Bull. Korean Math. Soc. **37** (2000), No. 3, pp. 507–517

ESTIMATION OF GIBBS SIZE FOR WAVELET EXPANSIONS

HONG-TAE SHIM

ABSTRACT. Existence of Gibbs' phenomenon has been well known in wavelet expansions. But the estimation of its size is another problem. Because of the oscillation of wavelets, it is not easy to estimate the Gibbs size of wavelet expansions. For wavelets defined via Fourier transforms, we give a new formula to calculate the size of overshoot. By using this we compute the size of Gibbs effect for Barttle-Lemarier wavelets.

1. Introduction

The Gibbs' phenomenon [3] in trigonometric expansion is well known. When a function is represented by the trigonometric series, one can see that the graphs of partial sums exhibit an overshoot or downshoot near the point of jump discontinuity of the function. This special quirk is called the Gibbs' phenomenon. At the beginning, this undesirable phenomenon was understood as the reason that the series expansion was approximated by a finite sum out of infinite series. To the contrary of the earlier guess, the overshoot (or downshoot) can not be removed. Instead, the ratio of overshoot to the jump converges to a certain constant, the Gibbs' constant, as the partial sum is taken to infinite series. But it is not unique to the trigonometric series. Foster [2] and Richard [5] demonstrated a Gibbs phenomenon using piecewise linear continuous and spline functions respectively. Kelly[4] showed that Daubechies' compactly supported wavelets exhibit this phenomenon at the origin and computed the size of them by using computer. It has also been

Received February 28, 2000. Revised April 7, 2000.
2000 Mathematics Subject Classification: 42C40, 42C15.
Key words and phrases: wavelet expansions, Gibbs phenomenon, Gibbs effect, distribution.
This paper was supported by the Sunmoon University Research Fund, 1998.

shown by Shim and Volkmer [6] that a Gibbs phenomenon occurs virtually all types of continuous orthogonal wavelets. In mathematical point of view, Gibbs effect is an obstacle to get a uniform convergence of the series. In image processing, the appearance of Gibbs' phenomenon can be useful for earlier detection of edges among different objects. But the existence of Gibbs' phenomenon is one thing and the calculation of overshoot is another thing. The bigger the size of Gibbs effect, the easier the detection of edges. So it is meaningful to try to estimate Gibbs' size corresponding to various wavelet expansions.

2. Background

A scaling function for a wavelet system is a square integrable function ϕ satisfying the following;

$$\begin{aligned} &\text{(i) } \{\phi(t-n)\}_{n\in\mathbb{Z}} \quad \text{is an orthonormal sequence,} \\ (2.1)\qquad &\text{(ii) } \phi(t) = \sum_{k=-\infty}^{\infty} c_k \phi(2t-k) \quad \text{for some} \quad c_k \in l^2, \\ &\text{(iii) the closed linear span of} \quad \{\phi(2^m t - n)\}_{m,n\in\mathbb{Z}} \text{ is } L^2(\mathbb{R}). \end{aligned}$$

Frequently, the above conditions are expressed in terms of their Fourier transforms. We give a sufficient condition for (2.1) as

$$\begin{aligned} &\text{(i) } \sum_k |\hat{\phi}(\omega + 2\pi k)|^2 = 1, \\ &\text{(ii) } \hat{\phi}(\omega) = (\frac{1}{\sqrt{2}} \sum_k c_k e^{ik\omega/2} \hat{\phi}(\frac{\omega}{2}) = m_0(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}), \\ (2.2)\qquad &\quad \text{where} \quad m_0(\frac{\omega}{2}) = \frac{1}{\sqrt{2}} \sum_k c_k e^{ik\omega/2} \in L^2(-2\pi, 2\pi), \\ &\text{(iii) } \hat{\phi}(\omega) \text{is continuous at } \omega = 0 \text{ and } \hat{\phi}(0) = 1. \end{aligned}$$

Most wavelet has an associated "multiresolution analysis" consisting of a nested sequence $\{V_m\}$ of subspaces of $L^2(\mathbb{R})$ where the space V_m is the closed linear span of $\{\phi(2^m t - n)\}_{n\in Z}$. A function f in $L^2(\mathbb{R})$ can be approximated by its projection $P_m f$ onto V_m;

$$(2.3)\qquad \begin{aligned}(P_m f)(x) = f_m(x) &= \int_{-\infty}^{\infty} q_m(x,y) f(y) dy \\ &= \sum_{n\in Z} \langle \phi_{mn}, f\rangle \phi_{mn}(x),\end{aligned}$$

where $q_m(x,y) = \sum_{n\in Z} \phi(2^m x - n)\phi(2^m y - n)$, $\phi_{mn}(x) = 2^{m/2}\phi(2^m x - n)$. For the convergence of this series, we refer to the work of Walter [8, p.12,116–128].

To study Gibbs phenomenon for wavelet expansions of $f \in L^2(\mathbb{R})$, we assume f is piecewise continuous and has jump discontinuity at dyadic rational number. So that we can take it to zero by translation. The spaces V_m are not translation invariant for irrational translations in general. We also assume the jump is in the positive direction, i.e., $f(0^+) > f(0^-)$. If there is a sequence $x_m \downarrow 0^+$ such that

$$(2.4)\qquad f_m(x_m) \longrightarrow \gamma > f(0^+) \quad \text{as} \quad m \to \infty,$$

then the wavelet series exhibits Gibbs phenomenon on the right hand side of 0 for the function (and similarly on the left hand side). The necessary and sufficient condition[4] for Gibbs phenomenon on the right (or on the left) to exist is

$$(2.5)\qquad \begin{aligned} G(x) := \int_0^{\infty} q(x,y) dy > 1 \quad &\text{for some} \quad x > 0 \\ \left(\text{or} \quad \int_0^{\infty} q(x,y) dy < 0 \quad \right. &\left.\text{for some} \quad x < 0\right),\end{aligned}$$

where $q(x,y) = q_0(x,y)$.

For wavelets defined via the Fourier transform, this formula can not be used for calculation.

3. Formula for Wavelets defined via the Fourier transform

To convert the function $G(x)$ in (2.5) in terms of the Fourier transform, we need following two facts. Theses two are already known but the proofs are not given (see[8, p. 191]). So we provide their proofs by using techniques of tempered distributions for the interest of completeness.

LEMMA 1. *For a test function ϕ of tempered distributions, infinitely smooth and rapidly decreasing, we have*

$$\int_{-\infty}^{\infty} \frac{cos\omega T}{\omega}\phi(\omega)d\omega \longrightarrow 0 \qquad \text{as} \qquad T \to \infty,$$

where the improper integral is in the sense of Cauchy *principal value.*

Proof. Now, the Cauchy principal value of the improper integral turns out to be

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\phi(\omega)}{\omega}\cos T\omega d\omega &= \lim_{\epsilon\to 0}\left(\int_{\epsilon}^{\infty} + \int_{-\infty}^{-\epsilon}\right)\frac{\phi(\omega)}{\omega}\cos T\omega d\omega \\ &= \lim_{\epsilon\to 0}\int_{\epsilon}^{\infty}\frac{\phi(\omega)-\phi(-\omega)}{\omega}\cos T\omega d\omega.\end{aligned}$$

By taking $\psi(\omega) := \frac{\phi(\omega)-\phi(-\omega)}{\omega}$ and defining $\psi(0) = lim_{\epsilon\to 0}\psi(\omega)$, we can see that ψ is integrable function over $(-\infty,\infty)$. Hence, by the Riemann-Lebesgue lemma, the integral converges to zero as $T \to \infty$.□

LEMMA 2. *Let h be the* Heaviside *functional*

$$h(t) = \begin{cases} 1 & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0. \end{cases}$$

Then we have

$$\hat{h}(\omega) = \pi\delta(\omega) + Pv\frac{1}{i\omega},$$

where $\hat{h}$ is the Fourier *transform of h, $\delta(\omega)$ is the delta functional and $Pv\frac{1}{i\omega}$ is the* Cauchy *principal value of $\int_{-\infty}^{\infty}\frac{\phi(\omega)}{i\omega}d\omega$ when it is applied to a test function ϕ.*

Proof. For any test function ϕ, infinitely smooth and rapidly decreasing, we have

$$\begin{aligned}\langle \hat{h}(\omega), \phi(\omega)\rangle &= \langle h(t), \hat{\phi}(t)\rangle \\ &= \int_{-\infty}^{\infty} h(t)\hat{\phi}(t)dt \\ &= \int_{0}^{\infty}\int_{-\infty}^{\infty} \phi(\omega)e^{-i\omega t}d\omega dt.\end{aligned}$$

By using the Fubini's theorem, the double integral in the final equality turns out to be

$$\begin{aligned}&\lim_{T\to\infty}\int_{0}^{T}\int_{-\infty}^{\infty} \phi(\omega)e^{-i\omega t}d\omega dt \\ &= \lim_{T\to\infty}\int_{-\infty}^{\infty}\phi(\omega)\left(\int_{0}^{T} e^{-i\omega t}dt\right)d\omega \\ &= \int_{-\infty}^{\infty}\frac{1}{i\omega}\phi(\omega)d\omega - \lim_{T\to\infty}\int_{-\infty}^{\infty}\frac{e^{-i\omega T}}{i\omega}\phi(\omega)d\omega \\ &=\langle \mathrm{Pv}\frac{1}{i\omega}, \phi\rangle - \lim_{T\to\infty}\int_{-\infty}^{\infty}\frac{e^{-i\omega T}}{i\omega}\phi(\omega)d\omega.\end{aligned} \tag{3.1}$$

The last integral can be written as

$$\lim_{T\to\infty}\int_{-\infty}^{\infty}\left(\frac{\phi(\omega)}{i\omega}\cos T\omega d\omega - \frac{\phi(\omega)}{i\omega}\sin T\omega d\omega\right). \tag{3.2}$$

By the following fact[8, p. 177]

$$\lim_{T\to\infty}\frac{\sin T\omega}{\pi\omega} \longrightarrow \delta(\omega),$$

the second term in (3.2) converges to $\pi\phi(0)$. By the lemma 1, the first term in (3.2) converges to zero. Hence we have

$$\begin{aligned}\langle \hat{h}(\omega), \phi(\omega)\rangle &= \langle \mathrm{Pv}\frac{1}{i\omega}, \phi(\omega)\rangle + \langle \pi\delta(\omega), \phi(\omega)\rangle \\ &= \langle \mathrm{Pv}\frac{1}{i\omega} + \pi\delta(\omega), \phi(\omega)\rangle.\end{aligned}$$

Therefore we have

$$\hat{h}(\omega) = \pi\delta(\omega) + \mathrm{Pv}\frac{1}{i\omega}. \qquad \square$$

DEFINITION. For a function $\phi \in S_r$, we define it as

$$|\phi^{(k)}(t)| \leq C_{pk}(1+|t|)^{-p}, \quad k = 0, 1, \cdots, r; \quad p \in \mathbb{Z}, \quad t \in \mathbb{R}.$$

THEOREM 1. *For a scaling function $\phi \in S_r$ and $\hat{\phi}(0) \geq 0$, we have*

$$\int_0^\infty q(x,y)dy = \frac{1}{2\pi}Pv\int_{-\infty}^\infty \frac{\bar{\hat{q}}(x,\omega)}{i\omega}d\omega + \frac{1}{2}.$$

Proof. For a scaling function $\phi \in S_r$, its Fourier transform $\hat{\phi}$ satisfies $\hat{\phi}(2n\pi) = \delta_{0n}$ if $\hat{\phi}(0) \geq 0$ (see [7, p. 41]). Now let

$$h(x) = \begin{cases} 1 & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0. \end{cases}$$

Then we have, by Parseval's identity,

$$\begin{aligned} \int_0^\infty q(x,y)dy &= \int_{-\infty}^\infty h(y)q(x,y)dy \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \hat{h}(\omega)\bar{\hat{q}}(x,\omega)d\omega \\ &= \frac{1}{2\pi}\langle \hat{h}(\cdot), \bar{\hat{q}}(x,\cdot)\rangle. \end{aligned}$$

By taking the Fourier transform of $q(x,y)$ with respect to y and using the Poisson's summation formula , we obtain

$$\begin{aligned} \bar{\hat{q}}(x,\omega) &= \bar{\hat{\phi}}(\omega)\sum_n \phi(x-n)e^{i\omega n} \\ &= \bar{\hat{\phi}}(\omega)\sum_n \hat{\phi}(\omega - 2\pi n)e^{i(\omega-2\pi n)x}. \end{aligned} \tag{3.3}$$

From the fact $\hat{\phi}(2n\pi) = \delta_{0n}$, we have $\bar{\hat{q}}(x,0) = 1$. By noticing $\hat{h}(\omega) = \mathrm{Pv}\frac{1}{i\omega} + \pi\delta(\omega)$, we have the result. $\square$

4. Gibbs size of Bartle-Lemarie family

We consider wavelets related to spline approximation. The k-th order cardinal B-spline $N^{[k]}$ is defined as the k - fold ccnvolution of the characteristic function of the interval $[0,1]$ for $k=2,3,\cdots$. These functions are not scaling functions in the sense of section 2 because the orthogonal condition is not satisfied. The corresponding orthogonalized scaling function $\phi^{[k]}$ leads to Battle-Lemarie wavelets. These are defined as the function whose Fourier transform is given by

$$\hat{\phi}^{[k]}(\omega)=\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{k}\sigma_k(\omega)^{-\frac{1}{2}}, \tag{4.1}$$

where (see [1, p.216])

$$\sigma_k(\omega)=\left(\sin\frac{\omega}{2}\right)^{2k}\sum_n\left(\frac{\omega}{2}+n\pi\right)^{-2k}. \tag{4.2}$$

We note that the space $V_0=V_0^{[k]}$, where V_0 is the closed span of $N^{[k]}(\cdot-n), n\in\mathbb{Z}$, consists of all square integrable and $k-2$ times continuously differentiable functions that agree with a polynomial function of degree at most $k-1$ on each interval $[n,n+1]$ for $n\in\mathbb{Z}$. By defining

$$r_k(\omega):=\sum_n\frac{1}{(\frac{\omega}{2}+n\pi)^k},$$

we summarize some formulas as follows;

(i) $\hat{\phi}^{[k]}(\omega)=e^{-i\omega k/2}\left(\frac{2}{\omega}\right)^k\frac{1}{\sqrt{r_{2k}(\omega)}}$.

(ii) $\sum_n\hat{\phi}^{[k]}(\omega-2n\pi)=e^{-ik\omega/2}\frac{r_k(\omega)}{\sqrt{r_{2k}(\omega)}}$.

(iii) $\overline{\hat{\phi}^{[k]}}(\omega)\sum_n\hat{\phi}^{[k]}(\omega-2n\pi)=\left(\frac{2}{\omega}\right)^k\frac{r_k(\omega)}{r_{2k}(\omega)}$.

(iv) For $x\in\mathbb{Z}$, $\frac{\overline{\hat{q}^{[k]}(x,\omega)}}{i\omega}=\overline{\hat{\phi}^{[k]}}(\omega)\sum_n\hat{\phi}^{[k]}(\omega-2n\pi)\frac{\cos\omega x+i\sin\omega x}{i\omega}$, The formula (iii) comes immediately from (i) and (ii). The formula (i) comes from (4.1) and (4.2) as follows

$$\hat{\phi}^{[k]}(\omega) = e^{-i\omega k/2}\left(\frac{\sin\omega/2}{\omega/2}\right)^k \sigma_k(\omega)^{-\frac{1}{2}} = e^{-i\omega k/2}\left(\frac{2}{\omega}\right)^k \frac{1}{\sqrt{r_{2k}(\omega)}}.$$

The formula (iv) follows when we replace x by an integer from the equation (3.3). The following calculation derives the formula (ii);

$$\begin{aligned}\sum_n \hat{\phi}^{[k]}(\omega - 2n\pi) &= \sum_n e^{-ik(\omega-2n\pi)/2}\left(\frac{\sin(\frac{\omega}{2}-n\pi)}{\frac{\omega}{2}-n\pi}\right)^k \sigma_k^{-\frac{1}{2}}(\omega-2n\pi)\\ &= \sigma_k^{-\frac{1}{2}}(\omega)e^{-ik\omega/2}\sum_n e^{ikn\pi}(-1)^k \sin^k(n\pi - \frac{\omega}{2})\frac{1}{(\frac{\omega}{2}-n\pi)^k}\\ &= e^{-ik\omega/2}\sigma_k{}^{-\frac{1}{2}}(\omega)\sin^k\frac{\omega}{2}\sum_n \frac{1}{(\frac{\omega}{2}-n\pi)^k}\\ &= e^{-ik\omega/2}\left(\sum_n \frac{1}{(\frac{\omega}{2}+n\pi)^{2k}}\right)^{-\frac{1}{2}}\sum_n \frac{1}{(\frac{\omega}{2}-n\pi)^k}\\ &= e^{-ik\omega/2}\frac{r_k(\omega)}{\sqrt{r_{2k}(\omega)}}.\end{aligned}$$

Now we take $Q_k(\omega)$ as $Q_k(\omega) := \overline{\hat{\phi}^{[k]}}(\omega)\sum_n \hat{\phi}^{[k]}(\omega - 2n\pi)$. Then by observing

$$r_k(-\omega) = \sum_n \frac{1}{(-\frac{\omega}{2}+n\pi)^k} = (-1)^k\sum_n \frac{1}{(\frac{\omega}{2}-n\pi)^k} = (-1)^k r_k(\omega),$$

we can see $Q_k(\omega)$ is an even function;

$$Q_k(-\omega) = (-1)^k(\frac{2}{\omega})^k\frac{(-1)^k r_k(\omega)}{r_{2k}(\omega)} = Q_k(\omega). \tag{4.3}$$

COROLLARY 1. *For $x \in \mathbb{Z}$, the principle part of the integral in* Theorem 1 *becomes*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\overline{\hat{q}^{[k]}}(x,\omega)}{i\omega}d\omega = \frac{1}{2\pi}\int_0^{\pi}\frac{r_k(\omega)r_{k+1}(\omega)}{r_{2k}(\omega)}\sin x\omega d\omega.$$

Proof. From the formula (ii) and $Q_k(\omega)$, we have

$$\frac{\overline{\hat{q}^{[k]}}(x,\omega)}{i\omega} = Q_k(\omega)\frac{\cos\omega x + i\sin\omega x}{i\omega}.$$

We also have $\int_{-\infty}^{\infty} Q_k(\omega)\frac{\cos\omega x}{\omega}d\omega = 0$ since $Q_k(\omega)$ is an even function. For the $\sin\omega x$ part, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} Q_k(\omega)\frac{\sin\omega x}{\omega}d\omega &= \frac{1}{2\pi}\sum_l \int_{2\pi l}^{2\pi(l+1)} \frac{1}{2}(\frac{2}{\omega})^{k+1}\frac{r_k(\omega)}{r_{2k}(\omega)}\sin x\omega d\omega \\ &= \frac{1}{4\pi}\int_0^{2\pi}\frac{r_k(\omega)}{r_{2k}(\omega)}\sum_l(\frac{1}{\frac{\omega}{2}+\pi l})^{k+1}\sin x\omega d\omega \\ &= \frac{1}{4\pi}\int_0^{2\pi}\frac{r_k(\omega)r_{k+1}(\omega)}{r_{2k}(\omega)}\sin x\omega d\omega.\end{aligned}$$

From the following facts

$$\begin{aligned}&r_{2k}(2\pi-\omega) = r_{2k}(\omega), \quad r_k(2\pi-\omega) = (-1)^k r_k(\omega), \\ &r_{k+1}(2\pi-\omega) = (-1)^{k+1}r_{k+1}(\omega), \quad \sin(2\pi-\omega)x = -\sin\omega x,\end{aligned}$$

we see that $\frac{r_k(\omega)r_{k+1}(\omega)}{r_{2k}(\omega)}\sin\omega x$ is symmetric about π. Therefore we obtained the corollary. □

5. Further discussions and numerical experiments for Gibbs size

Still it remains to determine for which integer x the integral does have maximum value. But it seems that it is not possible to decide it analytically. So we provide an easy algorithm to calculate the integral numerically.

We note

$$\sum_n \frac{1}{\frac{\omega}{2}+n\pi} = \cot\frac{\omega}{2},$$

$$r_2(w) = \sum_n \frac{1}{(\frac{\omega}{2}+n\pi)^2} = \csc^2\frac{\omega}{2}.$$

In fact, by differentiating $r_k(\omega)$, we obtain

$$r_k'(\omega) = -\frac{k}{2}r_{k+1}(\omega), \quad i.e., r_{k+1} = -\frac{2}{k}r_k'(\omega).$$

For $k = 2, x = 1$, $r_2 = \csc^2\frac{\omega}{2}$, $r_3(\omega) = \csc^3\frac{\omega}{2}\cos\frac{\omega}{2}$, $r_4(\omega) = \frac{1-\frac{2}{3}\sin^2\frac{\omega}{2}}{\sin^4\frac{\omega}{2}}$. So we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^\pi \frac{r_2(\omega)r_3(\omega)}{r_4(\omega)}\sin\omega d\omega &= \frac{1}{\pi}\int_0^\pi \frac{\cos^2\frac{\omega}{2}}{1-\frac{2}{3}\sin^2\frac{\omega}{2}} \\ &= \frac{3}{2\pi}\int_0^\pi \frac{1+\cos\omega}{2+\cos\omega}d\omega \\ &= \frac{3}{2} - \frac{\sqrt{3}}{2}.\end{aligned}$$

By adding $\frac{1}{2}$, Gibbs size equals $2 - \frac{\sqrt{3}}{2}$. For other $k's$ and $x's$, we provide a numerical result by Mathematica.

[Approximation of Gibbs size for Lemarie-Barttle family]

k	x=1	x=2	x=3	x=4	x=5	x=10
2	1.133975	0.964102	1.009619	0.997423	1.000691	0.99999
3	1.07115	0.975576	1.010249	0.995599	1.001895	0.99972
4	1.090053	0.955394	1.023602	0.987399	1.006741	0.99704
5	1.082607	0.962574	1.020242	0.9882	1.007072	0.99417
6	1.088664	0.953879	1.02842	0.98165	1.012024	0.998489
7	1.086197	0.957282	1.025574	0.983412	1.011199	0.998158
8	1.088765	0.953077	1.030266	0.979002	1.015011	0.996885
9	1.015011	0.989125	1.028359	0.980586	1.013896	0.996735
10	1.088924	0.952617	1.031158	0.97768	1.016649	0.995429
11	1.088302	0.953648	1.029905	0.978886	1.015633	0.995478
12	1.089048	0.952325	1.031668	0.976933	1.017605	0.99427
13	1.088673	0.952997	1.030825	0.977819	1.016778	0.994452
14	1.089138	0.952128	1.031993	0.976468	1.018207	0.99339
15	1.088895	0.952575	1.031408	0.977119	1.017556	0.99364

ACKNOWLEDGEMENT. The author sincerely thanks the anonymous referee for his valuable remarks and corrections on the first version of this paper.

References

[1] C. K. Chui, *An introduction to Wavelets*, Academic Press, New York, 1992.

[2] J. Foster and F. B. Richard, *A Gibbs phenomenon for piecewise linear approximation*, The Am. Math. Month (1991), 47-49.

[3] J. W. Gibbs, *Letter to the editor*, Nature **59** (1899), 606.

[4] S. Kelly, *Gibbs' phenomenon for wavelets*, App. Comp. Harmon. Anal **3** (1996), 72.

[5] F. B. Richard, *A Gibbs phenomenon for spline functions*, J. Approx. Theory **66** (1996), 334–351.

[6] H. -T. Shim and H. Volkmer, *On the Gibbs phenomenon for wavelet expansions*, J. Approx. Theory **84** (1995), 75–95.

[7] G. G. Walter, *Wavelets and Other Orthogonal Systems with Applications*, CRC Press, Boca Raton, FL, 1994.

[8] A. H. Zemanian, *Distribution Theory And Transform Analysis*, Dover Publications, INC., New York, 1965.

DEPARTMENT OF MATHEMATICS, SUN MOON UNIVERSITY, ASAN, CHOONGNAM, 337-840, KOREA

E-mail: hongtae@omega.sunmoon.ac.kr