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RIBBON CATEGORY AND MAPPING CLASS GROUPS

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ABSTRACT. The disjoint union of mapping class groups $\Gamma_{g,1}$ gives us a braided monoidal category so that it gives rise to a double loop space structure. We show that there exists a natural twist in this category, so it gives us a ribbon category. We explicitly express this structure by showing how the twist acts on the fundamental group of the surface $S_{g,1}$. We also make an explicit description of this structure in terms of the standard Dehn twists, as well as in terms of Wajnryb's Dehn twists. We show that the inverse of the twist τ_g for the genus g equals the Dehn twist along the fixed boundary of the surface $S_{g,1}$.

1. Introduction

Let $\Gamma_{g,k}$ be the mapping class group of compact orientation surface $S_{g,k}$ of genus g with k boundary components. It has been known (cf. [**6**]) that there is a binary operation on the disjoint union of $\Gamma_{g,1}$'s induced by a certain connected sum of the surfaces. In [**2**] it is shown that there is an interesting extra structure which is a braiding on the disjoint union of $\Gamma_{g,1}$'s. This gives rise to a braided monoidal category. In the homotopy theoretic point of view, from this we obtain a double loop space. This fact and the explicit expression of the braiding are important informations for the calculations of homology operations of mapping groups.

A braided monoidal category is a monoidal category equipped with a braiding which is a family of commutativity isomorphisms satisfying the Yang-Baxter equality. A ribbon category is a braided monoidal

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category equipped with a twist which is a family of self-isomorphisms compatible with the braiding and the duality. The twist is an algebraic imitation of the *twist* of ribbon graphs. A ribbon category (or a braided monoidal category) has been playing a key role in the theory of quantum groups and of quantum invariants of knots. A ribbon category gives rise to invariants of knots (cf. [7]). The ribbon category obtained in this paper, however, does not give an interesting invariants because the endomorphism set of the unit object is trivial.

In this paper we give a complete and explicit proof that the disjoint union of mapping class groups $\Gamma_{g,1}$ gives rise to a ribbon category. We express the twist in three different ways: as an element of the automorphism group of the free group, geometrically, and in terms of the standard Wajnryb generators. In the proof of Theorem 3.9 we explicitly describe the twist τ_g as an automorphism of the free group on $\{x_1, y_1, \cdots, x_g, y_g\}$, where $x_1, y_1, \cdots, x_g, y_g$ are the generators of $\pi_1 S_{g,1}$ which are induced by the standard Dehn twists of Figure 1. It is important to take the fundamental relator R of $\pi_1 S_{g,1}$ as $[y_1, x_1] \cdots [y_g, x_g]$, rather than $[x_1, y_1] \cdots [x_g, y_g]$ in our setting. Geometrically, the twist τ_g is equal to (the inverse of) the Dehn twist along the boundary of the surface $S_{g,1}$. We can describe τ_g as a product of the standard Dehn twists by using the classical tools by Lickorish ([5]). We can also describe it in terms of $2g+1$ Wajnryb generators. We believe that these descriptions of the Dehn twist along the boundary are of independent interest.

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2. Mapping Class Groups and Braided Monoidal Category

Let S_g be a closed connected orientable surface of genus g. Let $S_{g,1}$ be the surface obtained from S_g by removing an open disk from S_g. The mapping class group $\Gamma_{g,1}$ is the group of isotopy classes of orientation preserving self-diffeomorphisms of $S_{g,1}$ which are the identity maps on the boundary of $S_{g,1}$. The group $\Gamma_{g,1}$ is generated by $3g-1$ standard Dehn twists $a_1, \cdots, a_g, b_1, \cdots, b_g, \omega_1, \cdots, \omega_{g-1}$ of Figure 1 (cf. [1]).

The composition of Dehn twists or homeomorphisms of $S_{g,1}$ will be written from left to right. The mapping class group $\Gamma_{g,1}$ acts on $\pi_1 S_{g,1}$ on the *right*. Wajnryb showed in [8] that $\Gamma_{g,1}$ is generated by $2g+1$ Dehn twists $a_1, a_2, b_1, \cdots, b_g$, $\omega_1, \cdots$, ω_{g-1} and gave a complete form of presentations of them.

We recall the definition of braided monoidal category (cf. [4],[7]). We will deal with strict monoidal category. This does not harm the generality because according to MacLane's coherence theorem, any (braided) monoidal category is equivalent to a certain strict (braided) monoidal category.

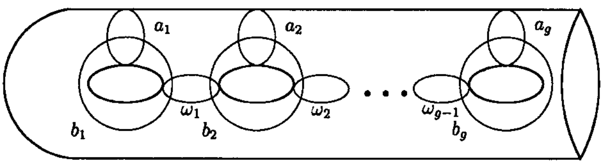

Figure 1.

DEFINITION 2.1. A *monoidal category* $(\mathcal{C}, \otimes, E)$ is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object E satisfying

(a) $\otimes$ is associative
(b) E is a two-sided unit for $\otimes$

The product $\otimes$ is usually called tensor product. E is called the unit object.

DEFINITION 2.2. A monoidal category $(\mathcal{C}, \otimes, E)$ is called a *braided monoidal category* if it has a *braiding* which means a natural family of isomorphisms

$$\beta = \{\beta_{A,B} \ : \ A\otimes B \longrightarrow B\otimes A\}$$

where A, B run over all objects of $\mathcal{C}$, such that the following diagrams commute:

$$\begin{tikzcd} A\otimes B\otimes C \arrow[rr, "\beta_{A\otimes B,C}"] \arrow[dr, "1_A\otimes\beta_{B,C}"'] & & C\otimes A\otimes B \\ & A\otimes C\otimes B \arrow[ur, "\beta_{A,C}\otimes 1_B"'] & \end{tikzcd}$$

$$
\begin{tikzcd}
A\otimes B\otimes C \arrow[rr, "\beta_{A,B\otimes C}"] \arrow[dr, "\beta_{A,B}\otimes 1_C"'] & & B\otimes C\otimes A \\
& B\otimes A\otimes C \arrow[ur, "1_B\otimes\beta_{A,C}"'] &
\end{tikzcd}
$$

The naturality of the braiding means that for any morphism $f : A \to A'$, $g : B \to B'$, we have

$$(g\otimes f) \circ \beta_{A,B} = \beta_{A',B'} \circ (f\otimes g) \ .$$

Note that the commutative diagrams of Definition 2.2 imply the Yang-Baxter equality:

$$(1_C\otimes\beta_{A,B})\circ(\beta_{A,C}\otimes 1_B)\circ(1_A\otimes\beta_{B,C}) = (\beta_{B,C}\otimes 1_A)\circ(1_B\otimes\beta_{A,C})\circ(\beta_{A,B}\otimes 1_C).$$

Let $\mathcal{C}(M) = \amalg_{g\geq 0}\Gamma_{g,1}$ be the disjoint union of mapping class groups $\Gamma_{g,1}$ for $g \geq 0$, that is, $\mathcal{C}(M)$ is a category whose objects are nonnegative integers g and morphisms are as follows:

$$\mathrm{Hom}(g,h) = \begin{cases} \Gamma_{g,1} & \text{if } g = h \\ \emptyset & \text{if } g \neq h. \end{cases}$$

It was shown in [**2**] that $\mathcal{C}(M)$ is a braided monoidal category. The tensor product of $\mathcal{C}(M)$ is defined by a certain connected sum (cf. [**6**]). More precisely, the product

$$\Gamma_{g,1} \times \Gamma_{h,1} \longrightarrow \Gamma_{g+h,1} \ , \quad (\alpha,\beta) \longmapsto \alpha\otimes\beta$$

is obtained by attaching a pair of pants, which is obtained from a sphere by removing three open disks from it, to the surfaces $S_{g,1}$ and $S_{h,1}$ along the boundary circles and extending the identity map on the boundary to the whole pants. The fundamental group of $S_{g,1}$ is isomorphic to the free group on generators $x_1, y_1, \cdots, x_g, y_g$ which are induced by the standard Dehn twists $a_1, b_1, \cdots, a_g, b_g$, respectively. $\Gamma_{g,1}$ can be identified with the subgroup of the automorphism group of $\pi_1 S_{g,1} \cong F_{\{x_1,y_1,\cdots,x_g,y_g\}}$ that consists of the automorphisms fixing the fundamental relator $R = [y_1,x_1]\cdots[y_g,x_g]$. The (g,h)-braiding $\beta_{g,h} \in \Gamma_{g+h,1}$ acts on $\{x_1, y_1, \cdots, x_{g+h}, y_{g+h}\}$ as follows:

$$\begin{array}{lll}
x_1 & \longmapsto & S x_{h+1} S^{-1} \\
y_1 & \longmapsto & S y_{h+1} S \\
 & \vdots & \\
x_g & \longmapsto & S x_{h+g} S^{-1} \\
y_g & \longmapsto & S y_{h+g} S^{-1} \\
x_{g+1} & \longmapsto & x_1 \\
y_{g+1} & \longmapsto & y_1 \\
 & \vdots & \\
x_{g+h} & \longmapsto & x_h \\
y_{g+h} & \longmapsto & y_h
\end{array}$$

where $S = [y_1, x_1][y_2, x_2] \cdots [y_h, x_h]$. It is easy to see that this braiding satisfies the diagrams of Definition 2.2.

The $(1,1)$-braiding $\beta_{1,1}$ is described in [**2**] in terms of the Dehn twists $a_1, b_1, a_2, b_2, \omega_1$ as follows:

$$\beta_{1,1} = (a_1 b_1 a_1)^4 (a_2 b_2 (a_1 b_1 a_1)^{-1} \omega_1 a_1 b_1 a_1^2 b_1)^{-3}. \tag{2.3}$$

Note that the Dehn twists a_1, b_1, a_2, b_2, ω_1 act on $\pi_1 S_{2,1} = F_{\{x_1, y_1, x_2, y_2\}}$ as follows:

$$\begin{array}{rl}
a_1: & y_1 \longmapsto y_1 x_1^{-1} \\
b_1: & x_1 \longmapsto x_1 y_1 \\
a_2: & y_2 \longmapsto y_2 x_2^{-1} \\
b_2: & x_2 \longmapsto x_2 y_2 \\
\omega_1: & x_1 \longmapsto z_1^{-1} y_2 x_2 y_2^{-1} \\
 & y_1 \longmapsto y_1 z_1 \\
 & y_2 \longmapsto z_1^{-1} y_2
\end{array} \tag{2.4}$$

where z_1 equals $x_1^{-1} y_2 x_2 y_2^{-1}$ which is induced by the Dehn twist ω_1. These automorphisms fix the generators that do not appear in the list.

Let $\beta_i = (a_i b_i a_i)^4 (a_{i+1} b_{i+1} (a_i b_i a_i)^{-1} \omega_i a_i b_i a_i^2 b_i)^{-3}$. Then the (g,h)-braiding $\beta_{g,h}$ can be expressed in terms of the standard Dehn twists as follows.

$$\beta_{g,h} = (\beta_h \beta_{h-1} \cdots \beta_1)(\beta_{h+1} \beta_h \cdots \beta_2) \cdots (\beta_{g+h-1} \cdots \beta_g) .$$

3. Ribbon category and the Dehn twist along the boundary

Let $\mathcal{C}(M) = \amalg_{g\geq 0}\Gamma_{g,1}$. In the previous section it has been shown that $\mathcal{C}(M)$ forms a braided monoidal category. In this section, we show that we can get a *ribbon category* by attaching to $\mathcal{C}(M)$ the mirror symmetry of $\mathcal{C}(M)$. The ribbon category, which plays a key role in the quantum theory and the invariants of 3-manifolds induced from it (cf. [**7**]), is a braided monoidal category equipped with some additional structures, which are twist and compatible duality.

The twist in the category $\mathcal{C}(M)$ is represented by an element of each $\Gamma_{g,1}$. In this section we describe the twist in two ways. We first show how it acts on the fundamental group of the surface. Secondly we explicitly express it in terms of the standard Dehn twists. We can also describe it in terms of Wajnryb's Dehn twists. The twist in $\Gamma_{g,1}$ turns out to be equal to the inverse of Dehn twist around the boundary of $S_{g,1}$.

DEFINITION 3.1. Let $\mathcal{C}$ be a braided monoidal category. A *twist* in $\mathcal{C}$ is a natural family of isomorphisms

$$\tau = \{\tau_A \ : \ A \longrightarrow A\}$$

where A runs over all objects of $\mathcal{C}$, such that for any two objects A, B of $\mathcal{C}$, we have

$$\tau_{A\otimes B} = \beta_{B,A} \circ \beta_{A,B} \circ (\tau_A \otimes \tau_B). \tag{3.2}$$

By the naturality of τ we mean that for any morphism $\alpha : A \to B$ in $\mathcal{C}$, we have $\tau_B \circ \alpha = \alpha \circ \tau_A$. From the naturality of τ, we get the following which is equivalent to (3.2):

$$\tau_{A\otimes B} = \beta_{B,A} \circ (\tau_B \otimes \tau_A) \circ \beta_{A,B} = (\tau_A \otimes \tau_B) \circ \beta_{B,A} \circ \beta_{A,B}\ . \tag{3.3}$$

Note that $\tau_E = 1_E$ for the unit object E.

DEFINITION 3.4. Let $\mathcal{C}$ be a monoidal category. We assume that for each object A of $\mathcal{C}$ there are an associated object A^* and two morphisms

$$e_A : A^* \otimes A \longrightarrow E, \quad ce_A : E \longrightarrow A \otimes A^* \tag{3.5}$$

which are called *evaluation* and *coevaluation*, respectively.

The rule $A \mapsto (A^*, e_A, ce_A)$ is called a *duality* in $\mathcal{C}$ if the following equalities hold:

$$(1_A \otimes e_A) \circ (ce_A \otimes 1_A) = 1_A, \tag{3.6.a}$$

$$(e_A \otimes 1_{A^*}) \circ (1_{A^*} \otimes ce_A) = 1_{A^*}. \tag{3.6.b}$$

For a braided monoidal category $\mathcal{C}$ equipped with the braiding β and the twist τ, we say that the duality in $\mathcal{C}$ is compatible with β and τ if the following equality holds for every object A of $\mathcal{C}$:

$$(\tau_A \otimes 1_{A^*}) \circ ce_A = (1_A \otimes \tau_{A^*}) \circ ce_A. \tag{3.7}$$

DEFINITION 3.8. A monoidal category $\mathcal{C}$ is called a *ribbon category* if $\mathcal{C}$ is equipped with a braiding β, a twist τ, and a compatible duality $(*, e, ce)$.

One of the fundamental examples of a ribbon category is the category of finite dimensional representations of a quantum group (cf. [**7**], chapter XI). Now we derive a ribbon category from $\mathcal{C}(M) = \amalg_{g\geq 0}\Gamma_{g,1}$ by attaching the mirror symmetry of $\mathcal{C}(M)$ to itself.

Let $\mathcal{D}(M)$ be the category which is a union of $\mathcal{C}(M)$ and its mirror symmetry $\mathcal{C}^*(M) = \amalg_{g\geq 0}\Gamma_{g^*,1}$. The objects of $\mathcal{D}(M)$ are of the form either g or g^* for a nonnegative integer g. The morphisms of $\mathcal{D}(M) = \mathcal{C}(M) \cup \mathcal{C}^*(M)$ are as follows:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}(M)}(g,h) &= \mathrm{Hom}_{\mathcal{C}(M)}(g,h)\\
\mathrm{Hom}_{\mathcal{D}(M)}(g,h^*) &= \emptyset = \mathrm{Hom}_{\mathcal{D}(M)}(g^*,h)\\
\mathrm{Hom}_{\mathcal{D}(M)}(g^*,h^*) &= \begin{cases} \Gamma_{g,1} & \text{if } g = h \\ \emptyset & \text{if } g \neq h \end{cases}.
\end{aligned}$$

$\mathrm{Hom}_{\mathcal{D}(M)}(0,0^*) = \{\phi\}$ and $\mathrm{Hom}_{\mathcal{D}(M)}(0^*,0) = \{\psi\}$, where ϕ and ψ are isomorphisms and are the inverses of each other.

The category $\mathcal{D}(M)$ is a monoidal category equipped with the tensor product as follows: Let g, h be nonnegative integers. The tensor product is defined to be commutative and defined as follows:

$$\begin{aligned}
g \otimes h &= g + h\\
g^* \otimes h &= 0^* = g \otimes h^*\\
g^* \otimes h^* &= (g+h)^* \quad \text{for } g, h > 0\\
g \otimes 0^* &= 0^* = g^* \otimes 0^*
\end{aligned}$$

for all nonnegative integers g and h.

It is easy to see that the assignment $g \mapsto g^*$, $h^* \mapsto h$ gives rise to a duality of $\mathcal{D}(M)$. For each object g the evaluation equals $\psi : g^* \otimes g = 0^* \to 0$, and for each h^* the evaluation also equals $\psi : h \otimes h^* = 0^* \to 0$. Likewise, the coevaluation of each object equals $\phi : 0 \to 0^*$. It is easy to see that these morphisms satisfy (3.6.a) and (3.6.b).

$\mathcal{D}(M)$ is a braided monoidal category with a duality. We now show that $\mathcal{D}(M)$ is a ribbon category, that is, it has a natural twist.

THEOREM 3.9. *$\mathcal{D}(M)$ is a ribbon category.*

Proof. It suffices to find a twist $\tau_g : g \to g$ for each positive integer g, since $\mathcal{C}^*(M)$ is symmetric to $\mathcal{C}(M)$. We regard $\Gamma_{g,1}$ as the subgroup of the automorphism group of the free group on $x_1, y_1, \cdots, x_g, y_g$ that consists of the automorphisms fixing the fundamental relator $R = [y_1, x_1] \cdots [y_g, x_g]$, where $x_1, y_1, \cdots, x_g, y_g$ are the generators of $\pi_1 S_{g,1}$ which are induced by the Dehn twists $a_1, b_1, \cdots, a_g, b_g$, respectively (Figure 1). Then we define τ_g to be the automorphism satisfying:

$$\begin{array}{ll} x_1 & \longmapsto R x_1 R^{-1} \\ y_1 & \longmapsto R y_1 R^{-1} \\ & \vdots \\ x_g & \longmapsto R x_g R^{-1} \\ y_g & \longmapsto R y_g R^{-1} \end{array}$$

where $R = [y_1, x_1] \cdots [y_g, x_g]$. We show that τ_g satisfies the equality (3.2), that is

$$\tau_{r+s} = \beta_{s,r}\beta_{r,s}(\tau_r \otimes \tau_s). \tag{3.10}$$

For positive integers m, n with $m < n$, let

$$R_{m,n} = [y_m, x_m] \cdots [y_n, x_n].$$

Let $x_1, y_1, \cdots, x_{r+s}, y_{r+s}$ be the generators of $\pi_1 S_{r+s,1}$. Let $R = R_{1,r+s}$ be the fundamental relator. Then we have

$$\begin{aligned} x_1 & \xmapsto{\tau_r \otimes \tau_s} R_{1,r} x_1 R_{1,r}^{-1} \\ & \xmapsto{\beta_{r,s}} R_{1,s} R_{s+1,s+r} R_{1,s}^{-1} R_{1,s} x_{s+1} R_{1,s}^{-1} R_{1,s} R_{s+1,s+r}^{-1} R_{1,s}^{-1} = R x_{s+1} R^{-1} \\ & \xmapsto{\beta_{s,r}} R x_1 R^{-1}. \end{aligned}$$

Similarly, we have

$$y_1 \mapsto Ry_1R^{-1}, \cdots, x_r \mapsto Rx_rR^{-1}, y_r \mapsto Ry_rR^{-1}.$$

For $x_{r+1}, y_{r+1}, \cdots, x_{r+s}, y_{r+s}$, we have

$$\begin{aligned} x_{r+1} &\xmapsto{\tau_r \otimes \tau_s} R_{r+1,r+s}x_{r+1}R_{r+1,r+s}^{-1} \\ &\xmapsto{\beta_{r,s}} R_{1,s}x_1R_{1,s}^{-1} \\ &\xmapsto{\beta_{s,r}} R_{1,r}R_{r+1,r+s}R_{1,r}^{-1}R_{1,r}x_{r+1}R_{1,r}^{-1}R_{1,r}R_{r+1,r+s}^{-1}R_{1,r}^{-1} = Rx_{r+1}R^{-1}. \end{aligned}$$

Similarly, we have

$$y_{r+1} \mapsto Ry_{r+1}R^{-1}, \cdots, x_{r+s} \mapsto Rx_{r+s}R^{-1}, y_{r+s} \mapsto Ry_{r+s}R^{-1}.$$

Hence the equality (3.10) holds. □

We now find the expression of the twist $\tau_g : g \to g$ in terms of the standard Dehn twists of Figure 1. We use the fact that the inverse of the twist in $\Gamma_{g,1}$ defined above equals the Dehn twist along the fixed boundary of the surface of genus g which acts on $\pi_1 S_{g,1}$ as $x_1 \mapsto R^{-1}x_1R, y_1 \mapsto R^{-1}y_1R, \cdots, x_g \mapsto R^{-1}x_gR, y_g \mapsto R^{-1}y_gR$. Let $(BD)_g$ be the Dehn twist along the fixed boundary of the surface $S_{g,1}$. We obtain the expression of $(BD)_g$ in terms of the standard Dehn twists using the tools of the classical paper of Lickorish ([5]).

We will first find $(BD)_1$ and $(BD)_2$, and then find the expression of $(BD)_g$ for any natural number g. Let a, b, c be the Dehn twists along the simple closed paths α, β, γ in Figure 2.

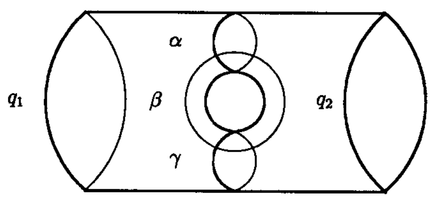

Figure 2.

Let q_1, q_2 be the circles which are two boundary components of the surface in Figure 2. Let h_{q_1} and h_{q_2} be the Dehn twists along q_1 and q_2, respectively. Then we have the following:

LEMMA 3.11 ([5], LEMMA 3). *Let $f = acbacb^2acbac$. Then h_{q_1} is isotopic to $h_{q_2}^{-1}f$.*

Note that $ac = ca$. Consider the surface $S_{g,1}$.

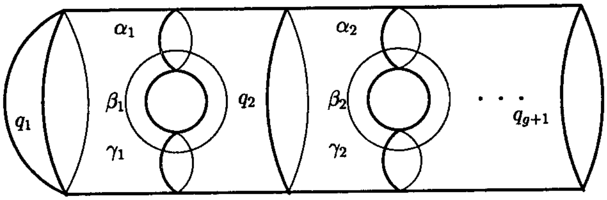

Figure 3.

The simple closed curves $\alpha_i, \beta_i, \gamma_i$ induce the Dehn twists a_i, b_i, c_i for $i = 1, \cdots, g$. If $g = 1$, we have $h_{q_2} \simeq (BD)_1$, where $\simeq$ means isotopic. Hence we have

$$(BD)_1 \simeq a_1^2 b_1 a_1^2 b_1^2 a_1^2 b_1 a_1^2 \tag{3.12}$$

since $a_1 \simeq c_1$ and $h_{q_1} \simeq 1$.

Let $f_i = c_i a_i b_i c_i a_i b_i^2 c_i a_i b_i c_i a_i$ for $i = 1, \cdots, g$. For $(BD)_2$, we first have $h_{q_2} \simeq h_{q_3}^{-1} f_2$. Since $(BD)_2 \simeq h_{q_3}$ and $f_1 \simeq h_{q_2}$, we have $(BD)_2 \simeq f_2 f_1^{-1}$. We should express c_i in terms of the standard Dehn twists of Figure 1.

Lemma 3.13 ([5], Lemma 5). *For $i = 1, \cdots, g$, we have*

$$c_i \simeq g_i^{-1} a_i g_i$$

where $g_i = b_i \omega_{i-1} b_{i-1} a_{i-1}^2 c_{i-1} b_{i-1} \omega_{i-1} b_i$.

From Lemma 3.13 we get the following.

Lemma 3.14. *The Dehn twist along the boundary for genus 2 is isotopic to*

$$(a_1 b_1 \omega_1)^4 b_2 (a_1 b_1 \omega_1)^4 b_2^2 (a_1 b_1 \omega_1)^4 b_2 (a_1 b_1 \omega_1)^4 (a_1^2 b_1 a_1^2 b_1^2 a_1^2 b_1 a_1^2)^{-1}. \tag{3.15}$$

Proof. Since $(BD)_2 \simeq f_2 f_1^{-1}$, from Lemma 3.13 we have

$$(BD)_2 \simeq (c_2 a_2 b_2 c_2 a_2 b_2^2 c_2 a_2 b_2 c_2 a_2)(a_1^2 b_1 a_1^2 b_1^2 a_1^2 b_1 a_1^2)^{-1} \tag{3.16}$$

where $c_2 = (b_2 \omega_1 b_1 a_1^2 b_1 \omega_1 b_2)^{-1} a_2 (b_2 \omega_1 b_1 a_1^2 b_1 \omega_1 b_2)$. We have the following equality of Wajnryb ([8]):

$$a_2 c_2 = a_2 (b_2 \omega_1 b_1 a_1^2 b_1 \omega_1 b_2)^{-1} a_2 (b_2 \omega b_1 a_1^2 b_1 \omega_1 b_2) = (a_1 b_1 \omega_1)^4.$$

Using this equality, we get the formula (3.15). Note that f_1 commutes with f_2. Hence we get the alternative expression of $(BD)_2$:

$$(BD)_2 \simeq (a_1^2b_1a_1^2b_1^2a_1^2b_1a_1^2)^{-1}(a_1b_1\omega_1)^4b_2(a_1b_1\omega_1)^4b_2^2(a_1b_1\omega_1)^4b_2(a_1b_1\omega_1)^4.$$

□

It is easy to check that the Dehn twist of (3.15) acts on the generators x_1, y_1, x_2, y_2 of $\pi_1 S_{2,1}$ as described in the proof of Theorem 3.9, that is, $x_1 \mapsto R^{-1}x_1R$, $y_1 \mapsto R^{-1}y_1R$, $x_2 \mapsto R^{-1}x_2R$, $y_2 \mapsto R^{-1}y_2R$, where $R = [y_1, x_1][y_2, x_2]$.

For an arbitrary genus $g (g \geq 3)$, we have the following:

THEOREM 3.17. *In $\Gamma_{g,1}$, the Dehn twist along the boundary of the surface $S_{g,1}$ equals*

$$(BD)_g \simeq f_g^{(-1)^g} \cdots f_3^{-1} f_2 f_1^{-1} \tag{3.18}$$

where

$$f_i = c_ia_ib_ic_ia_ib_i^2c_ia_ib_ic_ia_i \quad \text{for } i = 1, \cdots, g,$$

and

$$c_j = (b_j\omega_{j-1}b_{j-1}a_{j-1}^2c_{j-1}b_{j-1}\omega_{j-1}b_j)^{-1}a_j(b_j\omega_{j-1}b_{j-1}a_{j-1}^2c_{j-1}b_{j-1}\omega_{j-1}b_j)$$

for $j = 2, \cdots, g$.

Note that $(BD)_g$ equals the Dehn twist along the curve q_{g+1} (See Figure 3). We can get the expression of $(BD)_g$ in terms of the standard Dehn twists of Figure 1 by the inductive process. Moreover, we get another expression of (3.18) in terms of Wajnryb's Dehn twists $a_1, a_2, b_1, \cdots, b_g, \omega_1, \cdots, \omega_{g-1}$, since we have the following equality ([**3**]):

$$a_{i+2} = h_i^{-1}a_ih_i$$

where $h_i = b_i\omega_ib_{i+1}a_{i+1}\omega_{i+1}b_{i+2}b_{i+1}\omega_{i+1}\omega_ib_{i+1}b_i\omega_ia_{i+1}b_{i+1}\omega_{i+1}b_{i+2}$. We again note that τ_g^{-1} equals $(BD)_g$.

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