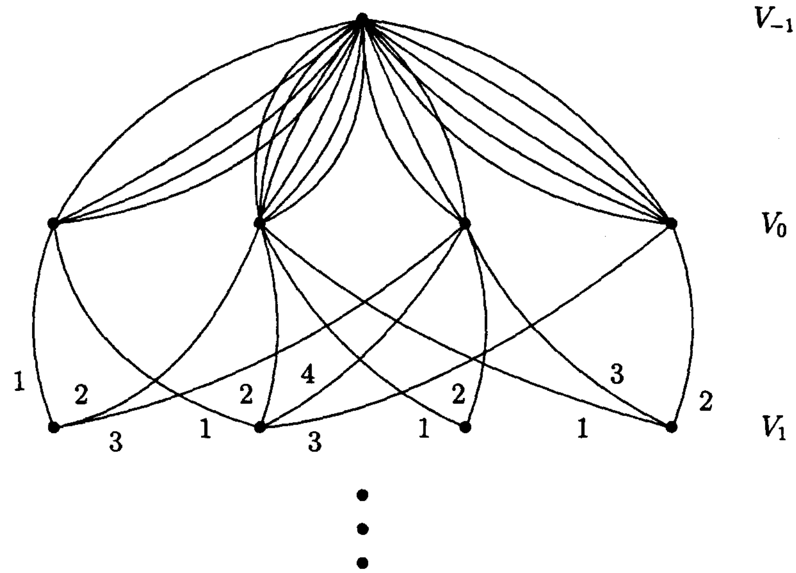

Figure 3. ($\mathcal{B}'$)

5.2 Construction of $(X_{\mathcal{B}''}, V_{\mathcal{B}''})$

We note that $R = \{abc, abdcbc, abdcbdcbc\}$ is the set of the return words to $c.a$ in the fixed point of τ'. We call the words a', b' and c' respectively. We get that

$a' =$ 0012 001212012,

$b' =$ 0012 00121201212012 001212012 and

$c' =$ 0012 00121201212012 00121201212012 0012012

We find the subset $R' = \{0012, 001212012, 00121201212012\}$ such that the words a', b' and c' are uniquely decomposed by the elements of R'. Therefore every word in R admits a concatenation of the elements of R'. We assign $\alpha = 0012$, $\beta = 001212012$ and $\gamma = 00121201212012$. We find a new substitution τ'' such that $\tau''\alpha = \alpha\beta$, $\tau''\beta = \alpha\gamma\beta$ and

$\tau''\gamma = \alpha\gamma\gamma\beta$. Since τ'' is proper and primitive, we construct a stationary and properly ordered Bratteli diagram $\mathcal{A}''$ using the same method as the above construction of $\mathcal{A}'$. Hence $(X_{\tau''}, T_{\tau''}) \cong (X_{\mathcal{A}''}, V_{\mathcal{A}''})$.

Let $\mathcal{B}''$ be the stationary properly ordered Bratteli diagram which is same as $\mathcal{A}''$ except the edges from the vertex V_{-1} to the each vertex in 0^{th} level V_0. We add the $|\alpha|(|\beta|, |\gamma|)$ edges between the top vertex V_{-1} and the vertex α (β, γ, respectively) in 0^{th} level V_0.

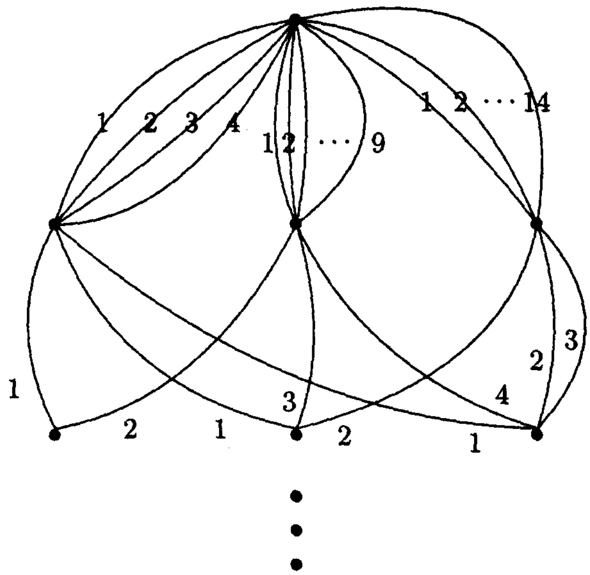

Figure 4. ($\mathcal{B}''$)

REMARK 10. $(X_{\mathcal{B}''}, V_{\mathcal{B}''}) \cong (X_\tau, V_\tau)$.

REMARK 11. We note that if we constract from V_{-1} to V_2 in the Bratteli diagram $\mathcal{B}$, then the resulting diagram is exactly the Bratteli diagram $\mathcal{B}''$.

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