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ON SINGULAR PLANE QUARTICS AS LIMITS OF SMOOTH CURVES OF GENUS THREE

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ABSTRACT. We compute $\lim_{t\to 0} C_t$ in $\overline{\mathcal{M}}_3$ of some family $\pi : \mathcal{C} \to \Delta$ of plane quartics whose general members are nonsingular.

1. Introduction

We consider a family of nonsingular plane quartics $\pi : \mathcal{C} \to \Delta^*$ over the punctured open disk $\Delta^* = \Delta - \{0\}$ of $\mathbb{C}$ degenerating to a singular plane quartic C_0. This family gives a morphism ϕ from Δ^* to the moduli space $\mathcal{M}_3$ of genus 3 smooth curves which extends uniquely to $\phi : \Delta \to \overline{\mathcal{M}}_3$ from Δ to the Deligne-Mumford compactification $\overline{\mathcal{M}}_3$ of $\mathcal{M}_3$ parametrizing of all genus three stable curves. The $\phi(0)$ is called the stable limit of $\{C_t\}_{t\in\Delta^*}$, or of a family $\pi : \mathcal{C} \to \Delta^*$, as $t \to 0$. The stable reduction theorem ([1, 11]) enables us to compute it. Stable limits depend on C_0 as well as a family $\{C_t\}$ degenerating to C_0. As mentioned in [6], various smoothings may produce many different stable limits.

In this paper we study the picture between C_0 and $\phi(0)$ where $\pi : \mathcal{C} \to \Delta^*$ is a generic smoothing of C_0 (section 3 and 4). We call this $\phi(0)$ the stable limit of the generic smoothing of C_0. Note that generic smoothing produces a very special stable limit of C_0. It is in general a difficult problem to find all possible stable limits of C_0.

One can find a nice introduction to stable reductions and stable limits in the recently published book ([6]). Brendan Hassett has recently made substantial progress on stable limits and related problems ([7, 8]). He has also studied the stable limits of the types in 2.4 in [7]. I

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This work was originally motivated by the attempt to construct a non-trivial one dimensional complete family of plane quartics $\pi : \mathcal{C} \to B$ which gives a morphism $\phi : B \to \mathcal{M}_3$ ([5, 12]).

In section 2, we explain the stable reduction process. In section 3 and 4, we compute stable limits of reduced and nonreduced quartics from their generic smoothings, respectively. We work over the field $\mathbb{C}$ of complex numbers.

2. Stable reduction

(2.1) DEFINITIONS. A semistable curve is a connected nodal, possibly reducible reduced curve with no smooth rational components meeting other components at less than two points, and a stable curve is a semistable curve without smooth rational components meeting other components at less than three points. By the genus $g(C)$ of an irreducible curve C we mean the geometric genus, the genus of its normalization. If C is an irreducible curve in a smooth surface S, then $g(C) = g_a(C) - \sum_{P\in C} \delta(P)$ where $g_a(C)$ is the arithmetic genus of C. Here $\delta(P)$ can be computed as follows [9]. Let

$$S_{n+1} \to S_n \to \cdots \to S_1 \to S_0 = S$$

be the sequence of blow-ups obtained to desingularize C at P, $f_i : S_i \to S_{i-1}$ a blow-up of S_{i-1} at a singular point of C_{i-1} which lies over P, C_i the proper transform of C_{i-1} under f_i and C_{n+1} smooth at all points lying over P, then

$$\delta(P) = \sum_{i=0}^{n} \sum_{P_{ij}\in f_{(i)}^{-1}(P)} \frac{m_{C_i}(P_{ij})(m_{C_i}(P_{ij})-1)}{2}$$

where $f_{(i)} = f_1 \circ f_2 \circ \cdots \circ f_i$ and $m_{C_i}(P_{ij})$ is the multiplicity of C_i at P_{ij}. If C is an irreducible plane curve of degree d, then $g_a(C) = \frac{(d-1)(d-2)}{2}$. The genus of a stable curve C (or a connected nodal curve) is its arithmetic genus here:

$$g(C) = \sum_{i=1}^{n} g_i + \delta - n + 1$$

if C has δ nodes and n irreducible components $C_1, C_2, \cdots, C_n$ of geometric genera $g_1, g_2, \cdots, g_n$ ([6]).

(2.2) We study the Euclidean algorithm of the greatest common divisor $d = (p, q)$ of two integers p and q. Assume that $p \leq q$. If we put $s_{-1} = q,\ s_0 = p$, then the Euclidean algorithm is

(2.2.1) $s_{i-1} = s_i r_{i+1} + s_{i+1},\ 0 \leq s_{i+1} < s_i,\ s_{k+1} = 0$ for $0 \leq i \leq k$.

Note that $r_{k+1} \geq 2$ if $k \geq 1$. Then $d = (p, q) = s_k$. Define two sequences $\{p_i\},\ \{q_i\}$ by

(2.2.2) $p_{-1} = 0,\ p_0 = 1, \cdots, p_i = p_{i-2} + p_{i-1} r_i$ for $1 \leq i \leq k+1$

(2.2.3) $q_{-1} = 1,\ q_0 = 0, \cdots, q_i = q_{i-2} + q_{i-1} r_i$ for $1 \leq i \leq k+1$.

Note that $p_1 = r_1,\ p_2 = 1 + r_2,\ q_1 = 1,\ q_2 = r_2$, etc. Then

(2.2.4) $s_i = (-1)^i (pp_i - qq_i)$ for $-1 \leq i \leq k+1$.

The case for $i = -1$ or 0 can be checked easily. Assuming (2.2.4) holds up to i, $s_{i+1} = s_{i-1} - s_i r_{i+1} = (-1)^{i-1}(pp_{i-1} - qq_{i-1}) - (-1)^i (pp_i - qq_i) r_{i+1} = (-1)^{i-1}(pp_{i+1} - qq_{i+1}) = s_{i+1}$. We introduce some formulas that will be needed in the stable reduction process of a curve with an isolated singular points of type $y^p = x^q$ and similar types.

(2.2.5) $r_{i+1}(qq_i + s_i) + pp_{i-1} = pp_{i+1}$ for $0 \leq i \leq k$ and i even

(2.2.6) $r_{i+1}(pp_i + s_i) + qq_{i-1} = qq_{i+1}$ for $0 \leq i \leq k$ and i odd

(2.2.7) $p_{k+1} = q/d, \qquad q_{k+1} = p/d.$

The formulas (2.2.5) and (2.2.6) can be proved easily by substituting (2.2.4). For (2.2.7), we use induction on k where $d = s_k$. It is trivial if

$k=0$. Let $s_i = \tilde{s}_{i-1}$. Then the equations (2.2.1) for $1 \le i \le k$ can be written as

$$\tilde{s}_{i-1} = \tilde{s}_i \tilde{r}_{i+1} + \tilde{s}_{i+1},\ 0 \le \tilde{s}_{i+1} < \tilde{s}_i,\ \tilde{s}_k = 0, \quad \text{for } 0 \le i \le k-1.$$

Let s'_i and p'_i are defined as p_i in (2.2.2) and as q_i in (2.2.3) respectively. Then by induction on i, $s'_i = q_{i+1}$. Now induction on k implies that $s'_k = p/d$, so that $p/d = q_{k+1}$. Since $0 = s_{k+1} = (-1)^k(pp_{k+1} - qq_{k+1})$, $p_{k+1} = q/d$ implies that $q_{k+1} = p/d$.

In the below we list some properties of p_i and q_i which are used in 2.4. We omit the proofs which follow from the induction and substitutions. For $0 \le i \le k+1$

$$\begin{aligned}
&p_i q_{i-1} - p_{i-1} q_i = (-1)^i \\
&(pp_i + q_i, pp_{i-1} + q_{i-1}) = 1 \\
&(qq_i + p_i, qq_{i-1} + p_{i-1}) = 1 \\
&(pp_i + p_i + q_i, pp_{i-1} + p_{-i} + q_{i-1}) = 1 \\
&(qq_i + q_i, qq_{i-1} + q_{i-1}) = q + 1 \\
&(pp_i + p_i, pp_{i-1} + p_{i-1}) = p + 1.
\end{aligned}$$

(2.3) STABLE REDUCTION PROCESS. Let $\pi : \mathcal{C} \to \Delta^*$ be a flat family of smooth curves of genus $g \ge 2$ degenerating to C_0. To compute the stable limit of a given family we apply the well known stable reduction process as follows. One may refer to section 3C of [6] or section 1 of [2]. If the total surface $\mathcal{C}$ is smooth at the singular point of C_0 we blow up $\mathcal{C}$ at the singular points of C_0 until we have a nodal curve over $t=0$. We still write the resulting family as $\pi : \mathcal{C} \to \Delta$. To remove the multiple components of C_0, we take a finite number of base changes (of the total order N, the least common multiple of the multiplicity of components of C_0) followed by the normalization of the total surface obtained from each base change. Then we have over $t=0$ a reduced curve with at most nodes. Blowing down smooth rational components of self intersection number -1 we get a semistable curve with the total surface smooth. This is the semistable reduction theorem. Then contracting smooth rational curves which meet other components less than three points we get a stable curve over $t=0$.

This is the stable limit of a family $\pi : \mathcal{C} \to \Delta^*$ degenerating to C_0. If $\mathcal{C}$ is not smooth at some points of C_0 we first desingularize $\mathcal{C}$ and then do the same process as before.

For 2.4 we introduce some terminologies. Let $f : X \to Y$ be a finite morphism of curves of degree n. We say that f is totally ramified(branched) at $P \in X$(at $Q \in Y$, resp.) if $f^*(Q) = nP$. If $f^*(Q) = \frac{n}{k}(P_1 + P_2 + \cdots + P_k)$, we say that f is evenly k-ramified at $Q \in Y$. We say that f is completely branched at a divisor D of Y if f is totally branched at all points $Q \in D$ with no other branch points. Similar terminology will be used for a finite morphism of surfaces.

(2.4) LEMMA. *Let $\pi : \mathcal{C} \to \Delta^*$ be a flat family of smooth projective curves of genus $g \geq 2$ degenerating to an irreducible curve C_0 with only one singular point P analytically equivalent to*

(a) $y^p = x^q$;
(b) $xy^p = x^{q+1}$;
(c) $y^{p+1} = x^q y$;
(d) $xy^{p+1} = x^{q+1}y$

where $1 < p \leq q$ in (a) *and $1 < p < q$ except* (a). *Suppose that the total surface $\mathcal{C}$ is smooth. Then the stable limit of each family $\pi : \mathcal{C} \to \Delta^*$ is as follows if $g(C_0) > 0$. We use the notations in 2.2 and write $p = p'd,\ q = q'd$.*

(a) *A union of the normalization $\bar{C}$ of C_0 and a smooth curve $\bar{E}$ of genus $(pq - p - q - d + 2)/2$ which meets $\bar{C}$ at d points lying over P. If $q = p$, then $\bar{E}$ is p to 1 cover of $\mathbb{P}^1$ totally ramified at p points that intersect $\bar{C}$. If $q = pr$ and $r > 1$, then $\bar{E}$ is q to 1 cover of $\mathbb{P}^1$ which is totally ramified at p points that intersect $\bar{C}$, and evenly p-ramified at one more branch point. If $p = 2$ and $q = 2r$, $\bar{E}$ is unique that is isomorphic to the normalization of $y^2 - 1 = x^q$. In general $\bar{E}$ can be visualized as a pq/d-fold cover of $\mathbb{P}^1$ which is totally ramified at d points that intersect $\bar{C}$, and has two more branch points at which it is evenly q-ramified and evenly p-ramified respectively. The branch points here are uniquely determined algebraically by the d branches of C at P. Moreover $\bar{E}$ is isomorphic to the normalization of the plane curve $z^{q(p'-q_k)}y^{pp_k}g(y,z) = x^{pq'}$ for odd integers k and to $z^{p(q'-p_k)}y^{qq_k}g(y,z) = x^{pq'}$ for even integers k, where $g(y,z)$ is a homogeneous polynomial of degree d with zeroes exactly at d totally* *branched points coming from d branches of C at P. If $(p,q)=1$, $\bar{E}$ is also isomorphic to the normalization of plane curve $y^p = x^q + 1$.*

(b) *A union of $\bar{C}$ and a smooth curve $\bar{E}$ of genus $(pq+p-q-d)/2$ which meet $\bar{C}$ at $d+1$ points lying over P. $\bar{E}$ is a $(pq+p)/d$-fold cover of $\mathbb{P}^1$ which is totally ramified at $d+1$ points that intersect $\bar{C}$ and evenly $(q+1)$-ramified at one more branch point (if $k=0$, the last branch point does not appear). Here $\bar{E}$ is isomorphic to the normalization of the plane curve $z^{(q+1)(p'-q_k)}y^{pp_k+q_k}g(y,z) = x^{p'(q+1)}$ for odd integers k and to $z^{p'(q+1)-(pp_k+q_k)}y^{qq_k+q_k}g(y,z) = x^{p'(q+1)}$ for even integers k, where $g(y,z)$ is a homogeneous polynomial of degree d with zeroes exactly at d totally branched points coming from d branches of C analytically equivalent to $y^p = x^q$ (also in* (c) and (d)*). If $p=2$ and $q=2r$, then $\bar{E}$ is isomorphic to the normalization to $y^2-1=x^{q+1}$.*

(c) *A union of $\bar{C}$ and a smooth curve $\bar{E}$ of genus $(pq-p+q-d)/2$ which meet $\bar{C}$ at $d+1$ points lying over P. $\bar{E}$ is a $(pq+q)/d$-fold cover of $\mathbb{P}^1$ totally ramified at $d+1$ points that intersect $\bar{C}$ and evenly $(p+1)$-ramified at one more branch point. Here $\bar{E}$ is isomorphic to the normalization of the plane curve $z^{q'(p+1)-(qq_k+p_k)}y^{pp_k+p_k}g(y,z) = x^{q'(p+1)}$ for odd integers k and to $z^{(p+1)(q'-p_k)}y^{qq_k+p_k}g(y,z) = x^{q'(p+1)}$ for even integers k.*

(d) *A union of $\bar{C}$ and a smooth curve $\bar{E}$ of genus $(pq+p+q-d)/2$ where $\bar{E}$ and $\bar{C}$ meet at $d+2$ points lying over P. $\bar{E}$ can be visualized as $(pq+p+q)/d$-fold cover of $\mathbb{P}^1$ completely ramified at $d+2$ intersection points with $\bar{C}$. Here $\bar{E}$ is isomorphic to the normalization of the plane curve $z^{(pq'+p'+q')-(qq_k+p_k+q_k)}y^{pp_k+p_k+q_k}g(y,z) = x^{pq'+p'+q'}$ for odd integers k and to $z^{(pq'+p'+q')-(pp_k+p_k+q_k)}y^{qq_k+p_k+q_k}g(y,z) = x^{pq'+p'+q'}$ for even integers k.*

Note that 2.4 is still valid when generic fibers are stable curves. We sometimes use affine equations since they are simple even though we mean projective curves.

Proof. We fix some notations. Let $\mathcal{C}_0 = \mathcal{C}$ and $P = P_0$. Let $f_i : \mathcal{C}_i \to \mathcal{C}_{i-1}$ be the blow-up of $\mathcal{C}_{i-1}$ at P_{i-1}, E_i the exceptional divisor, C_i the proper transform of C_{i-1} (therefore of C), $\pi_i = \pi \circ f_1 \circ f_2 \circ \cdots \circ f_i$, $Z_i =$ the central fiber over $t=0$ of $\pi_i : \mathcal{C}_i \to \Delta$, and $P_i =$ non-nodal singular point of Z_i if exists. Then $m(E_i) = m_{P_{i-1}}(Z_{i-1})$. Here $m(E_i)$ is the multiplicity of the components E_i and $m_{P_{i-1}}(Z_{i-1})$ is the multiplicity of Z_{i-1} at P_{i-1}. Write $(i) = r_1 + \cdots + r_i$ in subindex. Then on $\mathcal{C}_{(k+1)}$ which will be explained soon, the central fiber $Z_{(k+1)}$ is a nodal curve. Remember that r_i, p_i, q_i, s_i are defined in 2.2. Let $\bar{f} : \overline{\mathcal{C}} \to \mathcal{C}_{(k+1)}$ be the family obtained after the base change of order $m(E_{(k+1)})$ of $\mathcal{C}_{(k+1)}$ and a normalization. Call $\bar{Z}$ the central fiber of $\overline{\mathcal{C}}$ and $\bar{E} = \bar{f}^{-1}(E_{(k+1)})$.

(a) Since the total surface of the deformation space is smooth, C_1 and E_1 are given by $y^p = x^{p(r_1-1)+s_1}$ (by $y^p = 1$ if $q = p$) and $x = 0$ with $m(E_1) = p$ by the abuse of coordinates x and y. If $q = p$, then Z_1 is a nodal curve consisting of E_1 of multiplicity p and $\bar{C}$ which meet at p distinct points. Take a base change of order p to $\mathcal{C}_1 \to \Delta$. Call $\overline{\mathcal{C}}$ the total surface after a normalization. Then $\overline{\mathcal{C}}$ admits a p to 1 map to $\mathcal{C}$ completely branched on $\bar{C}$. (For, if some component is given locally by x^l where $(l,p) = 1$. Then the base change of order p changes the local equation by $x^l = t^p$ which is locally irreducible. So, it remains irreducible locally after normalization. Or, see ([6] p. 125) since we can divide any base change as a composition of base changes of prime order.) Therefore the inverse image $\bar{E}$ of E_1 is p to 1 cover of E_1 completely branched at p points: by Riemann-Hurwitz, $g(\bar{E}) = (p-1)(p-2)/2$. If $q = pr_1$ and $r_1 > 1$, then go to $\mathcal{C}_{r_1+\cdots+r_k}$ in the below where $k = 0$ and $E_{r_0} = \emptyset$.

We now assume that $k \geq 1$. Then C_1 and E_1 are not tangent at P_1 unless $r_1 = 1$. We now continuously blow up until we get $\mathcal{C}_{r_1}$. On $\mathcal{C}_{r_1}$, only two components E_{r_1} and C_{r_1} given by x and $y^p = x^{s_1}$ respectively pass tangentially at P_{r_1}. See *Figure* 1 with $E_{r_0} = \emptyset$. Note that only one more component E_{r_1-1} meets E_{r_1} transversely away from P_{r_1}. Note that

$$m(E_{r_1}) = r_1 p = pp_1.$$

Similarly on $\mathcal{C}_{(i)}$ for $i \geq 2$(see *Figure* 1), $E_{(i)}$, $E_{(i-1)}$ and $C_{(i)}$ are given by y, x, $y^{s_i} = x^{s_{i-1}}$ (or $y^{s_{i-1}} = x^{s_i}$) where

$$m(E_{r_1+r_2}) = r_2(m(E_{r_1}) + s_2) = r_2(pr_1 + s_1) = qr_2 = qq_1$$
$$m(E_{(i)-1}) = (r_i - 1)(m(E_{(i-1)}) + s_{i-1}) + m(E_{(i-2)})$$
$$m(E_{(i)}) = r_i(m(E_{(i-1)}) + s_{i-1}) + m(E_{(i-2)})$$

Claim. For $1 \leq i \leq k+1$,

$$m(E_{(i)}) = \begin{cases} pp_i & \text{for an odd integer } i \\ qq_i & \text{for an even integer } i. \end{cases}$$

The claim holds for $i = 1$. Assuming that it holds for all integer less than i,

$$\begin{aligned}&m(E_{(i)})\\=&r_i(m(E_{(i-1)}) + s_{i-1}) + m(E_{(i-2)})\\=&r_i(pp_{i-1} + s_{i-1}) + qq_{i-2}\\=&r_i(pp_{i-1} + qq_{i-1} - pp_{i-1}) + qq_{i-2} = qq_i\end{aligned}$$

if i is an even number. It can be proved similarly for odd numbers i.

On $\mathcal{C}_{(k)}$, $C_{(k)}$ at $P_{(k)}$ is given by $y^{s_k} - x^{s_{k-1}} = \prod_{l=1}^{d=s_k}(y - \xi^l x^{r_{k+1}}) = 0$ where ξ is a primitive d-th root of 1. So each branch of $C_{(k)}$ is smooth at $P_{(k)}$ with intersection number r_{k+1} with $E_{(k)}$. To separate these branches from $E_{(k)}$, we need to blow up $\mathcal{C}_{(k)}$ r_{k+1} times again. Finally on $\mathcal{C}_{(k+1)}$, the central fiber $Z_{(k+1)}$ is a nodal curve in *Figure* 2.

Now we need base changes to remove the multiple components of $Z_{(k+1)}$. Note that

$m(E_{(k+1)}) = (pp_{k+1} \text{ or } qq_{k+1}) = pq' = p'q$

$m(E_{(k+1)-1}) = p'q - m(E_{(k)}) - s_k = \begin{Bmatrix} q(p' - q_k) \text{ for odd } k \\ p(q' - p_k) \text{ for even } k \end{Bmatrix}.$

$m(E_{(i)+l}) = \begin{Bmatrix} lpp_i + pp_{i-1} \text{ for even } i \\ lqq_i + qq_{i-1} \text{ for odd } i \end{Bmatrix}$ for $1 \le l \le r_{i+1},\ 0 \le i \le k.$

Note if i is even and $1 \le l \le r_{i+1}$, then $E_{(i)+l}$ lies between E_1 and $E_{(k+1)}$:

$$(m(E_{(i)+l}), m(E_{(i)+l+1})) = p = (m(E_{(i+1)}), m(E_{(i+2)+l})).$$

If i is odd and $1 \le l \le r_{i+1}$, then $E_{(i)+l}$ lies between E_{r_1+1} and $E_{(k+1)}$:

$$(m(E_{(i)+l}), m(E_{(i)+l+1})) = q = (m(E_{(i+1)}), m(E_{(i+2)+l})).$$

Let E_1° and $E_{r_1+1}^\circ$ be the chain of exceptional curves up to $E_{(k+1)-1}$ from E_1 and E_{r_1+1} respectively and let E_1^e and $E_{r_1+1}^e$ be the components of E_1° and $E_{r_1+1}^\circ$ which meet $E_{(k+1)}$ respectively.

Next step is to take a base change of order of the least common multiple of all components of $Z_{(k+1)}$ to remove the multiple components of $Z_{(k+1)}$. We always use Riemann-Hurwitz formula to compute $g(X)$

of a finite morphism $f : X \to Y$ of curves. Remember that X is rational if $f : X \to \mathbb{P}^1$ is completely branched at two points. We take base changes of order d, p' and then q' continuously if $d > 1$ (if $d = 1$, base changes of order p and q automatically). Call the corresponding normalized surfaces $\mathcal{C}'$, $\mathcal{C}^{(1)}$, $\overline{\mathcal{C}}$ with the central fibers Z', $Z^{(1)}$, $\bar{Z}$. Then the degree d map from $\mathcal{C}^{(1)} \to \mathcal{C}_{(k+1)}$ is completely branched only on $\bar{C}$ where E_1° and $E_{r_1+1}^\circ$ split into d copies of multiplicity divided by d: if $m(E) = dm$, then the local equation of E after a base change of order d is $x^{dm} - t^d = \prod_{\xi:\xi^d=1}(x^m - \xi t)$. The degree p' map $\mathcal{C}^{(1)} \to \mathcal{C}'$ is totally branched on $\bar{C}$ and on d copies of $E_{r_1+1}^e$, and having p' copies of multiplicity divided again by p' over each d copies of E_1°, but having the same configuration as in E_{r+1}° over each of d copies of E_{r+1}° except multiplicity. The reason* for final argument is that if m is any prime divisor of p and the multiplicity of some component E in each copy of E_{r+1}° is a multiple of m, then two components meeting E have multiplicity prime to m, so these two components are branched and Riemann-Hurwitz keep the genus of E zero. Put $f^{(1)} : \mathcal{C}^{(1)} \to \mathcal{C}_{(k+1)}$ the degree p map that is the composition of degree d and degree p' maps in the above. Then similarly $f^{(2)} : \overline{\mathcal{C}} \to \mathcal{C}^{(1)}$ is branched on $\bar{C}$ and on p' copies of E_1^e, and is divided into q' copies over each of d copies of $E_{r_1+1}^e$. So, the curve $\bar{E}$ in $\overline{\mathcal{C}}$ is a pq/d-fold cover of $E_{(k+1)} \cong \mathbb{P}^1$ which is totally ramified at d points that intersect $\bar{C}$, evenly q-ramified over, say, ∞ and evenly p-ramified over, say, 0, and no other branch points. By Riemann-Hurwitz formula, we obtain $g(\bar{E}) = (pq-p-q-d+2)/2$. If one remembers $m(E_{(k+1)}) = m(E_{(k+1)-1}) + m(E_{(k)}) + d$ and the $g.c.d.$ of these multiplicities, the normalization of the plane curve given by

$$z^{m(E_{(k+1)-1})}y^{m(E_{(k)})}(y^d - z^d) = x^{m(E_{(k+1)})}$$

is one of such curves if $\bar{C}$ meets $E_{(k+1)}$ at the points satisfying $y^d = 1$. Due to the multiplicity of each components and $*$, all components except $\bar{C}$ and $\bar{E}$ keep the genus and intersection points, so are contracted after appropriate sequence of base changes. Uniqueness of the stable limit of a given family and the ramification information of $\bar{E}$ we have obtained in the proof should give the plane curve model of $\bar{E}$.

If $q = pr$, then $\bar{E}$ is degree q cover of $\mathbb{P}^1$ totally branched at p points and evenly p-ramified at only one more branch point (since $E_{(0)} = \emptyset$), so it is uniquely determined by the branch points. So, if $p = 2$ and

$q = 2r$, then it is always isomorphic to $x^q = y^2 - 1$ by fixing three branch points 1, -1, and ∞ up to $Aut(\mathbb{P}^1)$.

If $(p, q) = 1$, then $m(E_{(k+1)}) = pq$. So, if we take a base change of order q (here the inverse $(f^{(1)})^{-1}(E_{(k+1)})$ of $E_{(k+1)}$ keeps the genus 0 since it is branched at 2 points) and then a base change of order p, $\bar{E}$ becomes a degree p cyclic cover of $\mathbb{P}^1$ completely branched at $q+1$ points. On the other hand, if we take a base change of order p and then a base change of order q, $\bar{E}$ is a degree q cover of $\mathbb{P}^1$ completely branched at $p+1$ points. Then it is isomorphic to $y^p = x^q + 1$. For, the curve D which has a cyclic g_p^1 completely branched at $q+1$ points is realized as a plane curve

$$y^p = f(x) = x^q + a_{q-1}x^{q-1} + \cdots + a_1 x + a_0$$

where $f(x)$ has q distinct zeroes , since such curve is uniquely determined by the branch points and $(p, q) = 1$. Note that it is smooth except $(0 : 1 : 0)$. Now for D to have a g_q^1 completely branched at $p+1$ points it should have flex points (x_0, y_0) of highest order q with parallel tangent lines (in affine space). So, the Taylor expansion of D at (x_0, y_0) should be $y^p - \beta = (x - x_0)^q$ where $\beta = f(x_0)$. Note $\beta \neq 0$ since D is smooth except $(0 : 1 : 0)$. Now the projection from $(1 : 0 : 0)$ to the line $x = x_0$ gives a g_q^1 completely branched at $p+1$ points. Since $y^p - \beta = (x - x_0)^q$ is projectively equivalent to $y^p = x^q + 1$, we are done.

From the sequence of blow-ups at P, we see that

$$\delta(y^p = x^q) = \sum_{j=0}^{k} r_{j+1} \frac{s_j(s_j - 1)}{2} = \frac{pq - p - q + d}{2}.$$

This makes sure from the stable reduction theorem that all components over $t = 0$ in $\overline{C}$ except $\bar{E}$ and $\bar{C}$ should be contracted after an appropriate sequence of base changes from the formula of genus of stable curves. The same is true for the remaining cases, so it is enough to pay attention to the components which meet $E_{(k+1)}$.

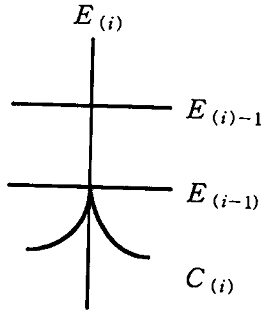
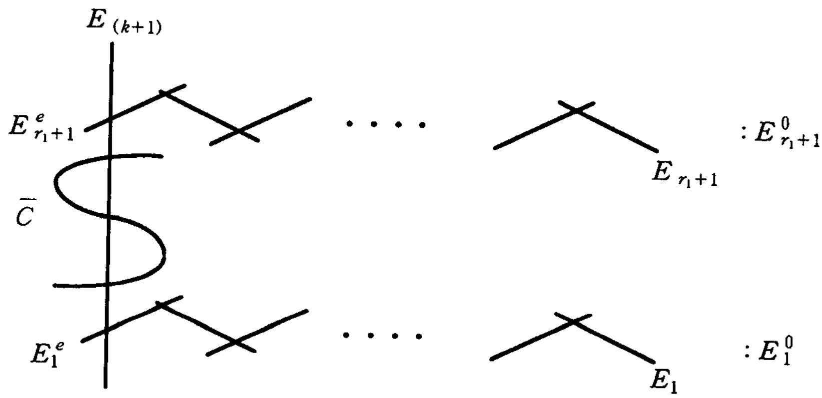


Figure 1



$$\begin{cases} E^e_1 = E_{(k)} \\ E^e_{r_1+1} = E_{(k+1)-1} \end{cases} k: \text{ odd}; \qquad \begin{cases} E^e_1 = E_{(k+1)-1} \\ E^e_{r_1+1} = E_{(k)} \end{cases} k: \text{ even}$$

Figure 2

(b) Again the configuration of E_i on $\mathcal{C}_{(k+1)}$ is same as in the case $y^p = x^q$ except that the multiplicity of each component is changed and that $\bar{C}$ meets E_1 at one point. Similarly as in (a), we have

$$m(E_{(i)}) = \begin{cases} pp_i + q_i \text{ for odd } i \\ qq_i + q_i \text{ for even } i. \end{cases}$$

$$m(E_{(k+1)}) = p'(q+1)$$

$$m(E_{(k+1)-1}) = \begin{cases} (q+1)(q_{k+1} - q_k) \text{ for odd } k \\ p'(q+1) - (pp_k + q_k) \text{ for even } k \end{cases}$$

$$m(E_{(i)+l}) = \left\{ \begin{array}{c} l(pp_i + q_i) + pp_{i-1} + q_{i-1} \text{ for even } i \\ l(qq_i + q_i) + qq_{i-1} + q_{i-1} \text{ for odd } i \end{array} \right\}$$

$$\text{for } 1 \leq l \leq r_{i+1},\ 0 \leq i \leq k.$$

Note that for $1 \leq l \leq r_{i+1} - 1$
$(m(E_{(i)+l}), m(E_{(i)+l+1})) = (m(E_{(i+1)}), m(E_{(i+2)+1})) = 1$ for even i
$(m(E_{(i)+l}), m(E_{(i)+l+1})) = (m(E_{(i+1)}), m(E_{(i+2)+1})) = q+1$ for odd i.
In particular, $(m(E_{(k+1)}), m(E_1^e)) = 1$. We now take base changes of order $q+1$ and then p' continuously and call the corresponding normalized surfaces $\mathcal{C}^{(1)}$, $\overline{\mathcal{C}}$. Then the degree $q+1$ map $f^{(1)} : \mathcal{C}^{(1)} \to \mathcal{C}_{(k+1)}$ is totally branched over $\bar{C}$ and E_1^e, while $E_{r_1+1}^\circ$ split into $q+1$ copies of multiplicity divided by $q+1$. The degree p' map $f^{(2)} : \overline{\mathcal{C}} \to \mathcal{C}^{(1)}$ is totally branched on $\bar{C}$, E_1^e and on the copies of $E_{r_1+1}^e$, while keeping the configuration and genus as in E_1° and $E_{r_1+1}^\circ$ as explained in (a). So, the curve $\bar{E}$ over $E_{(k+1)}$ is a $p'(q+1)$-fold cover of $E_{(k+1)} \cong \mathbb{P}^1$ which is totally ramified at $d+1$ points and evenly $q+1$-ramified over ∞. By Riemann-Hurwitz formula, we obtain $g(\bar{E}) = (pq+p-q-d)/2$. If one remembers $m(E_{(k+1)}) = m(E_{(k+1)-1}) + m(E_{(k)}) + d$ and *g.c.d.* of these multiplicities, one finds that $\bar{E}$ is isomorphic to the normalization of the plane curve given by

$$z^{m(E_{(k+1)-1})} y^{m(E_{(k)})} (y^d - z^d) = x^{m(E_{(k+1)})}$$

if $\bar{C}$ meets $E_{(k+1)}$ at the points satisfying $y^d = 1$. As in (a) all components except $\bar{C}$ and $E^{(2)}$ will be contracted after appropriate sequence of base changes. As E_1° contracted, we have one more intersection point between $\bar{C}$ and $\bar{E}$. If $(p,q) = 1$ here, $\bar{E}$ becomes a p-fold cover of $\mathbb{P}^1$ completely branched at $q+3$ points too. We also have

$$2\delta(xy^p = x^{q+1}) = pq + p - q + d.$$

(c) We may assume that $k \geq 1$, since $y(y^p - x^{pr})$ is analytically equivalent to $y^{p+1} - x^{(p+1)r}$. Again the configuration of $Z_{(k+1)}$ on $\mathcal{C}_{(k+1)}$ is same except the multiplicity and that $\bar{C}$ meets E_{r_1+1} at one point. Here

$$\begin{aligned} m(E_{(i)}) &= \begin{cases} pp_i + p_i \text{ for odd } i \\ qq_i + p_i \text{ for even } i \end{cases} \\ m(E_{(k+1)}) &= q'(p+1) \\ m(E_{(k+1)-1}) &= \begin{cases} (p+1)q' - (qq_k + p_k) \text{ for odd } k \\ (p+1)(p_{k+1} - p_k) \text{ for even } k \end{cases} \\ m(E_{(i)+l}) &= \left\{ \begin{matrix} l(pp_i + p_i) + pp_{i-1} + p_{i-1} \text{ for even } i \\ l(qq_i + p_i) + qq_{i-1} + p_{i-1} \text{ for odd } i \end{matrix} \right\} \\ &\quad \text{for } 1 \leq l \leq r_{i+1},\ 0 \leq i \leq k. \end{aligned}$$

Note that $(m(E_{(k+1)}), m(E^e_{r_1+1})) = 1$. By exchanging the roles of E°_1 and $E^\circ_{r_1+1}$ as well as p and q, we obtains (c). Here

$$2\delta(y^{p+1} = x^q y) = pq - p + q + d.$$

If $(p,q) = 1$, then $\bar{E}$ becomes a q-fold cover of $\mathbb{P}^1$ completely branched at $p+3$ points too.

(d) The similar argument can be applied here too. But $\bar{C}$ meets both E_1 and E_{r_1+1}. Since

$$\begin{aligned} m(E_{(i)}) &= \begin{cases} pp_i + p_i + q_i \text{ for odd } i \\ qq_i + p_i + q_i \text{ for even } i \end{cases} \\ m(E_{(k+1)}) &= pq' + p' + q' \\ m(E_{(k+1)-1}) &= \begin{cases} pq' + p' + q' - (qq_k + p_k + q_k) \text{ for odd } k \\ pq' + p' + q' - (pp_k + p_k + q_k) \text{ for even } k, \end{cases} \end{aligned}$$

all adjacent two components have relatively prime multiplicity. So if we take a base change of order $pq' + p' + q'$, then on $\overline{C}$ the curve $\bar{E}$ over $E_{(k+1)}$ becomes a $(pq' + p' + q')$-fold cover of $E_{(k+1)}$ completely branched at $d+2$ points while preserving the configurations and genus of two chains E_1° and E_{r+1}°. Now all components except $\bar{C}$ and $\bar{E}$ will be contracted after appropriate sequence of base changes so that $\bar{C}$ and $\bar{E}$ meet at $d+2$ points. Again $\bar{E}$ is uniquely determined once $d+2$ branch points are fixed. So, it is isomorphic to the normalization of the plane curve

$$z^{m(E_{(k+1)-1})}y^{m(E_{(k)})}g(y,z) = x^{m(E_{(k+1)})}$$

where $g(y,z)$ is a polynomial of degree d uniquely determined by d branches of C at P. Here

$$2\delta(xy^{p+1} = x^{q+1}y) = pq + p + q + d + 2. \qquad \square$$

REMARK. The referee pointed to the author that the tail in (a) when $p=2$ and $q=2r+1$ is in fact isomorphic to the normalization of $y^2z^{2r-1} = x^{2r+1} + z^{2r+1}$. He also suggested to prove 2.4 more generally as appeared here.

A point of a curve which is analytically equivalent to $y^2 = x^{2n}$ is called a node, a tacnode, or an oscnode if $n = 1$, 2, or 3 respectively. A unibranch singular point of a curve which is analytically equivalent to $y^2 = x^{2n+1}$ is called a cusp, a ramphoid cusp, or a keratoid cusp if $n = 1$, 2, or 3 respectively. The point analytically equivalent to $y^3 = x^4$ is called an ordinary cusp of multiplicity three. The complement of $\bar{C}$ in the stable limit is called as a tail ([6]). From now on we mean by $\bar{C}$ the partial normalization of C, i.e.,the normalization except nodes.

(2.5) COROLLARY. *Under the same assumption as in 2.4 except that P is an isolated (possibly reducible, but reduced) plane quartic singularity, the tail T of the stable limit of $C = C_0$ is as follows.*

(a) *If P is a cusp, then T is an elliptic curve of j-invariant 0.*

(b) *If P is a tacnode, then T is an elliptic curve isomorphic to the normalization of $y^2 = x^4 + 1$ while meeting $\bar{C}$ at two points. So, $j(T) = 1728$.*

(c) *If P is an ordinary triple point, then T is an elliptic curve of $j(T) = 0$ meeting $\bar{C}$ at three points.*

(d) *If P is a ramphoid cusp, then T is a genus 2 curve isomorphic to the normalization of $y^2 = x^5 + 1$.*

(e) *If P is a keratoid cusp, then T is a genus 3 curve isomorphic to the normalization of $y^2 = x^7 + 1$.*

(f) *If P is a oscnode, then T is a genus 2 curve which is isomorphic to the normalization of $y^2 = x^6 + 1$ while meeting $\bar{C}$ at two points.*

(g) *If P is a cusp with a smooth branch, then T is a genus 2 curve which is isomorphic to the normalization of $y^4 - y^3 = x^8$ while meeting $\bar{C}$ at two points.*

(h) *If P is an ordinary cusp of multiplicity 3, then T is a genus 3 smooth curve that is isomorphic to $y^3 = x^4 + 1$.*

(i) *If P is analytically equivalent to $x(y^2 - x^4)$, then T is a genus 2 curve isomorphic to the normalization of $y^2 = x^5 + 1$ while meeting $\bar{C}$ at two points.*

(j) *If P is analytically equivalent to $y(y^2 - x^3)$, then T is a genus 3 smooth curve isomorphic to $y^4 - y^3 = x^9$, which is trigonal totally ramified at 5 points, so it is isomorphic to a smooth plane quartic.*

(k) *If P is analytically equivalent to $y^2 - x^8$, then T is a genus 3 curve isomorphic to the normalization of $y^2 = x^8 + 1$ while meeting $\bar{C}$ at two points.*

(l) *If C is given by $xy(x-y)(ax-by)$, then T is a genus 3 curve isomorphic to the normalization of $xy(x-y)(ax-by) = z^4$, therefore the stable limit is T.*

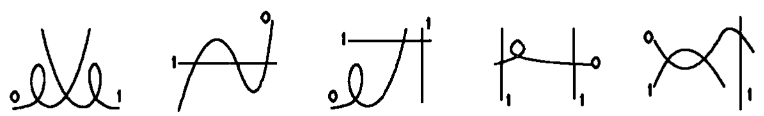
Moreover the point(s) of attachment of the tail in each case is (are) the totally ramified point(s) of some g^1_d as explained in 2.4. For example, the point of attachment of (d) is a fixed point for the hyperbolic involution as well as a ramification point for g^1_5 of $y^2 = x^5 + 1$ up to automorphisms.

Proof. All are the special cases of 2.4. In (a) and (c) $j(T) = 0$ since they are completely branched trigonal elliptic curves by 2.4(a). In (b), T is isomorphic to $y^2 = x^4 + 1$, so $j(T) = 1728$. For, the automorphism of $\mathbb{P}^1$ sending three branch points $e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}$ to $0, 1, \infty$ sends the remaining branch point $e^{7\pi i/4}$ to -1. $\square$

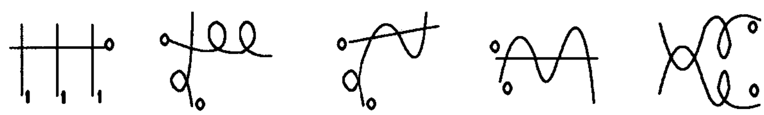
(2.6) The equisingular types of $\overline{\mathcal{M}}_3 - \mathcal{M}_3$. There are 41 equisingular types in the boundary of $\overline{\mathcal{M}}_3$ and the codimension of each equisingular types is the number of nodes of its general member. In *Figure* 3 we borrow all drawings from [3]. All numbers in *Figure* 3 are the genus of the corresponding components.

Δ1a Δ1b Δ2a Δ2b Δ2c

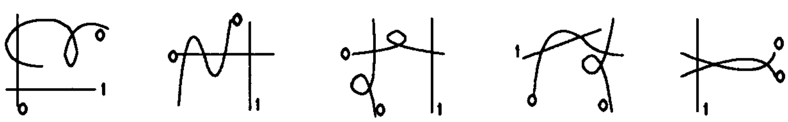
Δ2d Δ2e Δ3a Δ3b Δ3c



Δ3d Δ3e Δ3f Δ3g Δ3h



Δ3i Δ4a Δ4b Δ4c Δ4d



Δ4e Δ4f Δ4g Δ4h Δ4i

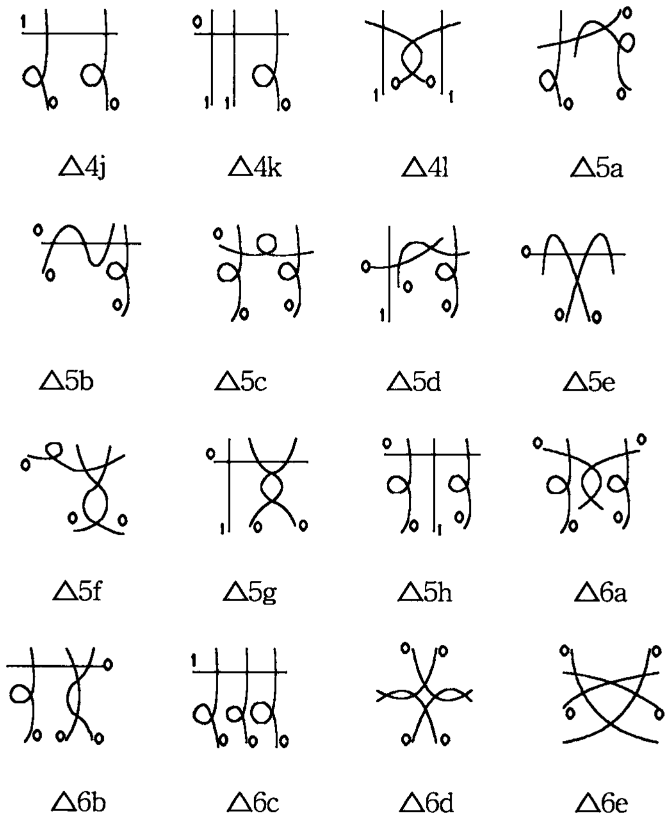


Figure 3

3. Reduced singular quartics

Let a family $\pi : \mathcal{C} \to \Delta^*$ of plane quartics be a generic smoothing of singular plane quartic C_0. We mean by a generic smoothing of C_0 a family obtained by taking small neighborhood of C_0 of the intersection of general 13 hypersurfaces of $\mathbb{P}^{14}$, the projective space parametrizing all plane quartics. In this section $C = C_0$ is a reduced singular quartic, then the total surface C can be chosen generically to be smooth at the singular points of C_0. We now determine the stable limit of each reduced singular quartic from its generic smoothing in 3.1 and 3.2.

(3.1) THEOREM. *The following is the list of equisingular types of stable limits of all irreducible singular quartics from their generic smoothings. For the special types of components of the tail, refer to* 2.5.

(C1a) *A quartic with one node is itself a stable curve* $\Delta 1a$.
(C2a) *A quartic with two nodes is itself a stable curve* $\Delta 2a$.
(C2b) *The stable limit of a quartic with one cusp is a curve* $\Delta 1b$.
(C3a) *A quartic with three nodes is itself a stable curve* $\Delta 3c$.
(C3b) *The stable limit of a quartic with a cusp and a node is a curve* $\Delta 2c$.
(C3c) *The stable limit of a quartic with a tacnode is a curve* $\Delta 2d$.
(C4a) *The stable limit of a quartic with a cusp and two nodes is* $\Delta 3b$.
(C4b) *The stable limit of a quartic with a tacnode and a node is* $\Delta 3c$.
(C4c) *The stable limit of a quartic with an ordinary triple point is* $\Delta 3e$.
(C4d) *The stable limit of a quartic with two cusps is a curve* $\Delta 2e$.
(C4e) *The stable limit of a quartic with a ramphoid cusp is a curve* $\Delta 1b$.
(C5a) *The stable limit of a quartic with two cusps and a node is* $\Delta 3g$.
(C5b) *The stable limit of a quartic with a cusp and a tacnode is a curve* $\Delta 3h$.
(C5c) *The stable limit of a quartic with a ramphoid cusp and a node is* $\Delta 2b$.
(C5d) *The stable limit of a quartic with an oscnode is a curve* $\Delta 1a$.
(C5e) *The stable limit of a quartic with a cusp with a smooth branch is* $\Delta 1a$.
(C6a) *The stable limit of a quartic with three cusps is a curve* $\Delta 3i$.
(C6b) *The stable limit of a quartic with a cusp and a ramphoid cusp is* $\Delta 1b$.
(C6c) *The stable limit of a quartic with a keratoid cusp is a smooth curve of genus three.*
(C6d) *The stable limit of a quartic with an ordinary cusp of multiplicity three is a smooth curve of genus three.*

Proof. From the genus formula of an irreducible plane curve, a quartic has at most three singular points. By fixing three points in $\mathbb{P}^2$ one can figure out equations of quartics listed above from which one can find out the codimensions too. We now show that the above is a complete list of all irreducible plane quartics. From the genus formula, $\delta(P) \leq 3$. Note that $\delta(P) = 1$ if and only if P is a node or a cusp. If $\delta(P) = 2$, then P is a double point and a point P' over P must have $\delta(P') = 1$. So P' is either a node or a cusp which implies P is either a tacnode or a ramphoid cusp. If $\delta(P) = 3$ and P is a double point of C, then $\delta(P') = 2$ where P' is a point over P. So, P is an oscnode or a keratoid cusp. If $\delta(P) = 3$ and P is a triple point of C, then we should have $\delta(P') = 0$ for any point P' over P. So, they are smooth. From what we have done, we cannot have tangential branches. Therefore P is an ordinary triple point (if it has three local branches), an ordinary cusp with a smooth branch (if it has two local branch) or an ordinary cusp of multiplicity three (if it is unibranch). Since $\sum_{P\in C} \delta(P) \leq 3$, a little combinatorics shows that the above list is complete.

Since the total surface of our family is smooth, we apply 2.5 to each singular point of C. By contracting the partial normalization $\bar{C}$ if it is smooth, rational, and meets the other components at less than three points, we get the corresponding stable curve. By a partial normalization we mean the desingularization of C except nodes of C.□

(3.2) THEOREM. *The following is the list of all reducible and reduced singular quartics and singular types of their corresponding stable limits from generic smoothings. For the special types of the components of the tail, see 2.5.*

(C3d) *A quartic of a cubic and a line is itself in $\Delta 3e$.*
(C4f) *A cubic plus a tangent line becomes a curve in $\Delta 2d$.*
(C4g) *A nodal cubic plus a line is itself a curve in $\Delta 4b$.*
(C4h) *A quartic of two conics is a stable curve in $\Delta 4c$.*
(C5f) *A cubic plus a flex line becomes a curve in $\Delta 1b$.*
(C5g) *A nodal cubic plus a tangent line becomes a curve in $\Delta 3c$.*
(C5h) *A nodal cubic plus a line through a node becomes a curve in $\Delta 3e$.*
(C5i) *A cuspidal cubic plus a line becomes a curve in $\Delta 4f$.*
(C5j) *Two conics meeting tangentially at one point become a curve*

in $\Delta 4i$.

(C5k) *A conic plus two lines is a stable curve in* $\Delta 5d$.

(C6e) *A nodal cubic plus a flex line becomes a curve in* $\Delta 2b$.

(C6f) *A nodal cubic plus a tangent line at a node becomes a curve in* $\Delta 1a$.

(C6g) *A cuspidal cubic plus a tangent line becomes a curve in* $\Delta 3h$.

(C6h) *A cuspidal cubic plus a line through a cusp becomes a curve in* $\Delta 1a$.

(C6i) *Two conics meeting tangentially at two points become a curve in* $\Delta 2d$.

(C6j) *A conic, a line plus a tangent line to a conic becomes a curve in* $\Delta 4i$.

(C6k) *A conic plus two lines which intersect on a conic becomes a curve in* $\Delta 3e$.

(C6l) *Four distinct lines become a stable curve of* $\Delta 6e$.

(C7a) *A cuspidal cubic plus a flex line becomes a curve in* $\Delta 1b$.

(C7b) *A cuspidal cubic plus the tangent line at a cusp becomes a smooth curve of genus three.*

(C7c) *Two conics with an intersection multiplicity 3 at one point become a curve in* $\Delta 1a$.

(C7d) *A conic plus two tangent lines becomes a curve in* $\Delta 2d$.

(C7e) *A conic, a line plus a tangent line through an intersection point of former two components becomes a curve in* $\Delta 1a$.

(C7f) *A line plus three concurrent lines becomes a curve in* $\Delta 3e$.

(C8a) *Two conics meeting only one point become a smooth curve.*

(C8b) *Four concurrent lines become smooth quartics.*

In 3.1 and 3.2, the numbering is chosen to emphasize the codimension of each equisingular stratum in $\mathbb{P}^{14}$. For example, C7 means the codimension seven in $\mathbb{P}^{14}$. In 3.2, all components meet transversely unless specified.

Proof. We first show that the above is a full list. Note that a singular irreducible cubic has one node or one cusp and that any cubic has a flex. By Bezout, the intersection number of a line and a cubic is three. These intersection points could be smooth or singular points of a cubic, and they meet transversely, tangentially, or meet at the flex point of a cubic. By Bezout's theorem and a little combinatorics will give the full list above. Remember that the total surface of a generic smoothing of C is smooth. So we apply 2.5 to each singular point of C and contract, if necessary, some smooth rational components of the partial normalization of C to get the stable limit. Here one has to be careful in chasing the different components of C. The singularities correspond as follows: C5f, C6e, C7a, C7c-oscnode; C6h-2.5(g); C6f and C7e-2.5(i); C7b-2.5(j); C8a-2.5(k); C8b-2.5(l). Others are easy to tell. Now all follow by applying 2.5. □

4. Non-reduced singular quartics

In this section we assume that C is a singular quartic with multiple components. Let $f(x,y,z)$ and
$F(x,y,z,t) = f(x,y,z) + tg_1(x,y,z) + t^2g_2(x,y,z) + t^3g_3(x,y,z) + \cdots$
be equations of C and $\mathcal{C}$, where $\pi : \mathcal{C} \to \Delta$ is a generic smoothing of $C = C_0$. Suppose that $f = h^n k, n = 2, 3$ or 4, and h, k with no multiple components. Then

$$\frac{\partial F}{\partial x} = nh^{n-1}\frac{\partial h}{\partial x}k + h^n\frac{\partial k}{\partial x} + t[\cdots] \quad \frac{\partial F}{\partial y} = nh^{n-1}\frac{\partial h}{\partial y}k + h^n\frac{\partial k}{\partial y} + t[\cdots]$$
$$\frac{\partial F}{\partial z} = nh^{n-1}\frac{\partial h}{\partial z}k + h^n\frac{\partial k}{\partial z} + t[\cdots] \quad \frac{\partial F}{\partial t} = g_1 + t[\cdots].$$

There are no common zeroes of the first three equations since C_t is a nonsingular quartic for $t \neq 0$. When $t = 0$, the above four equations are always zero at the common zeroes of h and g_1. Let S be the set of the isolated singular points of $\mathcal{C}$ and the intersection points of h and k. Since generic homogeneous polynomials miss the finite set S and meet the component of C defined by h transversely, we have

(4.1) PROPOSITION. *If C is a non-reduced plane quartic and $\pi : \mathcal{C} \to \Delta$ is an one dimensional generic smoothing of C, then $\mathcal{C}$ has $4deg(h)$ isolated singular points of types $x^n = yt$ for suitable coordinates on the multiple component of C of multiplicity n.*

(4.2) LEMMA. *Let X be a surface given by $x^n = yt$ in $\mathbb{C}^3$. Then the singular point $(0,0,0)$ of X can be resolved by $[\frac{n}{2}]$ times of blow-ups. In the nonsingular model, the divisor $t = 0$ defines a nodal curve consisting of the proper transform of the original curve $t = 0$ plus*

$n-1$ exceptional curves with multiplicity n, $n-1$, $n-2$, $\cdots$ and 1 respectively.

Proof. The singular point is type A_{n-1}. So, it is well known. For the multiplicity of each component of the divisor $t=0$, examine the divisor $(t=0)$ at each step of blow-up (for example, see [4]). $\square$

(4.3) THEOREM. *The following is the list of the stable limits of non-reduced plane quartics from their generic smoothings.*

(1) *Every hyperelliptic smooth curve of genus 3 is the stable limit of a double conic.*
(2) *Every hyperelliptic genus 2 curve with one node is the stable limit of a conic plus a double line.*
(3) *A union of any two elliptic curves meeting at two points is the stable limit of two double lines.*
(4) *A union of two elliptic curves meeting at two points is the stable limit of conic plus a tangential double line where j-invariant of one elliptic component is 0.*
(5) *Every hyperelliptic genus 2 curve with one node is the stable limit of two lines plus a double line.*
(6) *A union of two elliptic curves meeting at two points is the stable limit of three concurrent lines one of which is a double line where j-invariant of one elliptic component is 1728.*
(7) *Every trigonal curve of genus 3 that is completely branched at 5 points is the stable limit of a triple line plus a line. Such trigonal curve of genus 3 is isomorphic to the plane quartic $y^3=x(x-1)(x-\alpha)(x-\beta)$ for some α, $\beta\neq 0,1$.*
(8) *Every 4-gonal curve of genus 3 that is completely branched at 4 points is the stable limit of a quadruple line. Such a 4-gonal curve of genus 3 is isomorphic to the plane quartic $x(x-1)(x-\alpha)=y^4$ for some $\alpha\neq 0,1$.*

Proof. Since all singularities of C are of type A_n we obtain as in (4.2) the smoothing $\tilde{p}:\tilde{C}\to\Delta$ of $p:C\to\Delta$ and new central fiber $\tilde{p}^{-1}(0)$.

(1) Since C has eight singular points of type $u^2=tv$ on double conic, $\tilde{p}^{-1}(0)$ is a rational double curve with eight rational tails. A base change of order two and normalizations (and contractions of eight

tails) replace the central fiber with a smooth hyperelliptic curve of genus three. In this way we obtain all hyperelliptic curves of genus 3 depending on the eight singular points of C, so on the choice of generic smoothing. Fixing three branch points up to $\mathbb{P}^1$, the remaining 5 branch points determine the curve. In the remaining proof we do not mention this kind of argument since it is clear from the description.

(2) Here C has four singular points of type A_2 on the double line of C. So, $\tilde{p}^{-1}(0)$ is a union of C and four rational tails on a double line L. Note that the conic component of C meets L at two points. A base change of order two followed by a normalization replaces L with a genus 2 curve. After contractions of rational components including conic part we obtain an irreducible curve with one node.

(3) Here C has 8 singular points of type $u^2 = tv$ each four of which are on each double line. Then $\tilde{p}^{-1}(0)$ is a union of two rational double curves L_1, L_2 which meet at one point with 8 exceptional curves in the place of 8 singular points of C. Now we take a base change of order two. Let C' be the surface we get after the base change and normalization. Then there is a 2 to 1 map from C' to $\widetilde{C}$ branched along the eight exceptional curves. The curve L_i' over L_i is a double cover over L_i branched at four points for $i = 1, 2$: Riemann-Hurwitz says that $g(L_i') = 1$. Since the intersection point of L_1 and L_2 is not branched, there are two points over it. Thus L_1' and L_2' meet at two points. Since eight rational tails has self-intersection number -1, all are contracted.

(4) Call Q a conic and L a line part of C. Here C has four singular points of type $u^2 = tv$ on a double line. Therefore the fiber over $t = 0$ of $\tilde{p} : \widetilde{C} \to \Delta$ consists of C and four exceptional curves E_i on 4 singular points of C. We now blow up the total surface two times at the intersection point P of L and Q until the divisor $t = 0$ becomes a nodal curve. It consists of 6-tuple exceptional curve which meets each of three rational curves of multiplicity 3, 2 and 1 respectively at one point, where the multiplicity 2 component L has 4 rational tails E_i. If we take a base change of order 3 followed by a normalization, we obtain a curve over $t = 0$ of two rational double curves meeting at one point each of which has 4 rational tails which is $\tilde{p}^{-1}(0)$ in (3). If one takes a base change of order 2 first, one can show that one elliptic curve has j invariant 0.

(5) Since $\mathcal{C}$ has four singular points on a double line of type $u^2 = tv$, we obtain four more rational curves passing through these four singular points as well as components of C. After a base change of order two, a normalization (then the double components is replaced as 2 to 1 cover of $\mathbb{P}^1$ branched at 6 points) and contractions of rational curves of negative self intersections, we get an irreducible curve of genus two with one node.

(6) Call two simple lines L_1, L_2 and a double line D. Again $\mathcal{C}$ has four singular points on a double line of type $u^2 = tv$. So, we obtain four exceptional curves passing through these four singular points besides C over $t = 0$. Blow up $\widetilde{\mathcal{C}}$ at the intersection point P of three components of C. Then on the quadruple exceptional curve E three components of C are separated. If we take a base change of order 2 and a normalization, we have 2 to 1 map to $\widetilde{\mathcal{C}}$ completely branched on L_1, L_2 and 4 simple exceptional curves. The curve D' over D is a degree two cover over D branched at four points: $g(D') = 1$. The curve E' over E is a double cover branched at two points: $g(E') = 0$. And D' and E' meet at two points since the intersection points of D and E is not a branched point. One more base change of order two followed by a normalization will replace E' with an elliptic curve E''. Usual contractions gives two elliptic curves meeting two points. Here E'' admits g_4^1 totally ramified at 2 points and 2-evenly ramified at the remaining branch point, which is studied in 2.5(b). So, $j(E'') = 1728$. The other elliptic component can have any j-invariant depending on the choice of a smoothing.

(7) If C is a triple line M and a line L, we have four singular points of $\mathcal{C}$ on a triple line of type $u^3 = tv$. On $\widetilde{\mathcal{C}}$ we have four chains of exceptional curves as explained in 4.2 as well as L passing through M at one point. A usual base change of order three replaces M by a triple cover over a rational curve completely branched at 5 points: by Riemann-Hurwitz $g = 3$. Now standard semistable reduction process gives a smooth curve of genus three as a stable limit. By fixing three branch points at 0, 1, ∞, we obtain the equation.

(8) Here the singularities are of type A_4. So, the central fiber of $\widetilde{\mathcal{C}}$ is quadruple line plus 4 chains of exceptional curves as in 2.5. A base change of order 4 replace quadruple cover over a rational curve completely branched at 4 points:by Riemann-Hurwitz $g = 3$. Again,

standard semistable reduction process gives a smooth curve of genus three as a stable limit. By fixing three branch points at 0, 1, ∞, we obtain the equation. $\square$

The stable limit of a reduced plane quartic from a generic smoothing only depends on C, in particular the local equation at P. On the other hand, the stable limit of a non-reduced plane quartic depends on the choice of the equation of generic smoothing as well as C. Brendan Hassett gives the author an example of a plane quartic which has smooth limit from very special smoothing even if it does not admit smooth stable limit from the generic smoothings.

From what we have done, one can see

(4.4) COROLLARY ([10]). *Only eight equisingular types of plane quartics have smooth stable limits from the pencil of their generic smoothings. They are C6c, C6d, C7b, C8a, C8b, and the cases* $1, 7, 8$ *in* 4.3, *none of which are general members of* $\mathcal{M}_3$.

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