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# ON THE STABILITY OF 3-DIMENSIONAL QUADRATIC FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we investigate the Hyers-Ulam-Rassias stability of a quadratic functional equation $f(x+y+z)+f(x-y)+f(y-z)+f(z-x) = 3f(x)+3f(y)+3f(z)$ and prove the Hyers-Ulam stability of the equation on restricted (unbounded) domains.

## 1. Introduction

The quadratic function $f(x) = x^2$ $(x \in \mathbb{R})$ satisfies the functional equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y). \tag{1.1}$$

Hence, the above equation is called the quadratic functional equation, and every solution of the equation (1.1) is called a quadratic function. It is well known that a function $f : E_1 \to E_2$ between vector spaces is quadratic if and only if there exists a unique symmetric function $B : E_1 \times E_1 \to E_2$, which is additive in $x$ for each fixed $y$, such that $f(x) = B(x,x)$ for any $x \in E_1$ (see [1]). A Hyers-Ulam stability theorem for the functional equation (1.1) was proved by F. Skof for functions $f : E_1 \to E_2$ where $E_1$ is a normed space and $E_2$ a Banach space (see [8]). In [3], S. Czerwik proved the Hyers-Ulam-Rassias stability of the functional equation (1.1), and this result was generalized by J. M. Rassias, and C. Borelli and G. L. Forti (see [7], [2]).

Consider the following functional equations:

$$f(x+y+z)+f(x)+f(y)+f(z) = f(x+y)+f(y+z)+f(z+x) \tag{1.2}$$

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and

$$(1.3)\quad f(x+y+z)+f(x-y)+f(y-z)+f(z-x) = 3f(x)+3f(y)+3f(z).$$

The functional equation (1.2) was solved by Pl. Kannappan [6]. Recently, S.-M. Jung [5] investigated the Hyers-Ulam-Rassias stability of the equation (1.2) on restricted (unbounded) domains and applied the result to the study of an asymptotic behavior of the quadratic functions.

It is the main purpose of the present note to show that the above Kannappan and Jung's result is true for an equation (1.3). In particular, S.-M. Jung proved the Hyers-Ulam stability of the functional equation (1.2) under some additional conditions. In this paper, we shall prove the Hyers-Ulam-Rassias stability of the equation (1.3) without an additional condition that $f$ is approximately even.

By $\mathbb{N}$ and $\mathbb{R}$, we denote the set of positive integers and of real numbers, respectively.

## 2. Solutions of equation (1.3)

THEOREM 2.1. *If vector spaces $E_1$ and $E_2$ are common domain and range of the function $f$ in both the functional equations* (1.1) *and* (1.3), *then the functional equation* (1.3) *is equivalent to the functional equation* (1.1).

*Proof.* If we replace $x, y, z$ in (1.3) by 0, then we have $f(0) = 0$. By putting $y = z = 0$ in the equation (1.3), we see that every solution of equation (1.3) is even. Putting $z = 0$ in (1.3) and using the evenness of $f$ and $f(0) = 0$, we can transform the equation (1.3) into the equation (1.1).

Conversely, suppose that a function $f : E_1 \to E_2$ satisfies (1.1) for all $x, y, z \in E_1$. From (1.1), we get $f(x + y + z) + f(x) = 2f(x + \frac{y+z}{2}) + 2f(\frac{y+z}{2})$ and $f(y) + f(z) = 2f(\frac{y+z}{2}) + 2f(\frac{y-z}{2})$. According to (1.1) and the last two equalities, and using the fact $f(2x) = 4f(x)$, we obtain

$$\begin{aligned} f(x+y+z) &+ f(x-y) + f(y-z) + f(z-x) \\ &= f(x+y+z) + f(x) + f(y) + f(z) \\ &\qquad + f(x-y) + f(y-z) + f(z-x) - f(x) - f(y) - f(z) \\ &= 2f\left(x + \frac{y+z}{2}\right) + f(y+z) + 2f\left(\frac{y-z}{2}\right) \\ &\qquad + f(x-y) + f(y-z) + f(z-x) - f(x) - f(y) - f(z) \\ &= f(x+y) + f(y+z) + f(z+x) \\ &\qquad + f(x-y) + f(y-z) + f(z-x) - f(x) - f(y) - f(z) \\ &= 3f(x) + 3f(y) + 3f(z). \end{aligned}$$

This means the equivalence of the equations (1.1) and (1.3). □

## 3. Stability problems of equation (1.3)

In this section, let $E_1$ and $E_2$ be a normed space and a Banach space, respectively. Assume further that $\varepsilon > 0$ and $p \neq 2$ are given. From now on, we will use the following abbreviation

$$\begin{aligned} Df(x,y,z) \;:=\; & f(x+y+z) + f(x-y) + f(y-z) + f(z-x) \\ & - 3f(x) - 3f(y) - 3f(z). \end{aligned}$$

We can prove the Hyers-Ulam stability of the equation (1.3) as we shall see in the following theorem.

THEOREM 3.1. *Suppose that the function $f : E_1 \to E_2$ satisfies the following inequality*

$$\|Df(x,y,z)\| \;\leq\; \varepsilon \tag{3.1}$$

*for all $x, y, z \in E_1$, then there exists a unique quadratic function $g : E_1 \to E_2$ such that the inequality $\|f(x) - g(x)\| \;\leq\; \frac{9}{5}\varepsilon$ holds for all $x \in E_1$. If, moreover, $f$ is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$*

*for each fixed $x \in E_1$, then the function $g$ satisfies $g(tx) = t^2 g(x)$ for all $x \in E_1$ and $t \in \mathbb{R}$.*

*Proof.* Putting $x = y = z = 0$ in (3.1) yields

$$\|f(0)\| \le \frac{\varepsilon}{5}. \tag{3.2}$$

Replacing $y$ and $z$ in (3.1) by 0 we get

$$\|f(x) - f(-x)\| \le 2\varepsilon \tag{3.3}$$

for any $x \in E_1$. If we put $z = 0$ in (3.1), we have

$$\|f(x+y) + f(x-y) + f(-x) - 3f(x) - 2f(y) - 3f(0)\| \le \varepsilon \tag{3.4}$$

for all $x, y \in E_1$. It then follows from (3.4), (3.3) and (3.2) that the inequality

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\qquad \le \|f(x+y) + f(x-y) + f(-x) - 3f(x) - 2f(y) - 3f(0)\| \\ &\qquad\quad + \|f(x) - f(-x)\| + \|3f(0)\| \\ &\qquad \le \varepsilon + 2\varepsilon + \frac{3}{5}\varepsilon = \frac{18}{5}\varepsilon \end{aligned}$$

holds for all $x, y \in E_1$. Therefore, in view of the result of F. Skof [8], the assertion of our theorem is obvious. The remainder of the proof can be proved in the same way as S. Czerwik [3]. □

We shall prove the Hyers-Ulam-Rassias stability of the equation (1.3).

THEOREM 3.2. *Suppose that the function $f : E_1 \to E_2$ satisfies the inequality*

$$\|Df(x,y,z)\| \le \varepsilon \, (\|x\|^p + \|y\|^p + \|z\|^p). \tag{3.5}$$

*Then there exists a unique quadratic function $g : E_1 \to E_2$ such that*

$$\|g(x) - f(x)\| \le \begin{cases} \frac{4\varepsilon}{2^p - 4}\|x\|^p, & p > 2 \\ \frac{1}{2}\|f(0)\| + \frac{26+3\cdot 2^p}{4-2^p}\|x\|^p, & p < 2. \end{cases}$$

*If, moreover, $f$ is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then the quadratic function $g$ satisfies $g(tx) = t^2 g(x)$ for all $x \in E_1$ and $t \in \mathbb{R}$.*

*Proof.* In the case $p > 2$, putting $x = y = z = 0$ in (3.5) yields $f(0) = 0$. Letting $y = z = 0$ in (3.5) again we get

$$\|f(x) - f(-x)\| \leq \varepsilon \|x\|^p \tag{3.6}$$

for all $x \in E_1$. Furthermore, by letting $z = 0$ in (3.5) we obtain

$$\|f(x+y)+f(x-y)+f(-x)-3f(x)-2f(y)\| \leq \varepsilon (\|x\|^p+\|y\|^p) \tag{3.7}$$

for all $x, y \in E_1$. It then follows from (3.6), (3.7) that the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq 2\varepsilon (\|x\|^p + \|y\|^p)$$

holds for all $x, y \in E_1$. According to S. Czerwik [3](or see [2]), the last inequality implies that our theorem holds for $p > 2$.

In the case $p < 2$, putting $y = -x$ in (3.5), and replacing $z$ in (3.5) by $x$ and $-x$, separately, we have

$$\| -5f(x) + f(2x) + f(-2x) + f(0) - 3f(-x)\| \leq 3\varepsilon \|x\|^p \tag{3.8}$$

and

$$\| -5f(-x) + f(2x) + f(0) + f(-2x) - 3f(x)\| \leq 3\varepsilon \|x\|^p \tag{3.9}$$

for all $x \in E_1 \setminus \{0\}$. By (3.8) and (3.9) we obtain

$$\|f(x) - f(-x)\| \leq 3\varepsilon \|x\|^p \tag{3.10}$$

for all $x \in E_1 \setminus \{0\}$. If we set $z = -x$ in (3.5) then we obtain

$$\begin{aligned} \| -2f(y) + f(x-y) + f(x+y) + f(-2x) \\ - 3f(x) - 3f(-x)\| \leq \varepsilon (2\|x\|^p + \|y\|^p) \end{aligned} \tag{3.11}$$

for all $x, y \in E_1 \setminus \{0\}$. From (3.9) and (3.11), we get

(3.12)

$$\begin{aligned}&\|2f(y) - f(x-y) - f(x+y) - 2f(-2x) + 6f(x) \\ &\qquad + 8f(-x) - f(2x) - f(0)\| \ \leq \ \varepsilon\ (5\|x\|^p + \|y\|^p)\end{aligned}$$

for all $x, y \in E_1 \setminus \{0\}$. Using (3.10), (3.11) and (3.12),we arrive at the inequality

$$\begin{aligned}&\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\leq \| - 2f(y) + f(x-y) + f(x+y) + f(-2x) - 3f(x) - 3f(-x)\| \\ &\quad + \|f(x) + 3f(-x) - f(-2x)\| \\ &\leq \varepsilon(2\|x\|^p + \| y\|^p) + \|f(x) - f(-x)\| + \|4f(-x) - f(-2x)\| \\ &\leq \varepsilon(2\|x\|^p + \| y\|^p) + 3\varepsilon\|x\|^p + \frac{1}{2}\Big( \|2f(y) - f(x-y) - f(x+y) \\ &\quad - 2f(-2x) - f(2x) + 8f(-x) + 6f(x) - f(0)\| + \| - 2f(y) \\ &\quad + f(x-y) + f(x+y) + f(2x) - 6f(x)\| + \|f(0)\| \Big) \\ &\leq \varepsilon(5\|x\|^p + \|y\|^p) + \frac{1}{2}\Big( \varepsilon(5\|x\|^p + \|y\|^p) + \| - 2f(y) + f(x-y) \\ &\quad + f(x+y) + f(-2x) - 3f(x) - 3f(-x)\| + \|f(2x) - f(-2x)\| \\ &\quad + \|3f(-x) - 3f(x)\| + \|f(0)\| \Big) \\ &\leq \varepsilon(5\|x\|^p + \|y\|^p) + \frac{1}{2}\Big( \varepsilon(5\|x\|^p + \|y\|^p) + \varepsilon(2\|x\|^p + \|y\|^p) \\ &\quad + 3 \cdot 2^p \varepsilon\|x\|^p + 9\varepsilon\|x\|^p + \|f(0)\| \Big) \\ &= \varepsilon(5\|x\|^p + \|y\|^p) + \frac{1}{2}\varepsilon\Big( (16 + 3 \cdot 2^p)\|x\|^p + 2\|y\|^p \Big) + \frac{\|f(0)\|}{2} \\ &\leq \frac{26 + 3 \cdot 2^p}{2}\varepsilon(\|x\|^p + \|y\|^p) + \frac{\|f(0)\|}{2}\end{aligned}$$

for all $x, y \in E_1 \setminus \{0\}$. According to [3], this fact means the validity of the assertion in our theorem. The remainder of proof can be proved in the same way as S. Czerwik did in the paper [3]. We complete the proof of the theorem. □

The Hyers-Ulam-Rassias stability problem for the case of $p = 2$ was excluded in Theorem 3.2. In fact, the functional equation (1.3) is not stable for the case $p = 2$ as we shall see in the following remark.

REMARK. A slight modification of an example of Czerwik [3] shows that there exists a function $f : \mathbb{R} \to \mathbb{R}$ which satisfies the inequality (3.5) with $p = 2$, for all $x, y, z \in \mathbb{R}$, but for which there is no quadratic function $g : \mathbb{R} \to \mathbb{R}$ such that $| f(x) - g(x) |$ is bounded on $\mathbb{R} \setminus \{0\}$. (cf. [5]).

Indeed, let us define $\varphi(x) = \begin{cases} (1/132)\,\varepsilon & \text{for } | x | \geq 1, \\ (1/132)\,\varepsilon x^2 & \text{for } | x | < 1, \end{cases}$ and put $f(x) = \sum_{n=0}^{\infty} 9^{-n}\varphi(3^n x)$ for all $x \in \mathbb{R}$. By the similar way as in [4], we can easily verify that $f$ serves as a counterexample for the Hyers-Ulam-Rassias stability of the functional inequality (3.5) for the case $p = 2$.

By using the idea from the papers [5], we will prove the Hyers-Ulam stability of the equation (1.3) on restricted domains.

THEOREM 3.3. *Let $d > 0$ and $\varepsilon > 0$ be given. If a function $f : E_1 \to E_2$ satisfies the inequality (3.1) for all $x, y, z \in E_1$ with $\|x\| + \|y\| + \|z\| \geq d$, then there exists a unique quadratic function $g : E_1 \to E_2$ such that*

$$\|f(x) - g(x)\| \leq 27\varepsilon \tag{3.13}$$

*for all $x \in E_1$. If, moreover, $f$ is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then the function $g$ satisfies $g(tx) = t^2 g(x)$ for all $x \in E_1$ and $t \in \mathbb{R}$.*

*Proof.* Suppose that $\|x\| + \|y\| + \|z\| < d$. Choose a $t \in E_1$ with $\|t\| \geq 2d$. Then, it holds

$$\begin{aligned} \|x\| + \|y - t\| + \|z + t\| &\geq d; & \|x - y\| + \|t\| + \|t\| &\geq d; \\ \|x\| + \|z\| + \|t\| &\geq d; & \|z + t\| + \|x\| + \|x\| &\geq d; \\ \|x\| + \|t\| + \|t\| &\geq d; & \|y\| + \|z\| + \|t\| &\geq d; \end{aligned} \tag{3.14}$$

From (3.1), (3.14) and the following relation

$$
\begin{aligned}
&\|2f(x+y+z)+2f(x-y)+2f(y-z)+2f(z-x)\\
&\quad -6f(x)-6f(y)-6f(z)-10f(0)\|\\
&\le \|f(x+y+z)+f(x-y+t)+f(y-z-2t)+f(-x+z+t)\\
&\quad -3f(x)-3f(y-t)-3f(z+t)\|\\
&+\|f(x+y+z)+f(-x+y+t)+f(x-z-2t)+f(-y+z+t)\\
&\quad -3f(y)-3f(x-t)-3f(z+t)\|\\
&+\|-f(x-y)-f(x-y+t)-f(-2t)-f(-x+y+t)\\
&\quad +3f(x-y)+3f(-t)+3f(t)\|\\
&+\|-f(y-z)-f(y-z+t)-f(-2t)-f(-y+z+t)\\
&\quad +3f(y-z)+3f(-t)+3f(t)\|\\
&+\|-f(z-x)-f(-x+z+t)-f(-2t)-f(x-z+t)\\
&\quad +3f(-x+z)+3f(-t)+3f(t)\|\\
&+\|-f(y+z)-f(-y+t)-f(y-z-2t)-f(z+t)\\
&\quad +3f(0)+3f(y-t)+3f(z+t)\|\\
&+\|-f(x+z)-f(-x+t)-f(x-z-2t)-f(z+t)\\
&\quad +3f(0)+3f(x-t)+3f(z+t)\|\\
&+\|f(x-z+t)+f(x+z)+f(-z-t)+f(-x+t)\\
&\quad -3f(x)-3f(-z)-3f(t)\|\\
&+\|f(y-z+t)+f(y+z)+f(-z-t)+f(-y+t)\\
&\quad -3f(y)-3f(-z)-3f(t)\|\\
&+\|-f(z+t)-f(z+t)-f(0)-f(-z-t)\\
&\quad +3f(z+t)+3f(0)+3f(0)\|\\
&+\|-f(z+t)-f(-z-t)-f(z+t)-f(0)\\
&\quad +3f(0)+3f(z+t)+3f(0)\|\\
&+\|f(0)+f(-2t)+f(t)+f(t)-3f(-t)-3f(t)-3f(0)\|\\
&+\|2f(-2t)+2f(0)+2f(-t)+2f(t)-6f(-t)-6f(-t)-6f(0)\|\\
&+\|3f(z+t)+3f(z)+3f(-t)+3f(-z+t)-9f(z)-9f(0)-9f(t)\|\\
&+\|-3f(-z+t)-3f(z+t)-3f(-z)-3f(-t)\\
&\quad +9f(t)+9f(-z)+9f(0)\|\\
&+\|4f(t)+4f(t)+4f(0)+4f(-t)-12f(t)-12f(0)-12f(0)\|\\
&\le 24\varepsilon
\end{aligned}
$$

We obtain

(3.15)

$$\begin{aligned}\|Df(x,y,z)\| \leq \frac{1}{2}\ &\|2f(x+y+z)+2f(x-y)+2f(y-z)+2f(z-x)\\ &-6f(x)-6f(y)-6f(z)-10f(0)\| + \|5f(0)\|\\ \leq\ &\frac{1}{2}\cdot 24\varepsilon + 3\varepsilon \ = \ 15\varepsilon,\end{aligned}$$

since we get $\|f(0)\| \ \leq \ \frac{3}{5}\varepsilon$ by putting $y = z = 0$ in (3.1).

It is clear that the inequality (3.15) holds for all $x, y, z \in E_1$. Therefore, the assertions of our theorem are immediate consequence of Theorem 3.1. □

Define $S = \{(x,y,z) \in E^3 : \| x \| < d, \| y \| < d, \| z \| < d \}$ for some given $d > 0$. The fact $\{(x,y,z) \in E^3 : \| x \| + \| y \| + \| z \| \geq 3d \} \subset E^3 \setminus S$ implies that the following corollaries is a consequence of Theorem 3.3.

COROLLARY 3.4. *If a function $f : E_1 \to E_2$ satisfies the inequality* (3.1) *for all $(x,y,z) \in E^3 \setminus S$, then there exists a unique quadratic function $g : E_1 \to E_2$ which satisfies the inequality* (3.13) *for all $x \in E_1$. Moreover, if $f$ is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then the function $g$ satisfies $g(tx) = t^2 g(x)$ for all $x \in E_1$ and $t \in \mathbb{R}$.*

COROLLARY 3.5. *A function $f : E_1 \to E_2$ is a quadratic function if and only if $\|Df(x,y,z)\| \to 0$ as $\|x\| + \|y\| + \|z\| \to \infty$.*

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