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JORDAN DERIVATIONS IN NONCOMMUTATIVE BANACH ALGEBRAS

ICK-SOON CHANG

ABSTRACT. Our main goal is to show that if there exist Jordan derivations D, E and G on a noncommutative 2-torsion free prime ring R such that $(G^2(x)+E(x))D(x) = 0$ or $D(x)(G^2(x)+E(x)) = 0$ for all $x \in R$, then we have $D = 0$ or $E = 0$, $G = 0$.

1. Introduction

In this paper, R will represent an associative ring with center $C(R)$, and A will represent an algebra over a complex field $\mathbb{C}$. The Jacobson radical of A will be denoted by $rad(A)$. We write $[x,y]$ for $xy - yx$, and use the identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. Let I be any closed (2-sided) ideal of a Banach algebra A. Then we will let Q_I denote the canonical quotient map from A onto A/I. Recall that R is prime if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$. An additive mapping D from R to R is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And also, an additive mapping D from R to R is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$.

Singer and Wermer [6] proved that every continuous derivation on a commutative Banach algebra maps the algebra into its radical. They also made a very insightful conjecture, namely that the assumption of continuity was unnecessary. This became known as the Singer-Wermer conjecture and was proved in 1988 by Thomas [7]. The so-called non-commutative Singer-Wermer conjecture was proved that every derivation D on a Banach algebra A such that $[D(x), x] \in rad(A)$ for all

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$x \in A$ maps the algebra into its radical. Now it seems natural to ask, under additional assumptions, the range of product of continuous linear Jordan derivations on a noncommutative Banach algebra is contained in the radical. It is the purpose of this paper to show that if $(G^2(x) + E(x))D(x) \in rad(A)$ or $D(x)(G^2(x) + E(x)) \in rad(A)$ for all $x \in A$, then $D(A) \subseteq rad(A)$, or $E(A) \subseteq rad(A)$ and $G(A) \subseteq rad(A)$, where D, E and G are continuous linear Jordan derivations on a Banach algebra A.

2. The Results

To prove our main theorem, we shall need the following purely algebraic result.

LEMMA 2.1. *Let R be a 2-torsion free semiprime ring. If $D : R \to R$ is a Jordan derivation, then D is a derivation.*

Proof. See [1]. □

LEMMA 2.2. *Let R be a noncommutative 2-torsion free semiprime ring. Suppose that there exist Jordan derivations $E, G : R \to R$ such that $G^2(x) + E(x) = 0$ for all $x \in R$. Then we have $E = 0$, $G = 0$.*

Proof. By Lemma 2.1, E, G are derivations on R. Suppose now that

$$G^2(x) + E(x) = 0, \; x \in R. \tag{1}$$

Substituting xy for x in (1), we obtain

$$G^2(x)y + 2G(x)G(y) + xG^2(y) + E(x)y + xE(y) = 0, \; x, y \in R. \tag{2}$$

Then from (1) and (2), we get

$$2G(x)G(y) = 0, \; x, y \in R. \tag{3}$$

Since R is 2-torsion free, it follows (3) that

$$G(x)G(y) = 0, \; x, y \in R. \tag{4}$$

Replacing yx for y in (4), we have

$$G(x)G(y)x + G(x)yG(x) = 0, \ x, y \in R. \tag{5}$$

Combining (4) with (5), clearly,

$$G(x)yG(x) = 0, \ x, y \in R. \tag{6}$$

By semiprimeness of R, (6) gives

$$G(x) = 0, \ x \in R. \tag{7}$$

From (1) and (7), we have $E = 0$. Consequently, we obtain $E = 0, G = 0$. □

THEOREM 2.3. *Let R be a noncommutative 2-torsion prime ring. Suppose that there exist* Jordan *derivations D, E and G such that $(G^2(x) + E(x))D(x) = 0$ or $D(x)(G^2(x) + E(x)) = 0$ for all $x \in R$, then we have $D = 0$ or $E = 0, \ G = 0$.*

Proof. Without loss of generality, it suffices to prove the case that $(G^2(x) + E(x))D(x) = 0$ for all $x \in R$. By Lemma 2.1, D, E and G are derivations. Suppose that

$$(G^2(x) + E(x))D(x) = 0, \ x \in R \tag{8}$$

The linearization of (8) leads to

$$(G^2(y) + E(y))D(x) + (G^2(x) + E(x))D(y) = 0, \ x, y \in R. \tag{9}$$

Replacing $yD(x)$ for y in (9), we have

$$\begin{aligned} &\{G^2(y)D(x) + 2G(y)G(D(x)) + yG^2(D(x)) + E(y)D(x) \\ &+ yE(D(x))\}D(x) + (G^2(x) + E(x))D(y)D(x) + (G^2(x) \\ &+ E(x))yD^2(x) = 0, \ x, y \in R. \end{aligned} \tag{10}$$

Right multiplication of (9) by $D(x)$ leads to

$$(G^2(y) + E(y))D(x)^2 + (G^2(x) + E(x))D(y)D(x) = 0, \ x, y \in R. \tag{11}$$

From (10) and (11), we obtain

$$\begin{aligned}(12)\qquad &2G(y)G(D(x))D(x) + y(G^2(D(x)) + E(D(x)))D(x)\\ &+ (G^2(x) + E(x))yD^2(x) = 0,\ x, y \in R.\end{aligned}$$

Substituting $D(x)y$ for y in (12), we get

$$\begin{aligned}(13)\qquad &2D(x)G(y)G(D(x))D(x) + 2G(D(x))yG(D(x))D(x)\\ &+ D(x)y(G^2(D(x))D(x) + E(D(x))D(x)) + (G^2(x)\\ &+ E(x))D(x)yD^2(x) = 0,\ x, y \in R.\end{aligned}$$

Comparing (8) and (6), it is clear that

$$\begin{aligned}(14)\qquad &2D(x)G(y)G(D(x))D(x) + 2G(D(x))yG(D(x))D(x)\\ &+ D(x)y(G^2(D(x)) + E(D(x)))D(x) = 0,\ x, y \in R.\end{aligned}$$

Putting $D(x)y$ instead of y in (14), it follows that

$$\begin{aligned}(15)\qquad &2D(x)^2G(y)G(D(x))D(x) + 2D(x)G(D(x))yG(D(x))D(x)\\ &+ 2G(D(x))D(x)yG(D(x))D(x) + D(x)^2y(G^2(D(x))+\\ &E(D(x)))D(x) = 0,\ x, y \in R.\end{aligned}$$

Left multiplication of (14) by $D(x)$ gives

$$\begin{aligned}(16)\qquad &2D(x)^2G(y)G(D(x))D(x) + 2D(x)G(D(x))yG(D(x))D(x)\\ &+ D(x)^2y(G^2(D(x)) + E(D(x)))D(x) = 0,\ x, y \in R.\end{aligned}$$

From (15) and (16), we obtain

$$(17)\qquad 2G(D(x))D(x)yG(D(x))D(x) = 0,\ x, y \in R.$$

Since R is 2-torsion free, we get

$$(18)\qquad G(D(x))D(x)yG(D(x))D(x) = 0,\ x, y \in R.$$

But since R is prime, we have from (18)

$$G(D(x))D(x) = 0,\ x \in R. \tag{19}$$

From (14) and (19), we arrive at

$$D(x)y(G^2(D(x)) + E(D(x)))D(x) = 0,\ x, y \in R. \tag{20}$$

Left multiplication of (20) by $G^2(D(x)) + E(D(x))$ gives

$$\begin{aligned} &\Big(G^2(D(x)) + E(D(x))\Big)D(x)y\Big(G^2(D(x)) + E(D(x))\Big)D(x) \\ &= 0,\ x, y \in R. \end{aligned} \tag{21}$$

By primeness of R, it follows from (21) that

$$(G^2(D(x)) + E(D(x)))D(x) = 0,\ x \in R. \tag{22}$$

Thus from (12), (19) and (22), we obtain

$$(G^2(x) + E(x))yD^2(x) = 0,\ x, y \in R. \tag{23}$$

Writing $x + z$ instead of x in (23), we have

$$(G^2(x) + E(x))yD^2(z) + (G^2(z) + E(z))yD^2(x) = 0,\ x, y \in R. \tag{24}$$

Replacing $yD^2(z)u(G^2(x) + E(x))y$ instead of y in (24),

$$\begin{aligned} &(G^2(x) + E(x))yD^2(z)u(G^2(x) + E(x))yD^2(z) \\ &+ (G^2(z) + E(z))yD^2(z)u(G^2(x) + E(x))yD^2(x) \\ &= 0,\ x, y, z, u \in R. \end{aligned} \tag{25}$$

From (23) and (25),

$$(G^2(x)+E(x))yD^2(z)u(G^2(x)+E(x))yD^2(z) = 0,\ x, y, z, u \in R. \tag{26}$$

Since R is prime, (26) gives

$$(G^2(x) + E(x))yD^2(z) = 0,\ x, y, z \in R. \tag{27}$$

But also, by primeness of R, it is obvious from (27) that

$$G^2(x) + E(x) = 0, \ x \in R \tag{28}$$

or

$$D^2(z) = 0, \ z \in R. \tag{29}$$

Hence if (28) holds, then by Lemma 2.2, we get $E = 0, G = 0$. Thus suppose that (29) holds. Then we consider the case that $E = 0$ in Lemma 2.2. By Lemma 2.2, $D = 0$. Therefore we have $D = 0$ or $E = 0, G = 0$. □

THEOREM 2.4. *Let D and E, G be continuous linear* Jordan *derivations on a noncommutative* Banach *algebra A such that $(G^2(x) + E(x))D(x) \in rad(A)$ or $D(x)(G^2(x) + E(x)) \in rad(A)$ for all $x \in A$. Then $D(A) \subseteq rad(A)$, or $E(A) \subseteq rad(A)$ and $G(A) \subseteq rad(A)$.*

Proof. Let J be a primitive ideal of A. Since D, E and G are continuous, by [5, Theorem 2.2], we have $D(J) \subseteq J, E(J) \subseteq J$ and $G(J) \subseteq J$. Then we can define derivations D_J, E_J and G_J on A/J by

$$D_J(x + J) = D(x) + J, \ E_J(x + J) = E(x) + J, \ G_J(x + J) = G(x) + J$$

for all $x \in A$. The factor algebra A/J is prime and semisimple, since J is a primitive ideal. By Lemma 2.1 it is obvious that D_J, E_J and G_J are derivations on a prime Banach algebra A/J. Johnson and Sincliar [3] have proved that every derivation on a semisimple Banach algebra is continuous. Combining this result with Singer-Wermer theorem, we obtain that there are no nonzero derivations on a commutative Banach algebra. Hence in case A/J is commutative, we have $D_J = 0, E_J = 0$ and $G_J = 0$. It remains to show that $D_J = 0$ or $E_J = 0$ and $G_J = 0$ in the case when A/J is noncommutative. Note that the intersection of all primitive ideals is the radical. The assumption of the theorem

$$(G^2(x) + E(x))D(x) \in rad(A) \ (x \in A)$$

gives

$$(G_J^2(x + J) + E_J(x + J))D_J(x + J) = J \ (x \in A).$$

All the assumptions of Theorem 2.3 are fulfilled. Thus we have $D_J = 0$ or $E_J = 0$ and $G_J = 0$. Hence we see that $D(A) \subseteq J$, or $E(A) \subseteq J$ and $G(A) \subseteq J$, since J is a primitive ideal. Therefore since J was arbitrary, $D(A) \subseteq rad(A)$, or $E(A) \subseteq rad(A)$ and $G(A) \subseteq rad(A)$. □

THEOREM 2.5. *Let D and E, G be linear Jordan derivations on a noncommutative semisimple Banach algebra A such that $(G^2(x) + E(x))D(x) = 0$ or $D(x)(G^2(x) + E(x)) = 0$ for all $x \in A$. Then $D(A) = 0$, or $E(A) = 0$ and $G(A) = 0$.*

Proof. The arguments used in Theorem 2.4 carry over almost verbatim. □

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DEPARTMENT OF MATHEMATICS, CHUNGNAM NATIONAL UNIVERSITY, TAEJON 305-764, KOREA

E-mail: ischang@math.cnu.ac.kr