

PROJECTIVE SYSTEMS SUPPORTED ON THE COMPLEMENT OF TWO LINEAR SUBSPACES

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ABSTRACT. We discuss the class of projective systems whose supports are the complement of the union of two linear subspaces in general position. We express the weight enumerators of the codes generated by these projective systems using two simplex codes corresponding to given linear subspaces. We also prove these codes are uniquely determined upto equivalence by their weight enumerators.

1. Introduction and preliminaries

Let \mathbb{F}_q be the finite field of order q and \mathbb{P}^m the m -dimensional projective space over \mathbb{F}_q . We denote the dual space of \mathbb{P}^m by \mathbb{P}^{m*} which is the set of all hyperplanes in \mathbb{P}^m . A code $C \subset \mathbb{F}_q^n$ is said to be *degenerate*, if there is a position i ($1 \leq i \leq n$) such that $c_i = 0$ for any codeword $\mathbf{c} = (c_1, \dots, c_n) \in C$; otherwise C is said to be *nondegenerate*. Throughout this paper, a code means a nondegenerate code.

Let C be a nondegenerate $[n, k]_q$ -code with a generator matrix

$$G = (\mathbf{a}_1, \dots, \mathbf{a}_n),$$

where \mathbf{a}_i is a column vector of length k . Since C is nondegenerate, we can define a *positive 0-cycle* $\sum_{i=1}^n [\mathbf{a}_i]$ on the projective space \mathbb{P}^{k-1} , where $[\mathbf{a}_i]$

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is the point of the projective space represented by \mathbf{a}_i . Note that the support of the 0-cycle $\sum_{i=1}^n [\mathbf{a}_i]$ spans the whole space \mathbb{P}^{k-1} because the rank of G is k . If one chooses another generator matrix $G' = (\mathbf{a}'_1, \dots, \mathbf{a}'_n)$ of C , then the 0-cycle $\sum_{i=1}^n [\mathbf{a}'_i]$ is projectively equivalent to the previous one over \mathbb{F}_q . Therefore, for a given nondegenerate $[n, k]_q$ -code C , we can define a projective equivalence class of positive 0-cycles on \mathbb{P}^{k-1} whose supports span the whole space, and denote by \mathcal{X}_C one of the 0-cycles. We call \mathcal{X}_C a *projective system* associated to C after [4]. It is obvious that a code C is equivalent to C' if and only if \mathcal{X}_C is projectively equivalent to $\mathcal{X}_{C'}$. Moreover, there is one to one correspondence between the equivalence classes of nondegenerate $[n, k]_q$ -codes and the projective equivalence classes of positive 0-cycles of length n in \mathbb{P}^{k-1} whose supports span the whole space ([1], [2]).

We use the notation

$$N(m) := \frac{q^{m+1} - 1}{q - 1} = q^m + q^{m-1} + \dots + q^2 + q + 1 \text{ for } m \geq 1,$$

which is the number of all points on a projective space \mathbb{P}^m over \mathbb{F}_q . By convention, we let $N(0) = 1$, $N(-1) = 0$. Linear subspaces L_i , $i = 1, 2, \dots, r$ ($r \geq 2$) in \mathbb{P}^m are said to be *in general position* if, for any subset $\{i_1, \dots, i_s\} \subset \{1, \dots, r\}$, the dimension of the linear span $\langle \cup_{j=1}^s L_{i_j} \rangle$ of $\cup_{j=1}^s L_{i_j}$ is equal to $\min\{m, \sum_{j=1}^s \dim L_{i_j} + s - 1\}$.

The code corresponding to the 0-cycle $\sum_{P \in \mathbb{P}^{k-1}} P$ is called a *simplex code*, which is an $[N(k - 1), k, q^{k-1}]_q$ -code, and its weight enumerator is $1 + (q^k - 1)s^{q^{k-1}}$.

Let $C_i \subset \mathbb{F}_q^{n_i}$ be an $[n_i, k_i]_q$ -code for $i = 1, 2$. Then we define their direct sum by $C_1 \oplus C_2 = \{(\mathbf{c}_1, \mathbf{c}_2) \mid \mathbf{c}_1 \in C_1, \mathbf{c}_2 \in C_2\}$. Then $C_1 \oplus C_2$ is an $[n_1 + n_2, k_1 + k_2]_q$ -code and its weight enumerator is $W_{C_1 \oplus C_2}(s) = W_{C_1}(s) \cdot W_{C_2}(s)$.

We use the following well-known facts.

LEMMA 1.1([3], [4]).(Griesmer bound) *For any $[n, k, d]_q$ -code, we have*

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil,$$

where $\lceil x \rceil$ means the smallest integer greater than or equal to x .

Projective systems supported on the complement of two linear subspaces

LEMMA 1.2([3], [4]).(MacWilliams identity) Let C be an $[n, k]_q$ -code and C^\perp the dual code of C . Then

$$W_{C^\perp}(s) = \frac{1}{q^k} (1 + (q-1)s)^n \cdot W_C\left(\frac{1-s}{1+(q-1)s}\right).$$

LEMMA 1.3([3]). If H is a parity check matrix of a code of length n , then the minimum distance of the code is d if and only if every $d-1$ columns of H are linearly independent and some d columns are linearly dependent.

2. Weight enumerators

We compute the weight enumerators of codes corresponding to the projective systems supported on linear subspaces or their complements. For this we use the following lemmas.

LEMMA 2.1. Let S be a subset of \mathbb{P}^{k-1} , and assume that the linear spans of S and its complement S^c are the whole space \mathbb{P}^{k-1} . Let

$$\mathcal{X}_1 = \sum_{P \in S} P, \quad \mathcal{X}_2 = \sum_{P \in S^c} P,$$

and let C_i be the corresponding code to \mathcal{X}_i for $i = 1, 2$. Then we have

$$W_{C_2}(s) = 1 + (W_{C_1}\left(\frac{1}{s}\right) - 1) \cdot s^{q^{k-1}}.$$

Proof. Let n_i be the length of the code C_i for $i = 1, 2$, i.e., $n_1 = \# S$ and $n_2 = \# S^c$. Since $\#(H \cap S) + \#(H \cap S^c) = \# H = N(k-2)$ for any hyperplane H in \mathbb{P}^{k-1} , we have

$$\begin{aligned} n_2 - \#(H \cap S^c) &= \#S^c - \#(H \cap S^c) \\ &= (N(k-1) - \#S) - (N(k-2) - \#(H \cap S)) \\ &= q^{k-1} - (\#S - \#(H \cap S)) \\ &= q^{k-1} - (n_1 - \#(H \cap S)). \end{aligned}$$

Hence the sum of weights of codewords in C_1 and C_2 , corresponding to the same hyperplane H , is q^{k-1} , which implies the desired formula. \square

In the following lemmas, we deal with the case when the linear span of S is not a whole space.

LEMMA 2.2. *Let S be a subset of \mathbb{P}^{k-1} whose linear span is not the whole space \mathbb{P}^{k-1} . Then the linear span of S^c is the whole space.*

Proof. Choose arbitrary points P and Q in S and $\langle S \rangle^c$, respectively. Then the line PQ intersects $\langle S \rangle$ at only one point P , otherwise Q should be contained in $\langle S \rangle$. Thus the other q points on the line PQ is contained in $\langle S \rangle^c$, and hence in S^c . Thus the line PQ is contained in $\langle S^c \rangle$, and hence $P \in \langle S^c \rangle$. \square

LEMMA 2.3. *Let S be a subset of \mathbb{P}^{k-1} with $\dim \langle S \rangle = l - 1 < k - 1$. Let*

$$\mathcal{X}_1 = \sum_{P \in S} P, \quad \mathcal{X}_2 = \sum_{P \in S^c} P,$$

and let C_i be the corresponding nondegenerate code to \mathcal{X}_i for $i = 1, 2$. Then we have

$$W_{C_2}(s) = 1 + (q^{k-l} W_{C_1}(\frac{1}{s}) - 1) s^{q^{k-1}}.$$

Proof. Note that C_1 is an l -dimensional code. Let n_i be the length of the code C_i for $i = 1, 2$, i.e., $n_1 = \# S$ and $n_2 = \# S^c$. We may assume $\langle S \rangle = \mathbb{P}^{l-1} \subset \mathbb{P}^{k-1}$. Let H be any hyperplane in \mathbb{P}^{k-1} .

If $\langle S \rangle \subset H$, then

$$n_2 - \#(H \cap S^c) = (N(k-1) - \# S) - (N(k-2) - \# S) = q^{k-1}.$$

If $\langle S \rangle \not\subset H$, then

$$\begin{aligned} n_2 - \#(H \cap S^c) &= (N(k-1) - \# S) - (N(k-2) - \#(H \cap S)) \\ &= q^{k-1} - (\# S - \#(H_{\langle S \rangle} \cap S)) \\ &= q^{k-1} - (n_1 - \#(H_{\langle S \rangle} \cap S)), \end{aligned}$$

where $H_{\langle S \rangle} = H \cap \langle S \rangle$ is the hyperplane in $\langle S \rangle = \mathbb{P}^{l-1}$.

Projective systems supported on the complement of two linear subspaces

Since the number of hyperplanes in \mathbb{P}^{k-1} containing $\langle S \rangle$ is $N(k-1-l)$, we have

$$\begin{aligned} W_{C_2}(s) &= 1 + (N(k-l) - N(k-1-l))(W_{C_1}\left(\frac{1}{s}\right) - 1) \cdot s^{q^{k-1}} \\ &\quad + (q-1)N(k-1-l)s^{q^{k-1}} \\ &= 1 + (q^{k-l}W_{C_1}\left(\frac{1}{s}\right) - 1)s^{q^{k-1}}, \end{aligned}$$

as desired. □

REMARK 2.4.

- (1) If $l = k$ and $\dim \langle S^c \rangle = k - 1$, then the formula in Lemma 2.3 is same with that of Lemma 2.1.
- (2) If $l = k$ and $\dim \langle S^c \rangle < k - 1$, we may change the role of S and S^c and use Lemma 2.3.

Now we compute the weight enumerators of codes corresponding to 0-cycle whose support is the complement of linear subspaces in general position in \mathbb{P}^{k-1} .

THEOREM 2.5. *Let L_i , $i = 1, 2, \dots, r$, be linear subspaces of dimension s_i in \mathbb{P}^{k-1} , where s_i 's satisfy $0 \leq s_1 \leq s_2 \leq \dots \leq s_r < k - 1$ and $\sum_{i=1}^r s_i + r \leq k$. Assume that L_i 's are in general position. Let $T = \mathbb{P}^{k-1} \setminus \cup_{i=1}^r L_i$, $\mathcal{X} = \sum_{P \in T} P$ be a 0-cycle, and let C be the code corresponding to \mathcal{X} . Then*

$$W_C(s) = 1 + (q^{k-l}f\left(\frac{1}{s}\right) - 1)s^{q^{k-1}},$$

where $l - 1 = \dim(\cup_{i=1}^r L_i) = s_1 + s_2 + \dots + s_r + r - 1$ and

$$f(s) = (1 + (q^{s_1+1} - 1)s^{q^{s_1}}) \dots (1 + (q^{s_r+1} - 1)s^{q^{s_r}}).$$

Proof. Since the nondegenerate code corresponding to the linear subspace L_i is a simplex code, its weight enumerator is $1 + (q^{s_i+1} - 1)s^{q^{s_i}}$. Hence the weight enumerator of the code corresponding to the union of L_i 's is just $f(s)$, since it is a direct sum of the codes corresponding to L_i , $i = 1, 2, \dots, r$. Now Lemma 2.3 completes the proof. □

We prove a particular property of the codes in Theorem 2.5.

THEOREM 2.6. *If $r < q$, then the code in Theorem 2.5 achieves the Griesmer bound in Lemma 1.1.*

Proof. The length of the code is $n = N(k-1) - \sum_{i=1}^r N(s_i)$. For the minimum distance d , we have

$$\begin{aligned} d &= n - \max\{\#(H \cap T) \mid H \text{ is a hyperplane in } \mathbb{P}^{k-1}\} \\ &= n - (\#H - \min\{\#(H \cap (\cup_{i=1}^r L_i)) \mid H \text{ is a hyperplane in } \mathbb{P}^{k-1}\}) \\ &\geq \left(N(k-1) - \sum_{i=1}^r N(s_i)\right) - \left(N(k-2) - \sum_{i=1}^r N(s_i-1)\right) \\ &= q^{k-1} - \sum_{i=1}^r q^{s_i}, \end{aligned}$$

and equality holds if and only if there exists a hyperplane containing no L_i , $i = 1, 2, \dots, r$. To prove the existence of such a hyperplane, we compute

$$\begin{aligned} &\#(\text{all hyperplanes}) - \#(\text{hyperplanes } \supset L_i \text{ for some } i = 1, 2, \dots, r) \\ &\geq N(k-1) - \sum_{i=1}^r N(k-1-s_i-1) \\ &\geq N(k-1) - rN(k-2) \\ &\geq N(k-1) - qN(k-2) = 1. \end{aligned}$$

Thus $d = q^{k-1} - \sum_{i=1}^r q^{s_i}$.

Since $r < q$ by assumption, we have

$$\sum_{s_i < j} q^{s_i-j} \leq \sum_{s_i < j} \frac{1}{q} \leq \frac{r}{q} < 1,$$

for all $j = 1, 2, \dots, k-1$, and hence

$$\begin{aligned} \left\lceil \frac{d}{q^j} \right\rceil &= \left\lceil q^{k-1-j} - \sum_{s_i \geq j} q^{s_i-j} - \sum_{s_i < j} q^{s_i-j} \right\rceil \\ &= q^{k-1-j} - \sum_{s_i \geq j} q^{s_i-j}. \end{aligned}$$

Hence

(the right hand side of Griesmer bound)

$$\begin{aligned}
 &= (q^{k-1} - \sum_{i=1}^r q^{s_i}) + [q^{k-2} - \sum_{i=1}^r q^{s_i-1}] + \dots + [1 - \sum_{i=1}^r q^{s_i-(k-1)}] \\
 &= (q^{k-1} - \sum_{i=1}^r q^{s_i}) + (q^{k-2} - \sum_{s_i \geq 1} q^{s_i-1}) + (q^{k-3} - \sum_{s_i \geq 2} q^{s_i-2}) \\
 &\quad + \dots + (q - \sum_{s_i \geq k-2} q^{s_i-(k-2)}) + 1 \\
 &= N(k-1) - \sum_{i=1}^r N(s_i) = n.
 \end{aligned}$$

Thus the proof is complete. \square

The following example shows that the condition $r < q$ is necessary in Theorem 2.6.

EXAMPLE 2.1. In the projective space \mathbb{P}^3 over \mathbb{F}_2 , let S be the union of two skew lines. Let

$$\mathcal{X} = \sum_{P \in \mathbb{P}^3 \setminus S} P,$$

and C be the code corresponding to 0-cycle \mathcal{X} . Then we can easily prove that C is $[9, 4, 4]_2$ -code but it does not satisfy the equality in Griesmer bound.

3. Uniqueness for $r = 2$

In this section we prove that any two codes with the same weight enumerator appeared in Theorem 2.5 are equivalent to each other.

THEOREM 3.1. *Let S be a subset of \mathbb{P}^n with $\#S = N(s) + N(t)$, $s \geq t$, and $s + t = n - 1$. Suppose that for any hyperplane H in \mathbb{P}^n ,*

$$\#(H \cap S) = N(s) + N(t-1), \quad N(s-1) + N(t), \quad \text{or} \quad N(s-1) + N(t-1),$$

and

$$\begin{cases} \#\{H \in \mathbb{P}^{n*} \mid \#(H \cap S) = N(s) + N(t - 1)\} = N(t), \\ \#\{H \in \mathbb{P}^{n*} \mid \#(H \cap S) = N(s - 1) + N(t)\} = N(s). \end{cases}$$

Then S is the union of two linear subspaces in general position of dimension s and t , respectively.

Proof. We divide the proof into five claims.

Claim 1. If a line contains at least 3 points of S , then it is contained in S .

Proof of Claim 1. Let

$$\begin{cases} P, \text{ a point not contained in } S \\ M, \text{ a line through } P \\ \check{P}, \text{ the set of all hyperplanes containing } P \\ \check{M}, \text{ the set of all hyperplanes containing } M. \end{cases}$$

Then $\#(\check{P} \setminus \check{M}) = N(n - 1) - N(n - 2) = q^{n-1}$. Let $\#(M \cap S) = u$. By assumption, $\#(H \cap S) \geq N(s - 1) + N(t - 1)$ for all hyperplane H , so we have

$$\begin{aligned} \sum_{H \in \check{P} \setminus \check{M}} \#(H \cap S) &\geq \#(\check{P} \setminus \check{M}) \cdot (N(s - 1) + N(t - 1)) \\ &= q^{n-1} \cdot (N(s - 1) + N(t - 1)). \end{aligned}$$

On the other hand, for any point $Q \in S \setminus M$,

$$\begin{aligned} \#(\text{all hyperplanes in } \check{P} \setminus \check{M} \text{ containing } Q) &= \#((\check{P} \cap \check{Q}) \setminus (\check{M} \cap \check{Q})) \\ &= N(n - 2) - N(n - 3) \\ &= q^{n-2}. \end{aligned}$$

Hence we have

$$\begin{aligned} \sum_{H \in \check{P} \setminus \check{M}} \#(H \cap S) &= \#(S \setminus M) \cdot \#((\check{P} \cap \check{Q}) \setminus (\check{M} \cap \check{Q})) \\ &= (N(s) + N(t) - u) \cdot q^{n-2}. \end{aligned}$$

Comparing above two formulas, we have

$$(N(s) + N(t) - u) \cdot q^{n-2} \geq q^{n-1} \cdot (N(s - 1) + N(t - 1)),$$

which is equivalent to $u \leq 2$. Thus Claim 1 is proved.

Projective systems supported on the complement of two linear subspaces

Claim 2. Let H_0 be a hyperplane such that $\#(H_0 \cap S) = N(s-1) + N(t)$. Then every line containing two points in $S \setminus H_0$ meets H_0 at a point in S .

Proof of Claim 2. Suppose not. Then there exists a line M such that

$$\begin{cases} \#(M \cap (S \setminus H_0)) \geq 2 \\ M \cap H_0 \notin S. \end{cases}$$

Let $P = M \cap H_0$. By Claim 1, $\#(M \cap S) = 2$. In the computation in the proof of Claim 1, we have equality, which means $\#(H \cap S) = N(s-1) + N(t-1)$ for any hyperplanes $H \in \check{P} \setminus \check{M}$. This contradicts the choice of H_0 .

Claim 3. S contains a linear subspace of dimension s . More precisely, S contains the linear span $L_0 = \langle H_0^c \cap S \rangle$ of $H_0^c \cap S$ and $\dim L_0 = s$.

Proof of Claim 3. By Claim 1 and 2, we conclude that L_0 is contained in S . Thus $H_0^c \cap L_0 = H_0^c \cap S$. Let $\dim L_0 = r$. Then

$$\begin{aligned} N(r) = \#(L_0) &= \#(H_0^c \cap L_0) + \#(H_0 \cap L_0) \\ &= \#(H_0^c \cap S) + \#(H_0 \cap L_0) \\ &= (N(s) - N(s-1)) + N(r-1), \end{aligned}$$

since $H_0^c \cap L_0 = H_0^c \cap S$ and L_0 is not contained in H_0 .

Thus we have $r = s$ and we have proved Claim 3.

Claim 4. Let $S' = S \setminus L_0$. Then $\#(S') = N(t)$ and

$$\#(H \cap S') = N(t) \text{ or } N(t-1)$$

for any hyperplane H in \mathbb{P}^n . Moreover, there exist H_1 and H_2 such that

$$\begin{cases} \#(H_1 \cap S') = N(t) \\ \#(H_2 \cap S') = N(t-1). \end{cases}$$

Proof of Claim 4. For any hyperplane H , note that

$$\begin{cases} \#(H \cap S) = \#(H \cap L_0) + \#(H \cap S') \\ \#(H \cap L_0) = N(s) \text{ or } N(s-1), \end{cases}$$

and by assumption of the theorem

$$\#(H \cap S) = N(s) + N(t-1), N(s-1) + N(t), \text{ or } N(s-1) + N(t-1).$$

If $s > t$, then $N(s)$ is bigger than $N(s-1) + N(t)$ and $N(s-1) + N(t-1)$. Hence, for a hyperplane H with $\#(H \cap L_0) = N(s)$, we have $\#(H \cap S) = N(s) + N(t-1)$. Indeed,

$$\begin{aligned} \#\{H \in \mathbb{P}^{n*} \mid \#(H \cap L_0) = N(s)\} &= \#\{H \in \mathbb{P}^{n*} \mid L_0 \subset H\} \\ &= N(n-s-1) = N(t), \end{aligned}$$

which is equal to

$$\#\{H \in \mathbb{P}^{n*} \mid \#(H \cap S) = N(s) + N(t-1)\}.$$

Thus $\#(H \cap S') = N(t)$ or $N(t-1)$ for any $H \in \mathbb{P}^{n*}$ and

$$\begin{cases} \#\{H \in \mathbb{P}^{n*} \mid \#(H \cap S') = N(t)\} = N(s) \\ \#\{H \in \mathbb{P}^{n*} \mid \#(H \cap S') = N(t-1)\} = N(n) - N(s). \end{cases}$$

If $s = t$, then $N(s) > N(s-1) + N(t-1)$ and $N(s) + N(t-1) = N(s-1) + N(t)$. Counting the number of hyperplanes again, we have

$$\begin{cases} \#\{H \in \mathbb{P}^{n*} \mid \#(H \cap S') = N(t)\} = N(s) \\ \#\{H \in \mathbb{P}^{n*} \mid \#(H \cap S') = N(t-1)\} = N(n) - N(s). \end{cases}$$

Thus we have proved the Claim 4.

Claim 5. S' is a t -dimensional linear subspace of \mathbb{P}^n .

Proof of Claim 5. We modify the proof of Claim 1 to show that any line through two points in S' is contained in S' .

Let

$$\begin{cases} P, \text{ a point not contained in } S' \\ M, \text{ a line through } P \\ \check{P}, \text{ the set of all hyperplanes containing } P \\ \check{M}, \text{ the set of all hyperplanes containing } M. \end{cases}$$

Then $\#(\check{P} \setminus \check{M}) = N(n-1) - N(n-2) = q^{n-1}$. Let $\#(M \cap S') = u$. By assumption, $\#(H \cap S') \geq N(t-1)$ for all hyperplane H , so we have

$$\begin{aligned} \sum_{H \in \check{P} \setminus \check{M}} \#(H \cap S') &\geq \#(\check{P} \setminus \check{M}) \cdot N(t-1) \\ &= q^{n-1} \cdot N(t-1). \end{aligned}$$

Projective systems supported on the complement of two linear subspaces

On the other hand, for any point $Q \in S' \setminus M$,

$$\begin{aligned} \#(\text{all hyperplanes in } \check{P} \setminus \check{M} \text{ containing } Q) &= \#((\check{P} \cap \check{Q}) \setminus (\check{M} \cap \check{Q})) \\ &= N(n-2) - N(n-3) \\ &= q^{n-2}. \end{aligned}$$

Hence we have

$$\begin{aligned} \sum_{H \in \check{P} \setminus \check{M}} \#(H \cap S') &= \#(S' \setminus M) \cdot \#((\check{P} \cap \check{Q}) \setminus (\check{M} \cap \check{Q})) \\ &= (N(t) - u) \cdot q^{n-2}. \end{aligned}$$

Comparing above two formulas, we have

$$(N(t) - u) \cdot q^{n-2} \geq q^{n-1} \cdot N(t-1)$$

which is equivalent to $u \leq 1$. That means that any line not contained in S' has at most one point of S' , or equivalently, any line containing two points in S' is contained in S' . Hence S' is a linear subspace, and $\dim S' = t$ since $\#(S') = N(t)$.

Combining above 5 claims, we conclude that S is the union of two linear subspaces of dimension s and t , and they are in general position since there is no hyperplane containing both of them and $s+t = n-1$. \square

Finally we prove a uniqueness theorem using the following lemma.

LEMMA 3.2. *Let C and C' be nondegenerate codes with the same weight enumerator. If every coefficient in the 0-cycle \mathcal{X}_C is ≤ 1 , i.e., $\mathcal{X}_C = \sum_{P \in \text{Supp}(\mathcal{X}_C)} P$, then $\mathcal{X}_{C'} = \sum_{P \in \text{Supp}(\mathcal{X}_{C'})} P$. In this case, we can identify 0-cycle with its support.*

Proof. The assumption $\mathcal{X}_C = \sum_{P \in \text{Supp}(\mathcal{X}_C)} P$ means that any two columns of a generator matrix of C are linearly independent, which is equivalent to the minimum distance of C^\perp being ≥ 3 , by Lemma 1.3. Since the weight enumerators of C^\perp and C'^\perp are same by MacWilliams identity, we conclude that the minimum distance of C'^\perp is ≥ 3 , and hence the lemma is proved. \square

THEOREM 3.3. *If a code C' over \mathbb{F}_q has the same weight enumerator with the code in Theorem 2.5 with $r = 2$, then C' is equivalent to C .*

Proof. By Lemma 3.2, we have $\mathcal{X}_{C'} = \sum_{P \in \text{Supp}(\mathcal{X}_{C'})} P$, and we identify the 0-cycle $\mathcal{X}_{C'}$ with its support. Let $T = \text{Supp}(\mathcal{X}_{C'})$ and let $S = T^c$ in \mathbb{P}^{k-1} . Then, by Lemma 2.3, we can check that S satisfies the conditions in Theorem 3.1 where $n = s + t + 1$. Hence S is the union of two linear subspaces in general position in \mathbb{P}^n . Therefore, we can find a projective automorphism on \mathbb{P}^{k-1} which maps S onto $L_1 \cup L_2$ in Theorem 2.5. Thus the code C' is equivalent to C . \square

The following example shows that a code appeared in Theorem 2.5 is not determined by its parameters $[n, k, d]$ alone.

EXAMPLE 3.1. In the projective space \mathbb{P}^3 over \mathbb{F}_2 , let S, \mathcal{X}, C be same as Example 2.1. Let S_1 be a punctured plane. Let

$$\mathcal{X}_1 = \sum_{P \in \mathbb{P}^3 \setminus S_1} P,$$

and let C_1 be a code corresponding to 0-cycle \mathcal{X}_1 . Then we can prove easily that C_1 is $[9, 4, 4]_2$ -code, same as C . However their weight enumerators are

$$\begin{cases} W_C(s) = 1 + 9s^4 + 6s^6 \\ W_{C_1}(s) = 1 + 6s^4 + 8s^5 + s^8. \end{cases}$$

Note that the weight enumerator $W_C(s)$ is the polynomial appeared in Theorem 2.5.

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Projective systems supported on the complement of two linear subspaces

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