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MAYER-VIETORIS SEQUENCE AND TORSION THEORY

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ABSTRACT. This work presents a new construction of Mayer-Vietoris sequence using techniques from torsion theory and including the classical case as an example.

1. Introduction

Any reader with a basic grounding in local cohomology will recall the important role that the Mayer-Vietoris sequence can play in that subject. There is an analogue of the Mayer-Vietoris sequence in torsion theory. It is our intention in this paper to present the basic theory of the Mayer-Vietoris sequence in torsion theory. In the following R will always be a commutative Noetherian ring and we denote by $\mathfrak{C}(R)$ the category of modules over R. We will begin with a quick introduction to torsion theories, looking for the tools we will use in our construction. In section 3 we construct the Mayer-Vietoris sequence relative to a pair of torsion functors σ and τ.

2. Preliminaries

A *torsion theory* $(\mathcal{T}, \mathcal{F})$ in $\mathfrak{C}(R)$ is a pair of non-empty classes of R-modules satisfying:

(1) $\mathrm{Hom}_R(M, N) = 0$ for all $M \in \mathcal{T}$ and each $N \in \mathcal{F}$;
(2) if $\mathrm{Hom}_R(X, N) = 0$ for all $N \in \mathcal{F}$, then $X \in \mathcal{T}$;
(3) if $\mathrm{Hom}_R(M, Y) = 0$ for all $M \in \mathcal{T}$, then $Y \in \mathcal{F}$;
(4) $\mathcal{T}$ is closed under submodules.

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The R-modules in $\mathcal{T}$ are called *torsion* and those in $\mathcal{F}$ are called *torsion free*. From the definition, it follows that $\mathcal{T}$ is closed under quotients, direct sums, extensions and submodules and $\mathcal{F}$ is closed under submodules, direct products, extensions and essential extensions.

It is possible to define a torsion theory from a single non-empty class $\mathcal{T}$ of R-modules, closed under quotients, direct sums, extensions and submodules (such a class is called *torsion class*), by setting

$$\mathcal{F} = \{N \in \mathfrak{C}(R) : \mathrm{Hom}(M, N) = 0 \text{ for all } M \in \mathcal{T}\}.$$

The pair $(\mathcal{T}, \mathcal{F})$ is then a torsion theory. In a similar way, we can obtain a torsion theory from a non-empty *torsion free class*, i.e., a class closed under submodules, direct products, extension and essential extension.

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory, then we can consider for each R-module M the submodule

$$\sigma(M) = \sum\{N \subseteq M : N \in \mathcal{T}\}.$$

It is easy to prove that $\sigma(-)$ defines a subfunctor of the identity functor in $\mathfrak{C}(R)$ satisfying the following properties : for any R-module M and any submodule N of M we have $\sigma(N) = N \cap \sigma(M)$ and $\sigma(M/\sigma(M)) = 0$. Such a functor is called *torsion functor.* On the other hand, given a torsion functor τ, the classes

$$\mathcal{T}_\tau = \{M \in \mathfrak{C}(R) : \tau(M) = M\}$$

and

$$\mathcal{F}_\tau = \{M \in \mathfrak{C}(R) : \tau(M) = 0\}$$

define a torsion theory. With this development it is possible to prove that there exists a bijective correspondence between torsion theories and torsion functors.

Moreover there is a one-to-one correspondence between torsion theories and idempotent filters [4, (0.4)]. We note that Gabriel's definition of idempotent filter [4, Page 7] becomes shorter when one is working in commutative algebra. We call the set of ideals Δ an idempotent filter over R if it satisfies the following conditions :

(1) $R \in \Delta$;
(2) if $I \in \Delta$ and $I \subseteq J$, then $J \in \Delta$;
(3) if $I \in \Delta$ and J is an ideal of R such that $(J : a) \in \Delta$ for all $a \in I$, then $I \cap J \in \Delta$.

To any torsion theory $(\mathcal{T}, \mathcal{F})$ we associated the set

$$\Delta = \{I \text{an ideal of } R : R/I \in \mathcal{T}\}.$$

Then Δ is an idempotent filter. Conversely, given any idempotent filter Δ, we call the R-module M torsion if $(0:_R m) \in \Delta$ for all $m \in M$. In this way we obtain a one-to-one correspondence between torsion theories and idempotent filters. It is easy to see that, if σ is a torsion functor and Δ is the corresponding idempotent filter, then, when M is an R-module,

$$\sigma(M) = \{x \in M : Ix = 0 \text{ for some } I \in \Delta\}.$$

There is a one-to-one correspondence between torsion theories in $\mathfrak{C}(R)$ and partitions of Spec(R) into two sets, one of them closed under specialization (see [1, Lemma (1.1)]). By saying that $T \subseteq \mathrm{Spec}(R)$ is closed under specialization, we main that, whenever $\mathfrak{p},\mathfrak{q} \in \mathrm{Spec}(R)$ and $\mathfrak{p} \subseteq \mathfrak{q}$ then $\mathfrak{p} \in T$ implies that $\mathfrak{q} \in T$. If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathfrak{C}(R)$, then the associated partition (T, F) of Spec(R) is given by

$$T = \{\mathfrak{p} \in \mathrm{Spec}(R) : R/\mathfrak{p} \in \mathcal{T}\}$$

and $F = \mathrm{Spec}(R) - T$. The prime ideals in T are called *torsion-prime* and the prime ideals in F are called *free-prime*. To every partition (T, F) of Spec(R) in which T is closed under specialization, we assign the torsion theory in which an R-module M is torsion if and only if $\mathrm{Ass}_R(M) \subseteq T$.

Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in $\mathfrak{C}(R)$ and σ its corresponding torsion functor. It can easily be shown that σ is an additive, covariant, R-linear and left exact functor. Hence the right derived functors of σ may be formed. For each integer $i \geq 0$, we denote the i-th right derived functor of σ by H^i_σ.

3. Mayer-Vietoris sequence

Throughout the paper, R will always denote a commutative Noetherian ring with identity. The Mayer-Vietoris sequence in torsion theory involves two torsion functors, and so, throughout this paper, we fix our notations as following.

Let $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{T}_0, \mathcal{F}_0)$ be two torsion theories in $\mathfrak{C}(R)$. Suppose that σ, τ and (T, F), (U, V), where T and U are closed under specialization, denote respectively torsion functors and partitions of Spec(R) corresponding to $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{T}_0, \mathcal{F}_0)$.

Set $T' = T \cap U$ and $T'' = T \cup U$. It is clear that T' and T'' are closed under specialization. Let $(\mathcal{T}', \mathcal{F}')$ and σ' be torsion theory and torsion functor corresponding to partition (T', F') of Spec(R). Also, we denote torsion theory and torsion functor corresponding to partition (T'', F'') of $\mathrm{Spec}(R)$ by $(\mathcal{T}'', \mathcal{F}'')$ and σ''.

LEMMA 3.1. *Let M be an R-module. Then $\sigma(M)$ and $\tau(M)$ are submodules of $\sigma''(M)$ and $\sigma'(M)$ is a submodule of $\sigma(M) \cap \tau(M)$.*

Proof. Let M be an R-module and $x \in \sigma(M)$. As we mentioned earlier, there exists an ideal I in $L(\sigma) = \{I$ an ideal of $R : R/I \in \mathcal{T}\}$ such that $Ix = 0$. Since $R/I \in \mathcal{T}$ thus $\mathrm{Ass}_R(R/I) \subseteq T$ (see [3, Proposition 1.4]). Now, it is clear that $\mathrm{Ass}_R(R/I) \subseteq T \cup U = T''$. Hence $R/I \in \mathcal{T}''$ and $x \in \sigma''(M)$. As above we can show that $\tau(M)$ is a submodule of $\sigma''(M)$. In order to prove the second part, let $x \in \sigma'(M)$. Therefore, there exists $J \in L(\sigma')$ such that $Jx = 0$. Since $L(\sigma') \subseteq L(\sigma) \cap L(\tau)$ thus $J \in L(\sigma) \cap L(\tau)$ and so $x \in \sigma(M) \cap \tau(M)$. $\square$

In order to prove Propositions 3.3 and 3.4, we need to following remark.

REMARK 3.2. Let $0 \longrightarrow M \xrightarrow{\alpha} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \dots \longrightarrow E^i \xrightarrow{d^i} \dots$ be a minimal injective resolution for R-module M. Then

$$0 \longrightarrow \sigma(E^0)/\sigma'(E^0) \xrightarrow{f^0} \sigma(E^1)/\sigma'(E^1) \xrightarrow{f^1} \dots \longrightarrow \sigma(E^i)/\sigma'(E^i) \xrightarrow{f^i} \dots$$

and

$$0 \longrightarrow \sigma''(E^0)/\tau(E^0) \xrightarrow{g^0} \sigma''(E^1)/\tau(E^1) \xrightarrow{g^1} \dots \longrightarrow \sigma''(E^i)/\tau(E^i) \xrightarrow{g^i} \dots$$

are two complexes of R-modules and R-homomorphisms, where for all $x \in \sigma(E^i)$ and $y \in \sigma''(E^i)$

$$\begin{aligned} &f^i : \sigma(E^i)/\sigma'(E^i) \longrightarrow \sigma(E^{i+1})/\sigma'(E^{i+1}) \\ &f^i(x + \sigma'(E^i)) = \sigma(d^i)(x) + \sigma'(E^{i+1}) \\ &g^i : \sigma''(E^i)/\tau(E^i) \longrightarrow \sigma''(E^{i+1})/\tau(E^{i+1}) \\ &g^i(y + \tau(E^i)) = \sigma''(d^i)(y) + \tau(E^{i+1}). \end{aligned}$$

PROPOSITION 3.3. *Let $E^\bullet : 0 \longrightarrow M \xrightarrow{\alpha} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \dots \longrightarrow E^i \xrightarrow{d^i} \dots$ be a minimal injective resolution for R-module M. Then*

$$0 \longrightarrow \sigma'(E^\bullet) \longrightarrow \sigma(E^\bullet) \longrightarrow \sigma(E^\bullet)/\sigma'(E^\bullet) \longrightarrow 0$$

is an exact sequence of complexes which results the following long exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0_{\sigma'}(M) \longrightarrow H^0_{\sigma}(M) \longrightarrow \ker f^0 \longrightarrow H^1_{\sigma'}(M) \longrightarrow \dots \\ &\longrightarrow H^i_{\sigma'}(M) \longrightarrow H^i_{\sigma}(M) \longrightarrow \ker f^i/Imf^{i-1} \longrightarrow H^{i+1}_{\sigma'}(M) \longrightarrow \dots . \end{aligned}$$

Proof. We are concerned with a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma'(E^0) & \xrightarrow{\subseteq} & \sigma(E^0) & \xrightarrow{\text{nat}} & \sigma(E^0)/\sigma'(E^0) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma'(d^0)} & & \downarrow{\scriptstyle \sigma(d^0)} & & \downarrow{\scriptstyle f^0} & & \\
0 & \longrightarrow & \sigma'(E^1) & \xrightarrow{\subseteq} & \sigma(E^1) & \xrightarrow{\text{nat}} & \sigma(E^1)/\sigma'(E^1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma'(E^i) & \xrightarrow{\subseteq} & \sigma(E^i) & \xrightarrow{\text{nat}} & \sigma(E^i)/\sigma'(E^i) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma'(d^i)} & & \downarrow{\scriptstyle \sigma(d^i)} & & \downarrow{\scriptstyle f^i} & & \\
0 & \longrightarrow & \sigma'(E^{i+1}) & \xrightarrow{\subseteq} & \sigma(E^{i+1}) & \xrightarrow{\text{nat}} & \sigma(E^{i+1})/\sigma'(E^{i+1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

in which the rows are exact and the columns are 0-sequences. Thus by [5, Theorem 4.6.5] we get the long exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0_{\sigma'}(M) \longrightarrow H^0_{\sigma}(M) \longrightarrow \ker f^0 \longrightarrow H^1_{\sigma'}(M) \longrightarrow \dots \\ &\longrightarrow H^i_{\sigma'}(M) \longrightarrow H^i_{\sigma}(M) \longrightarrow \ker f^i/Imf^{i-1} \longrightarrow H^{i+1}_{\sigma'}(M) \longrightarrow \dots \ \square \end{aligned}$$

PROPOSITION 3.4. *Let* $E^\bullet : 0 \longrightarrow M \xrightarrow{\alpha} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \dots \longrightarrow E^i \xrightarrow{d^i} \dots$ *be a minimal injective resolution for R-module M. Then*

$$0 \longrightarrow \tau(E^\bullet) \longrightarrow \sigma''(E^\bullet) \longrightarrow \sigma''(E^\bullet)/\tau(E^\bullet) \longrightarrow 0$$

is an exact sequence of complexes which results the following long exact sequence

$$0 \longrightarrow H^0_\tau(M) \longrightarrow H^0_{\sigma''}(M) \longrightarrow \ker g^0 \longrightarrow H^1_\tau(M) \longrightarrow \dots$$
$$\longrightarrow H^i_\tau(M) \longrightarrow H^i_{\sigma''}(M) \longrightarrow \ker g^i/Img^{i-1} \longrightarrow H^{i+1}_\tau(M) \longrightarrow \dots .$$

Proof. The proof of (3.4) is similar to that of Proposition 3.3. □

In order to prove our main result we need the following useful lemma.

LEMMA 3.5. *Let E be an injective R-module. If $x \in \sigma''(E)$ then there exist $y \in \sigma(E)$ and $z \in \tau(E)$ such that $x = y + z$.*

Proof. Let $E = \bigoplus_{\mathfrak{p}\in \mathrm{Spec}(R)} E(R/\mathfrak{p})$ be a decomposition of E as a direct sum of indecomposable modules. As we mentioned the class $\mathcal{F}''$ is closed under passage to injective envelopes and direct sums; hence, if $\mathfrak{p} \in F''$ then $E(R/\mathfrak{p}) \in \mathcal{F}''$. So that $\sigma''(E) = \bigoplus_{\mathfrak{p}\in T\cup U} E(R/\mathfrak{p})$ which is a submodule of $(\bigoplus_{\mathfrak{p}\in T} E(R/\mathfrak{p}))\bigoplus(\bigoplus_{\mathfrak{p}\in U} E(R/\mathfrak{p})) = \sigma(E)\bigoplus\tau(E)$. Now, for each $x \in \sigma''(E)$ there exist $y \in \sigma(E)$ and $z \in \tau(M)$ such that $x = y + z$. □

We now come to the main theorem of this paper.

THEOREM 3.6. *For any R-module M, there is a long exact sequence (called the* Mayer-Vietoris sequence *for M with respect to σ and τ)*

$$0 \longrightarrow H^0_{\sigma'}(M) \longrightarrow H^0_\sigma(M)\bigoplus H^0_\tau(M) \longrightarrow H^0_{\sigma''}(M) \longrightarrow H^1_{\sigma'}(M) \longrightarrow \dots$$
$$\longrightarrow H^i_{\sigma'}(M) \longrightarrow H^i_\sigma(M)\bigoplus H^i_\tau(M) \longrightarrow H^i_{\sigma''}(M) \longrightarrow H^{i+1}_{\sigma'}(M) \longrightarrow \dots .$$

Proof. Let $i \geq 0$. We define

$$h_i : \sigma(E^i)/\sigma'(E^i) \longrightarrow \sigma''(E^i)/\tau(E^i)$$
$$h_i(x + \sigma'(E^i)) = x + \tau(E^i)$$

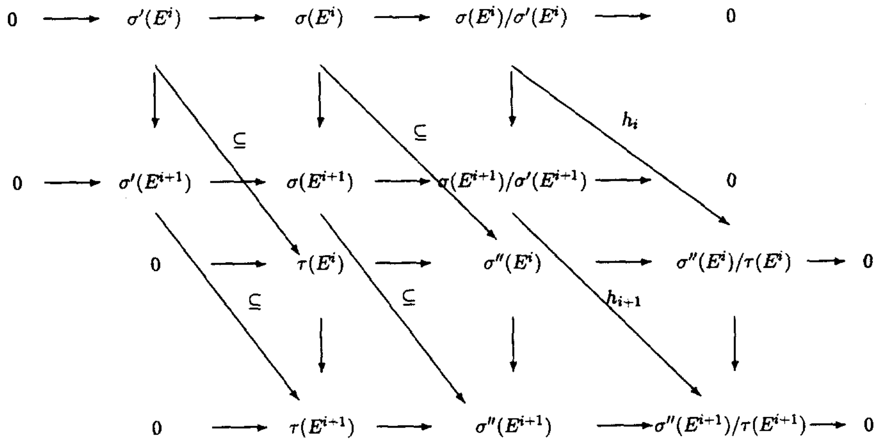
for all $x \in \sigma(E^i)$. We show that h_i is an isomorphism. It is clear that h_i is an R-homomorphism. Let $x \in \sigma''(E^i)$. By Lemma 3.5 there exist $y \in \sigma(E^i)$ and $z \in \tau(E^i)$ such that $x = y + z$ also

$$h_i(y + \sigma'(E^i)) = y + \tau(E^i) = y + z + \tau(E^i) = x + \tau(E^i)$$

thus h_i is an epimorphism. Now, let $x \in \sigma(E^i)$ and

$$h_i(x + \sigma'(E^i)) = x + \tau(E^i) = 0.$$

Thus there are $I \in L(\sigma)$ and $J \in L(\tau)$ such that $Ix = Jx = 0$. Since $R/I \in \mathcal{T}$ and $R/J \in \mathcal{T}_0$ we have $V(I) \subseteq T$ and $V(J) \subseteq U$. Hence $V(I+J) \subseteq U \cap T$ and so $R/I+J \in \mathcal{T}'$. Therefore, from $I+J \in L(\sigma')$ and $(I+J)x = 0$ it follows $x \in \sigma'(E^i)$, which is the required, i.e. h_i is an isomorphism. Furthermore, by using the commutative diagram



we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & H^0_{\sigma'}(M) & \to & H^0_{\sigma}(M) & \to & \ker f^0 & \to & H^1_{\sigma'}(M) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \bar{h}_o & & \downarrow & & \\
0 & \to & H^0_{\tau}(M) & \to & H^0_{\sigma''}(M) & \to & \ker g^0 & \to & H^1_{\tau}(M) & \to & \cdots \\
 & \to & H^i_{\sigma'}(M) & \to & H^i_{\sigma}(M) & \to & \ker f^i/Imf^{i-1} & \to & H^{i+1}_{\sigma'}(M) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \bar{h}_i & & \downarrow & & \\
 & \to & H^i_{\tau}(M) & \to & H^i_{\sigma''}(M) & \to & \ker g^i/Img^{i-1} & \to & H^{i+1}_{\tau}(M) & \to & \cdots
\end{array}$$

with exact rows and $\bar{h}_i$ is an isomorphism, for each $i \geq 0$. Now, the reader will have no difficulty in showing that

$$0 \to H^0_{\sigma'}(M) \to H^0_{\sigma}(M) \bigoplus H^0_{\tau}(M) \to H^0_{\sigma''}(M) \to H^1_{\sigma'}(M) \to \cdots$$
$$\to H^i_{\sigma'}(M) \to H^i_{\sigma}(M) \bigoplus H^i_{\tau}(M) \to H^i_{\sigma''}(M) \to H^{i+1}_{\sigma'}(M) \to \cdots$$

is an exact sequence. □

Theorem 3.6 has some immediate consequences we record here.

COROLLARY 3.7. *Let I and J be ideals of R. Then for any R-module M, there is a long exact sequence (called* Mayer-Vietoris *sequence with* *respect to I and J)*

$$0 \longrightarrow H^0_{I+J}(M) \longrightarrow H^0_I(M) \bigoplus H^0_J(M) \longrightarrow H^0_{I\cap J}(M) \longrightarrow H^1_{I+J}(M) \longrightarrow \dots$$
$$\longrightarrow H^i_{I+J}(M) \longrightarrow H^i_I(M) \bigoplus H^i_J(M) \longrightarrow H^i_{I\cap J}(M) \longrightarrow H^{i+1}_{I+J}(M) \longrightarrow \dots .$$

Proof. Let I be an ideal of R. Then $V(I)$ is closed under specialization; hence, the partition $(V(I), \mathrm{Spec}(R) - V(I))$, determines a torsion theory in $\mathfrak{C}(R)$ which is denote by $(\mathcal{T}_I, \mathcal{F}_I)$. On the other hand Γ_I, the local cohomology functor with respect to I corresponds to the same partition of $\mathrm{Spec}(R)$. Hence Γ_I is the torsion functor corresponding to $(\mathcal{T}_I, \mathcal{F}_I)$ (see [1, Remark (3.3)]). Therefore, if we consider $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{T}_0, \mathcal{F}_0)$ as torsion theories corresponding to partitions $(V(I), \mathrm{Spec}(R) - V(I))$ and $(V(J), \mathrm{Spec}(R) - V(J))$ of $\mathrm{Spec}(R)$, then it is easy to see that $(\mathcal{T}', \mathcal{F}')$ and $(\mathcal{T}'', \mathcal{F}'')$ are corresponding to partitions $(V(I+J), \mathrm{Spec}(R) - V(I+J))$ and $(V(I\cap J), \mathrm{Spec}(R) - V(I\cap J))$ of $\mathrm{Spec}(R)$. Now, the statement follows by Theorem 3.6. □

COROLLARY 3.8. *Let M be an R-module. Then*

$$\sigma'\text{-}\mathrm{depth}_R\, M \geq \max\{\sigma\text{-}\mathrm{depth}_R\, M, \tau\text{-}\mathrm{depth}_R\, M\}$$

and

$$\sigma''\text{-}\mathrm{depth}_R\, M = \min\{\sigma\text{-}\mathrm{depth}_R\, M, \tau\text{-}\mathrm{depth}_R\, M\}.$$

Proof. The first inequality follows by definition of $\sigma'\text{-}\mathrm{depth}_R\, M$ (see [2. Definition 1.1]) and $\sigma'(M)$ is a submodule of $\sigma(M) \cap \tau(M)$. The second part follows by Theorem 3.6 and

$$\sigma''\text{-}\mathrm{depth}_R\, M = \inf\{i \geq 0 : H^i_{\sigma''}(M) \neq 0\}$$

for an R-module M (see [1, Lemma (1.3)]). □

DEFINITION AND REMARK 3.9. The non-empty set Φ of ideals of R is said to be a *system of ideals* if whenever $\mathfrak{a}, \mathfrak{b} \in \Phi$, then there is an ideal $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$.

Let Φ be a system of ideals of R. Certainly the set of all ideals of R is an idempotent filter containing Φ. Let $(\Delta_i)_{i\in I}$ be the family of all idempotent filters each of which contains Φ and put $\Delta = \bigcap_{i\in I} \Delta_i$. Then $\Phi \subseteq \Delta$ and it is clear that Δ is an idempotent filter. We call Δ the idempotent filter generated by Φ. In [1, Proposition (3.11)] it is shown that

$$\Delta = \{I \text{ an ideal of } R : \ J \subseteq I \text{ for some ideal } J \in \Phi\}$$

and $\Gamma_\Phi = \Gamma_\Delta$, where Γ_Φ is the general local cohomology functor and for any R-module M,

$$\Gamma_\Phi(M) = \{m \in M : \ Im = 0 \text{ for some } I \in \Phi\}.$$

Moreover, in [1, Proposition (3.11)] it is shown that Δ is the set of dense ideals corresponding to the torsion theory defined by Γ_Φ. That is

$$\begin{aligned}\Delta &= \{I \text{ an ideal of } R : \ \Gamma_\Phi(R/I) = R/I\} \\ &= \{I \text{ an ideal of } R : \ V(I) \subseteq T\},\end{aligned}$$

where $T = \{\mathfrak{p} \in \operatorname{Spec}(R) : \Gamma_\Phi(R/\mathfrak{p}) = R/\mathfrak{p}\}$.

Suppose that Ψ is an another system of ideals and Δ_0 idempotent filter generated by Ψ. Hence, as above we can show that

$$\Delta_0 = \{J \text{ an ideal of } R : \ V(J) \subseteq U\},$$

where $U = \{\mathfrak{p} \in \operatorname{Spec}(R) : \Gamma_\Psi(R/\mathfrak{p}) = R/\mathfrak{p}\}$. Set

$$\Phi' = \{I \text{an ideal of } R : \ V(I) \subseteq T \cap U\}$$

and

$$\Phi'' = \{I \text{an ideal of } R : \ V(I) \subseteq T \cup U\}.$$

It is easy to see that Φ' and Φ'' are idempotent filters and so systems of ideals.

COROLLARY 3.10. *Let Φ and Ψ be systems of ideals and Δ, Δ_0, Φ' and Φ'' be as* Remark 3.9. *Then for any R-module M, there is a long exact sequence*

$$\begin{aligned}&0 \to H^0_{\Phi'}(M) \to H^0_\Phi(M) \bigoplus H^0_\Psi(M) \to H^0_{\Phi''}(M) \to H^1_{\Phi'}(M) \to \dots \\ &\to H^i_{\Phi'}(M) \to H^i_\Phi(M) \bigoplus H^i_\Psi(M) \to H^i_{\Phi''}(M) \to H^{i+1}_{\Phi'}(M) \to \dots .\end{aligned}$$

Proof. Let Δ be an idempotent filter. It is easy to see that, if σ is a torsion functor corresponding to Δ then, when M is an R-module, $\sigma(M) = \Gamma_\Delta(M)$. Also, by Remark 3.9 $\Gamma_\Delta(M) = \Gamma_\Phi(M)$. Therefore, for all $i \geq 0$ and for all R-module M, $H^i_\sigma(M) = H^i_\Phi(M)$. Now, by Theorem 3.6 the result follows. □

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References

[1] M. H. Bijan-Zadeh, *Torsion theories and local cohomology over commutative Noetherian rings*, J. London Math. Soc. **2**, (19)(1979), 402-410.
[2] M. H. Bijan-Zadeh and Sh. Payrovi, *A generalization of Cohen-Macaulay modules by torsion theory*, Honam Math. J. **20** (1)(1998), 1-14.
[3] P. J. Cahen, *Torsion theory and associated primes*, Proc. Amer. Math. Soc. **38** (1973), 471-476.
[4] J. Lambek, *Torsion theory, additive semantics, and rings of quotients*, (Lecture Nots in Maths. No.177)(Springer-Verlag, Berlin-Heideberg- New-York, 1971).
[5] D. G. Northcott, *An introduction to homological algebra*, 1st (Cambridge University Press, 1962).

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